WIENER INDEX OF PLANAR MAPS

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ABSTRACT

In trees with n vertices, the Wiener index of tree is minimized by stars and maximized by paths, both uniquely. In this paper, we give an inequality similar in the case of planar maps.

Keywords: Wiener index, maps, trees

1. INTRODUCTION

In this paper we will present some useful definitions related to our work as follows: an undirected graph G is a triplet (V, E, δ) where V is the set of vertices of the graph, E is the set of edges of the graph and δ is the application δ : E → P(u), where u and v are end vertices of the edge. We notice that the set {u, v} as a multiset (if u = v, the same vertex appears twice in δ(u)). A loop is an edge with u = v, if δ(u) = δ(v). A graph which contains neither multiple edges nor loops is called a simple graph.

The degree of a vertex u noted deg(u) is the number of edges incident to it. The sum of the degrees of all vertices of a graph is equal to twice the number of its edges i.e. \(\sum \text{deg}(u) = 2|E|\).

In a graph G, a path is a sequence of vertices and edges p = u_0, u_1, u_2, ..., u_n such that δ(u_i) = {u_i, u_i-1}. We say that this path attached both ends u_0 and u_n. A cycle is a path such that u_n = u_0. A graph G is called connected if any two of its vertices may be connected by a path.

The distance between two distinct vertices u_i and u_j of a graph G, denoted by d(u_i, u_j) is equal to the length of (number of edges in) the shortest path that connects u_i and u_j. Conventionally, d(u_i, u_i) = 0. We define a complete vertex by the vertex u such that d(u, u_i) = 1 for each u_i ∈ V. In a complete graph all the vertices are completes.

The graphs that we consider are in most cases connected but may contain multiple edges.

A map C is a graph drawn on a surface X or embedded into it (that is, a compact variety orientable 2-dimensional) in such a way that:

- The vertices of graph are represented as distinct points of the surface.
- The edges are represented as curves on the surface that intersect only at the vertices.
- If we cut the surface along the graph thus drawn, what remains (that is, the set X \ C) is a disjoint union of connected components, called faces, each homeomorphic to an open disk (for more information on the faces of a map see [4]).

A planar map is a map drawn on the plane. We define a simple map as a map without loops and without multiple edges. In all the following, map means a planar map, connected and simple.

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Fig. 1. One graph gives two planar maps

A tree is a connected graph without cycle. A plan tree is a tree designed on the plane [4]. The Wiener index of a connected graph is the sum of
distances between all pairs of vertices [2], [3], [6], the Wiener index of a connected graph G is defined as:

\[ W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) \]

We define W (u, G) (Wiener index of vertex u in G) the sum of distances of vertex u to each vertex of vertices of G i.e.

\[ W(u,G) = \sum_{v \in V(G)} d(u,v) \]

Let \( T_n \) a tree with n vertices, then the Wiener index of a tree \( T_n \) is minimized by that of the star tree with n vertices and maximized by that of the path with n vertices [5]. The goal of this work is to give an inequality similar in the case of maps.

2. CALCULATION OF THE WIENER INDEX OF MAPS

In the same way as graphs, we define the Wiener index for maps as follows:

\[ W(u,C) = \sum_{v \in V(C)} d(u,v) \]

We notice that:

\[ W(C) = \frac{1}{2} \sum_{u \in V(C)} \sum_{v \in V(C)} d(u,v) = \frac{1}{2} \sum_{u \in V(C)} W(u,C) \]

**Example 1.** Let \( C_3 \) be a map with \(|V|=5\), we have: \( \deg(u_1) = 4, \deg(u_2) = 2 \), \( W(u_1,C_3) = 6 \), \( u_1 \) and \( u_3 \) are complete vertices (see Fig 2).

![Fig. 2. An example of map C3](image)

**Example 2.** In the maps \( E_2 \), \( E_3 \) and \( E_4 \) the vertices are all completes (we say that the map is complete) (see Fig 3).

