ON THE MATRIX OF CHROMATIC JOINS

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ABSTRACT

This paper, considers various Graph Theory terms and definitions. The Chromatic Join of two partitions P and Q of S_n and the matrix M (n) associated with such a Chromatic Joins. The order of auxiliary matrix A (n, P) and also recursion formula for det{A(n, P)}. Further it discussed the formula for evaluation of the determinants of det{A(n, P)}.

Keywords: *Chromatic Numbers, Chromatic Polynomials, Partition, Beraha Numbers, Beraha Polynomials, Refines, Chromatic Joins, Catalan Number, P - flow, θ - function, φ – function*

1. INTRODUCTION:

A The well known four color problems has formed the very basis for the development of planarity in Graph Theory and Combinatorial Topology. Planarity has also its applications in psychology. Problems of linear programming and operational research, such as maritime traffic problem can be tackled by the theory of flows in networks, office management problems. Such as the personal assignment problems can be dealt with by matching in graphs.

The study of simplified complex can be associated with the study of graph theory. Many more such problems can be added to the above list. An elaborate review of the contents in the paper [12] has been made.

2. DEFINITION:

2.1. Chromatic Polynomial: A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of polynomial. This polynomial is calls the chromatic polynomials of G.

2.2. Chromatic Numbers: A Graph G that requires k different colour for its proper colorings and the number k is called the chromatic number of G.

2.3. Beraha Numbers: The Beraha number $B_n = 2 + 2\cos\left(2\pi/n\right)$, where n=1,2,3… The Beraha numbers turn up in odd corners of the theory of chromials. In particular expressing constrained chromials in terms of free ones at least when n is 4, 5, 6 or 7.

2.4. Refines: Let P and Q be partitions of S_n. We say that P refines Q if each part of Q must be a union of parts of P. We note that each partition refines itself.

2.5. Isthmus or Bridge: The graph G(P,Q) is on whose deletion increases the number of components of its graph is called on Isthmus or Bridge.

3. PLANAR AND NON – PLANAR PARTITION:

A partition P is said to be non-planar, if two vertices of one part separates two vertices of another in the cyclic sequence. If there is no such separation then P is called a planner.

Example 1: Consider S_4= (V_1,V_2,V_3,V_4,V_1), The partition of S_4 are [1234], [123,4], [134,2], [142,3], [234,1], [12,34], [13,24], [14,23], [12,3,4], [13,2,4], [14,2,3], [23,1,4], [24,1,3], [34,1,2] and [1,2,3,4]. In this fifteen partition of S_4, the fourteen partitions are planar except [13, 24]. Because [13, 24], the vertices 2 and 3 separate two vertices of another in the cyclic sequence. So the partitions [13, 24] are called non – planar.

3.1. Chromatic Join: A partition J(P,Q) of S_n, whose parts correspond to the components of G(P, Q). Each part is the set of edges of the corresponding component. We call the partition J(P,Q) is the chromatic Joins of P and Q. We write the number of parts of J(P, Q), which is the number of components of G(P, Q). Simply as h(P,Q). We know G(P, Q) is isomorphic with G(Q, P).

Example 2: Consider S_4= (V_1,V_2,V_3,V_4,V_1), Take P: [13,2,4], Q: [1,23,4]

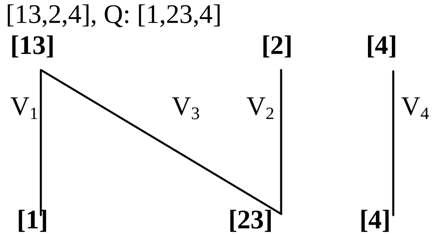

Fig. 1. The chromatic join of J (P, Q) = 2.

3.2. Matrix M (n): A Square matrix M(n) whose rows correspond to the planar partitions of S_n in some order and whose column correspond to the same partitions in the same order. In M (n) we write row (P) for the row correspond to the partition P and col(Q) for the column of partition Q. The element at the intersection of this row and column is $\xi(P,Q)$.

That is $\xi(P,Q) = \lambda^{h(P,Q)}$ **(1)**

For each P and Q where λ is an indeterminate.

