WIENER INDEX OF PLANAR MAPS

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ABSTRACT

In trees with n vertices, the Wiener index of tree is minimized by stars and maximized by paths, both uniquely. In this paper, we give an inequality similar in the case of planar maps.

Keywords: *Wiener index, maps, trees*

1. INTRODUCTION

In this paper we will present some useful definitions related to our work as follows : an undirected graph G is a triplet (V, E, δ) where V is the set of vertices of the graph, E is the set of edges of the graph and δ is the application $\delta : E \rightarrow P(\upsilon)$, $e_i \rightarrow \delta(e_i) = \{\upsilon_i, \upsilon_k\}$ with υ_i and υ_k are end vertices of the edge e_i. We notice that the set $\{\upsilon_j, \upsilon_k\}$ as a multiset (if $\upsilon_j = \upsilon_k$, the same vertex appears twice in $(\delta(e_i))$. A loop is an edge $e_i \in E$ with $\upsilon_j = \upsilon_k$, if $\delta(e_i) = \delta(e_j)$ with $i \neq j$ then the edges e_i and e_j are called multiple. A graph which contains neither multiple edges nor loops is called a simple graph.

The degree of a vertex υ noted deg(υ) is the number of edges incident to it. The sum of the degrees of all vertices of a graph is equal to twice the number of its edges i.e. $\sum_{\upsilon \in V} \deg(\upsilon) = 2|E|$.

In a graph G, a path is a sequence of vertices and edges $p = \upsilon_0, e_1, \upsilon_1, e_2, \ldots, \upsilon_{n-1}, e_n, \upsilon_n$ such that $\delta(e_i) = \{\upsilon_{i-1}, \upsilon_i\}$. We say that this path attached both ends υ_0 and υ_n. A cycle is a path such that $\upsilon_0 = \upsilon_n$. A graph G is called connected if any two of its vertices may be connected by a path [1]-[4]-[5].

The distance between two distinct vertices υ_i and υ_j of a graph G, denoted by $d(\upsilon_i, \upsilon_j)$ is equal to the length of (number of edges in) the shortest path that connects υ_i and υ_j. Conventionally, $d(\upsilon_i, \upsilon_j) = 0$ [7]. We define a complete vertex by the vertex υ_0 such that $d(u, \upsilon_0) = 1$ for each $u \in V$. In a complete graph all the vertices are completes

The graphs that we consider are in most cases connected but may contain multiple edges.

A map C is a graph drawn on a surface X or embedded into it (that is, a compact variety orientable 2-dimensional) in such a way that:

- The vertices of graph are represented as distinct points of the surface.
- The edges are represented as curves on the surface that intersect only at the vertices.
- If we cut the surface along the graph thus drawn, what remains (that is, the set X \ C) is a disjoint union of connected components, called faces, each homeomorphic to an open disk (for more information on the faces of a map see [4]).

A planar map is a map drawn on the plane. We define a simple map as a map without loops and without multiple edges. In all the following, map means a planar map, connected and simple.

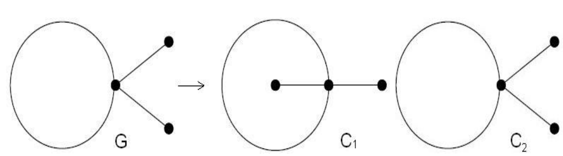

Fig. 1. One graph gives two planar maps

A tree is a connected graph without cycle. A plan tree is a tree designed on the plane [4]. The Wiener index of a connected graph is the sum of distances between all pairs of vertices [2], [3], [6], the Wiener index of a connected graph G is defined as:

$$W(G) = \sum_{\{v_i,v_j\} \subseteq V(G)} d(v_i, v_j)$$

We define W (u, G) (Wiener index of vertex u in G) the sum of distances of vertex u to each vertex of vertices of G i.e.

$$W(u,G) = \sum_{u \in V(C)} d(u,v)$$

Let T_n a tree with n vertices, then the Wiener index W (T_n) is minimized by that of the star tree with n vertices and maximized by that of the path with n vertices [5]. The goal of this work is to give an inequality similar in the case of maps.