Example 3: Consider S_3 = (V_1,V_2, V_3, V_1), the Partitions P_1=[V_1,V_2,V_3], P_2=[V_1V_2,V_3], P_3=[V_1V_3,V_2], P_4=[V_2V_3,V_1], P_5=[$V_1V_2V_3$].

$$M(3) = \begin{pmatrix} \lambda^{h(P_1,P_1)} & \lambda^{h(P_1,P_2)} & \lambda^{h(P_1,P_3)} & \lambda^{h(P_1,P_4)} & \lambda^{h(P_1,P_5)} \\ \lambda^{h(P_2,P_1)} & \lambda^{h(P_2,P_2)} & \lambda^{h(P_2,P_3)} & \lambda^{h(P_2,P_4)} & \lambda^{h(P_2,P_5)} \\ \lambda^{h(P_3,P_1)} & \lambda^{h(P_3,P_2)} & \lambda^{h(P_3,P_3)} & \lambda^{h(P_3,P_4)} & \lambda^{h(P_3,P_5)} \\ \lambda^{h(P_4,P_1)} & \lambda^{h(P_4,P_2)} & \lambda^{h(P_4,P_3)} & \lambda^{h(P_4,P_4)} & \lambda^{h(P_4,P_5)} \\ \lambda^{h(P_5,P_1)} & \lambda^{h(P_5,P_2)} & \lambda^{h(P_5,P_3)} & \lambda^{h(P_5,P_4)} & \lambda^{h(P_5,P_5)} \end{pmatrix}$$

$$M(3) = \begin{pmatrix} \lambda^3 & \lambda^2 & \lambda^2 & \lambda^2 & \lambda^1 \\ \lambda^2 & \lambda^2 & \lambda^1 & \lambda^1 & \lambda^1 \\ \lambda^2 & \lambda^1 & \lambda^2 & \lambda^1 & \lambda^1 \\ \lambda^2 & \lambda^1 & \lambda^1 & \lambda^2 & \lambda^1 \\ \lambda^1 & \lambda^1 & \lambda^1 & \lambda^1 & \lambda^1 \end{pmatrix}$$

3.3. Beraha Polynomial: For each non negative integer k we define a polynomial q(k, z) in an indeterminate z. We define q(0,z) and q(1,z) as follows.

q(0,z) = 0 **(2)**
q(1,z) = 1 **(3)**

we extend the definition to larger suffixes by imposing the recursion formula.

q(k, z) = q(k-1,z) – z q(k-2,z), where k>1 **(4)**
q(2,z) = 1 **(5)**
q(3,z) = 1 – z **(6)**
q(4,z) = 1 – 2z **(7)**
q(5,z)=1– 3z + z^2 **(8)**
q(6,z)=1–4z + $3z^2$ **(9)**
q(7,z)=1–5z + $6z^2$ – z^3 **(10)**
and so on.

3.4. Reversed Beraha polynomial: The polynomial $C(k,\lambda) = \lambda^{[(k-1)/2]} q(k,1/\lambda)$ **(11)**
in the variable as the Beraha polynomial,

C(1,λ) = 1 **(12)**
C(2,λ) = 1 **(13)**
C(3,λ) = λ -1 **(14)**
C(4,λ) = λ - 2 **(15)**
C(5,λ) = λ^2 - 3λ + 1 **(16)**
C(6,λ) = λ^2 - 4λ + 3 **(17)**
C(7,λ) = λ^3 - $5\lambda^2$ + 6λ - 1 **(18)**

The equation **(12)** to **(18)** we call q(k, z) as Reversed Beraha Polynomial using mathematical induction method:

q(k, z) = 1 – the ascending powers of z. **(19)**

Theorem 1: Let P be a member of W(n, P). Then no two of the first s+1 vertices belong to the same part of P.

Proof: We know W(n, P) is the set of all those planar partitions of Sn that satisfy the following conditions. If P = 2S and P = 2S+1 no one of the first S vertices is a singleton of P and no one of the first S+1 vertices is a consecutive pairs is contractive in P. Hence the theorem.

Theorem 2: The set W(n,n-1) has just one member E. The parts of E are the pairs {V_k, W_k}, when k ranges from 1 to r together with the singleton {V_{r+1}} = {W_{r+1}}, if n=2r+1.

Proof: Let Q be any member of W(n,n-1). Then first r vertices are non singletons in Q. By the previous theorem these must be paired r vertices W_1 to Wr to make r parts of Q. There is only one way to make this pairing while preserving planarity. Hence Q = E. The completes the proof of the theorem.

4. CATALAN NUMBER:

Let us write t(n) for the numbers of planar partitions of S_n. This is the cardinality of W(n,0) and the order of A(n,0) = M(n).

From Number theory $t(n) = \frac{^{2n-2}C_{n-1}}{n}$ t(n) **(20)**

Replacing n by n+1, we get

$$t(n) = \frac{2n!}{n!(n+1)!} \quad \textbf{(21)}$$

We make the definition

$$a(n,h) = \frac{(h+2)(2n-h-1)!}{(n-h-1)!(n+1)!} \quad \textbf{(22)}$$

We know a(n,0) = a(n,-1) = t(n) **(23)**

Theorem 3: The order of A(n, P) is a(n, P).

Proof: To prove the theorem using contradiction
a(1,0) = 1 and a(N-1,P-1)+a(N,P+1)=a(N, P) **(24)**
But this is the value required by the proposed formula to the choice of N and P. This contradiction establish the theorem.

Theorem 4: If P is any partition in W(n,P) and Q any partition in W(n, P+1) then

$$(-1)^P F(P,Q) = q(P+2,z)e(P,Q)$$
$$-q(P+3,z)u(P,Q). \quad \textbf{(25)}$$

Proof: Case (i): Let us write σ = 1 if P = 2S and σ = 0 if P = 2S+1, we can calculate

$$F(P,Q) = \lambda^{h(P,Q)+\sigma} \lambda^{-s+1+i} \{zq(2i-1,z)$$
$-q(2i,z)+q(2i+1,z)\}$, and
e(P,Q) = u(P,Q) = 0.
Hence the theorem.

Case(ii): When i = S+1 and P = 2S + 1, then we get

$$F(P,Q) = \lambda^{h(P,Q)} \{-q(2i+1,z)\}$$
$$= (-1)^P q(P+2,z)e(P,Q).$$

Hence the theorem.

Case (iii): Let us consider the case in which H(P,Q) has structure [i]. Then

$$F(P,Q) = \lambda^{h(P,Q)+\sigma} \lambda^{-s-2+i} \{-zq(2i-2,z)$$
$+q(2i-1,z)-q(2i,z)\}$ for i =1, we should omit the term for q(i-1,Q).
But q(2i-2,z) = q(0,z) = 0.
So omission makes no difference to the result.

Case (iv): In this case V_i is an isthmus of G(P,Q). Hence e(P,Q) = u(P,Q) = 0. Then theorem is hold contrary to the definition of 1. The theorem is hold when H(P,Q) has structure [i].

Case (v): Finally H (P, Q) has structure [0]. If P = 2S, we have

$$(-1)^P F(P,Q) = zq(2s+1,z)\lambda^{h(P,Q)},$$

If P = 2S+1, we have

$$(-1)^P F(P,Q) = q(2s+2,z)\lambda^{h(P,Q)-1}$$

In either case

$$(-1)^P F(P,Q) = \lambda^{h(P,Q)} \{q(P+2,z)$$
$$-q(P+3,z)\}$$
$$= q(P+2,z)e(P,Q) - q(P+3,z)u(P,Q)$$

The theorem is also true in this case.
In the other cases of u(P,Q), e(P,Q), e(P,f(i,Q)), e(P,q(i,Q)) can be non zero. In all other cases the theorem is trivially true.

5. EVALUATION OF DETERMI- NANTS:

Using the known theorems we can replace B(n, P) in F(P,Q) by one of the matrix A(k, m), we get

$$\det\{A(n,P)\} =$$
$$\det[\beta A(n-1,P-1)].\det\{\alpha A(n,P+1)\}.$$

Where $\alpha = \left(q(P+3,z)/q(P+2,z)\right)$. **(26)**

5.1. Definition of θ function: The function θ establish some auxiliary result concerning them.

$$\theta(n,2t) = \frac{(t+1)(2n-2t-1)!}{n!(n-2t-1)!}, \quad \textbf{(27)}$$

for any integer such that 2t+1<n.