2. CALCULATION OF THE WIENER INDEX OF MAPS

In the same way as graphs, we define the Wiener index for maps as follows:

$$W(u,C) = \sum_{u \in V(C)} d(u,v)$$

We notice that:

$$W(C) = \frac{1}{2} \sum_{u \in V(C)} \sum_{v \in V(C)} d(u,v) = \frac{1}{2} \sum_{u \in V(C)} W(u,C)$$

Example 1. Let C_5 be a map with |V|=5, we have: deg(v_4) = 4, deg(v_2) = 2, W(v_4,C_5) = 4, W(v_2, C_5) = 6, v_1 and v_4 are complete vertices (see Fig 2).

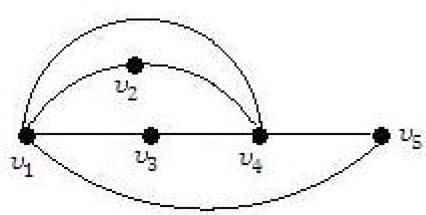

Fig. 2. An example of map C_5

Example 2. In the maps E_2, E_3 and E_4 the vertices are all completes (we say that the map is complete) (see Fig 3).

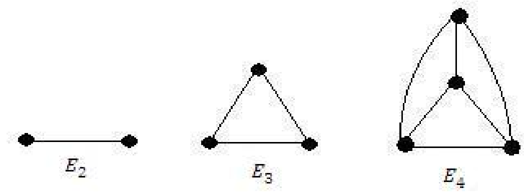

Fig. 3. The maps E_2, E_3 and E_4

Let C be a map and let e be an edge of map C (e ∈ E(C)), we denote by C − e the map obtained after deleting the edge e of the map C and the resulting map is connected.

Lemma 1. Let C be a map and let e_1, e_2 be two edges of map C that connect the vertices v_1 and v_2 (multiple edges of C), then

$$W(C - e_i) = W(C), \quad i = 1, 2$$

where C− e_i is the map obtained by deleting the edge e_i of the map C.

We notice that delete a multiple edge does not affect the Wiener index; through this paper, we consider only the simple maps (without loops and without multiple edges).

Lemma 2. Let C_n be a simple map with n vertices and let υ be a complete vertex of C, then

$$\deg(v) = n - 1 \text{ and } W(v, C_n) = n - 1$$

Lemma 3. Let C_n be a simple map with n vertices (n ≥ 2) and let υ a vertex not complete of a map C, then

$$W(v, C_n) \geq n$$

Remark 1.

1. Let C be a simple map and let υ a vertex of map C_n, then W(υ, C_n) ≥ n-1.

2. Let C_n be a simple map with n vertices, e be an edge of C_n and let $C_n - e$ be the map obtained by deleting the edge e such that the map $C_n - e$ remains connected, then W($C_n - e$) ≥ W(C_n).

Theorem 1. Let C_n be a simple map with n vertices, then

$$W(C_n) \geq \frac{n(n-1)}{2}$$

Proof: Let C_n be a map with n vertices and let υ be a vertex of C_n. From Remark 1, we have $W(\upsilon, C_n) \geq n-1$

$$W(C_n) = \frac{1}{2}\sum_{\upsilon \in V(C_n)} W(\upsilon, C_n)$$
$$\geq \frac{1}{2}\sum_{\upsilon \in V(C_n)} (n-1)$$
$$\geq \frac{1}{2}(n-1)\sum_{\upsilon \in V(C_n)} 1$$
$$\geq \frac{n(n-1)}{2}$$

Let E_n be a family of maps that has:

- n vertices, two complete vertices of degree n-1, two vertices of degree 3 and n-4 vertices of degree 4.
- 2(n-2) faces of degree 3.
- 3(n-2) edges.

Remark 2.

$$\forall u, \upsilon \in V(E_n), d(u,\upsilon) \leq 2$$

The maps E_2, E_3 and E_4 are presented in the example 2. For E_5, E_6, E_7 and E_n (see Fig 4)

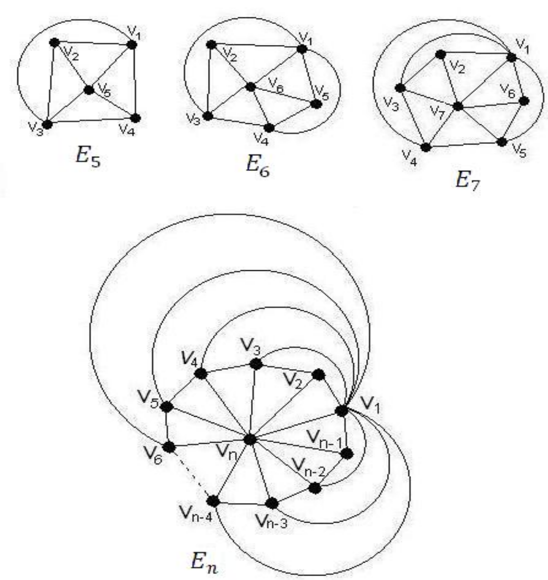

Fig. 4. The maps E_5, E_6, E_7 and E_n

Proposition 1. Let E_n be the map defined above $(n \geq 3)$ and let υ a vertex of E_n, then we have:

1. $W(\upsilon, E_n) = 2n - \deg(\upsilon) - 2$
2. $W(E_n) = (n-2)^2 + 2$

Proof:

1. $$W(\upsilon, E_n) = \sum_{u \in V(E_n)} d(u,\upsilon)$$ We use Remark 2

$$= \sum_{\substack{u \in V(E_n) \\ d(u,\upsilon)=1}} d(u,\upsilon) + \sum_{\substack{u \in V(E_n) \\ d(u,\upsilon)=2}} d(u,\upsilon)$$
$$= \deg(\upsilon) + 2\sum_{\substack{u \in V(E_n) \\ d(u,\upsilon)=2}} 1$$
$$= \deg(\upsilon) + 2(n - \deg(\upsilon) - 1)$$
$$= 2n \quad \deg(\upsilon) \quad 2$$

2. $$W(E_n) = \frac{1}{2}\sum_{\upsilon \in V(E_n)} W(\upsilon, E_n)$$

$$= \frac{1}{2}\sum_{\upsilon \in V(E_n)} (2n - \deg(\upsilon) - 2)$$

From 1

$$= (n-1)\sum_{\upsilon \in V(E_n)} 1 - \frac{1}{2}\sum_{\upsilon \in V(E_n)} \deg(\upsilon)$$
$$= n(n-1) - \frac{1}{2} \times 2\,|E(E_n)|$$
$$= (n-2)^2 + 2$$

Lemma 4. Let C_n be a simple map with n vertices $(n \geq 2)$ and let υ a vertex of map C_n, then

$$W(\upsilon, C_n) \geq 2n - \deg(\upsilon) - 2$$

Proof:

$$W(\upsilon, C_n) = \sum_{u \in V(C_n)} d(u,\upsilon)$$
$$= \sum_{\substack{u \in V(C_n) \\ d(u,\upsilon)=1}} d(u,\upsilon) + \sum_{\substack{u \in V(C_n) \\ d(u,\upsilon)=2}} d(u,\upsilon)$$
$$\geq \deg(\upsilon) + 2(n - \deg(\upsilon) - 1)$$
$$\geq 2n - \deg(\upsilon) - 2$$

Theorem 2. Let C_n be a simple map with n vertices, then

$$W(E_n) \leq W(C_n) \leq W(P_n)$$

Proof: By Remark 1, in each deleted edge of C_n, we expand the Wiener index. The connected map obtained after deleting all possible edges is a spanning tree of C_n, On the other hand in the trees the Wiener index is maximized by the path P_n with n vertices, hence $W(C_n) \leq W(P_n)$.
Remains to show that $W(E_n) \leq W(C_n)$

- for n = 2, 3 and 4 :

$$W(E_n) = \frac{1}{2} n(n-1)$$

and as :

$$W(C_n) \geq \frac{1}{2} n(n-1) = W(E_n)$$

hence the result.

- for n $\geq$ 5 :

Since $W(\upsilon, C_n) \geq 2n - \deg(\upsilon) - 2$, we have :

$$\begin{aligned} W(C_n) &= \frac{1}{2} \sum_{\upsilon \in V(E_n)} W(\upsilon, E_n) \\ &\geq \frac{1}{2} \sum_{\upsilon \in V(E_n)} (2n - \deg(\upsilon) - 2) \\ &\geq n(n-1) - \frac{1}{2} \sum_{\upsilon \in V(E_n)} \deg(\upsilon) \\ &\geq n(n \quad 1) \quad |E(E_n)| \\ &\geq W(E_n) \end{aligned}$$

3. CONCLUSION:

For a graph G with n vertices, W (G_n) ≤ W (P_n) (P_n is the path with n vertices). The lower bound of W (G_n) is not yet known [5], [8]. But in the case of maps we have given in this paper the upper bound that is W (E_n).

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