We get θ(n,0)= θ(n,-1). **(28)**

5.2. Definition of φ function: The function φ establish some auxiliary result concerning them.

$$\varphi(m,i) = (m-2i)\frac{(m-1)!}{i!(m-i)!}, \quad \textbf{(29)}$$

with conversion that φ(m,i) = 0, if i, is negative.
φ(m,0) = 1 φ(m,1) = m-2. **(30)**

Theorem 5: The Function θ satisfies the identities

$$\theta(n,2t+1) = \theta(n-1,2t) + \theta(n,2t+2) \quad \textbf{(31)}$$

and $\theta(n,2t) = a(n-1,2t-1) + \theta(n-1,2t-1)$
$$+\theta(n,2t+1) \quad \textbf{(32)}$$

Proof: The sum S on the right of **(31)** is
S= $\theta(n-1,2t) + \theta(n,2t+2)$ =

$$\frac{(2n-2t-3)!}{n!(n-2t-3)!}.\{n(2t+3) - 2(t+1)(t+2)\}.$$

Hence S = θ(n, 2t+1).

The Sum T on the right of
T =
$$a(n-1,2t-1)+\theta(n-1,2t-1)+\theta(n,2t+1)$$
= uv
where
$$u=\frac{(2n-2t-3)!}{(n-1)!(n-2t-1)!}$$
$$v=(t+1)(2n-2t-1)(2n-2t-2)$$
Hence T = θ(n, 2t). Hence the theorem.

Theorem 6: The function φ satisfies the identify φ(m, i) = φ(m-1, i-1) + φ(m-1, i-1). (33)

Proof: Suppose i>0, the sum S of right of **(33)** is
S = φ(m-1, i-1)+φ(m-1, i-1)
$$S=\frac{(m-2)!}{i!(m-i)!}(m-2i)(m-1)=\varphi(m,i).$$
Hence the theorem.

Theorem 7: The function φ satisfies the identity
φ(m,i) - φ(m,i-2) - φ(m-1,i) + φ(m-1, i-3) - φ(m-2,i-1) - φ(m-2, i-3) = 0. **(34)**

Proof: Using above theorem **(6),** we get the result **(34)**. Hence the theorem.

Theorem 8: The determinants **det{A(n,P)}** are evaluated by the formula
$$\det\{A(n,P)\}$$
$$=\lambda^{\theta(n,p)}\prod_{i=0}^{k-2}\left\{\frac{q(n+1-i,z)}{q(k-1-i,z)}\right\}^{\varphi(n+k,i)} \quad (35)$$

Proof: Assume that the theorem is false. Let N be the least value of n for which it fails and P be the greatest value of p for which it fails at this value of n and K be the corresponding value of k. Thus P=N-K.

Case (i): Let n=1 and k=1, we have P=0 and S=0. Then det{A(1,0)} = λ and θ(1,0) = 1. So the theorem holds in this case. It follows N ≥ 2.

Case (ii): Take integral k=1 and n=p+1, we have n=2S+1 and p=2S and n=2S+2 and p=2S+1. So, the theorem is true in this case thus we have K ≥ 2. If we split the expression on the right of (35) in two factors. The first is A(n,P) that is λ to the power θ(n, P). We denote the second complementary factor by Q(n, k). It is a product of powers, positive or negative or reversed Beraha polynomials.

The matrix M(n) is A(n,0), by the above theorems and results its determinant is given as
$$\det M(n)=\lambda^{\theta(n,0)}\prod_{i=0}^{n-2}\left\{\frac{q(n+1-i,z)}{q(n-1-i,z)}\right\}^{\varphi(2n,i)}.$$
That is the number θ(n, P) is the degree of the polynomial det{A(n, P)}.

6. CONCLUSION:

In this paper we have given the definitions of various terms in graph theory and the Chromatic Joins of two partitions P and Q of S_n and the matrix M(n) associated with such a Chromatic Joins has been discussed elaborately. Then we study the order of auxiliary matrix A(n,P) and also recursion formula for det{A(n,P)}. Finally we have discussed the formula for evaluation of the determinants of det{A(n, P)}.

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