![Fig. 3. The maps E2, E3 and E4](image)

Let C be a map and let e be an edge of map C (\( e \in E(C) \)), we denote by \( C - e \) the map obtained after deleting the edge e of the map C and the resulting map is connected.

**Lemma 1.** Let C be a map and let \( e_1, e_2 \) be two edges of map C that connect the vertices \( u_1 \) and \( u_2 \) (multiple edges of C), then

\[ W(C - e_i) = W(C), \quad i = 1, 2 \]

where \( C - e_i \) is the map obtained by deleting the edge \( e_i \) of the map C.

We notice that delete a multiple edge does not affect the Wiener index; through this paper, we consider only the simple maps (without loops and without multiple edges).

**Lemma 2.** Let \( C_n \) be a simple map with n vertices and let \( u \) be a complete vertex of C, then

\[ \deg(u) = n - 1 \text{ and } W(u,C_n) = n - 1 \]

**Lemma 3.** Let \( C_n \) be a simple map with n vertices \((n \geq 2)\) and let \( u \) a vertex not complete of a map C, then

\[ W(u,C_n) \geq n \]

**Remark 1.**

1. Let C be a simple map and let \( u \) a vertex of map \( C_n \), then \( W(u,C_n) \geq n - 1 \).

2. Let \( C_n \) be a simple map with n vertices, \( e \) be an edge of \( C_n \) and let \( C_n - e \) be the map obtained by deleting the edge \( e \) such that the map \( C_n - e \) remains connected, then \( W(C_n - e) \geq W(C_n) \).

**Theorem 1.** Let \( C_n \) be a simple map with n vertices, then

\[ W(C_n) \geq \frac{n(n-1)}{2} \]
Proof: Let $C_n$ be a map with $n$ vertices and let $u$ be a vertex of $C_n$. From Remark 1, we have $W(u, C_n) \geq n - 1$.

$$W(C_n) = \frac{1}{2} \sum_{v \in V(C_n)} W(u, C_n) \geq \frac{1}{2} \sum_{v \in V(C_n)} (n - 1) \geq \frac{1}{2} (n - 1) \sum_{v \in V(C_n)} 1 \geq \frac{n(n - 1)}{2}.$$ 

Let $E_n$ be a family of maps that has:

- $n$ vertices, two complete vertices of degree $n-1$, two vertices of degree 3 and $n-4$ vertices of degree 4.
- $2(n-2)$ faces of degree 3.
- $3(n-2)$ edges.

Remark 2.

$$\forall u, v \in V(E_n), d(u, v) \leq 2$$

The maps $E_2$, $E_3$ and $E_4$ are presented in the example 2. For $E_5$, $E_6$, $E_7$ and $E_n$ (see Fig 4).

![Graphs](image)

Fig. 4. The maps $E_2$, $E_3$, $E_4$, $E_5$, $E_6$, $E_7$ and $E_n$.

Proposition 1. Let $E_n$ be the map defined above ($n \geq 3$) and let $u$ a vertex of $E_n$, then we have:

1. $W(u, E_n) = 2n - \deg(u) - 2$
2. $W(E_n) = (n - 2)^2 + 2$

Proof:

1. $W(u, E_n) = \sum_{v \in V(E_n)} d(u, v)$
   $$= \sum_{v \in V(E_n)} d(u, v) + \sum_{v \in V(E_n)} d(u, v)$$
   $$= \deg(u) + 2 \sum_{v \in V(E_n)} 1$$
   $$= \deg(u) + 2(n - \deg(u) - 1)$$
   $$= 2n - \deg(u) - 2$$

2. $W(E_n) = \frac{1}{2} \sum_{v \in V(E_n)} W(u, E_n)$
   $$= \frac{1}{2} \sum_{v \in V(E_n)} (2n - \deg(u) - 2)$$

From 1

$$= (n - 1) \sum_{v \in V(E_n)} 1 - \frac{1}{2} \sum_{v \in V(E_n)} \deg(v)$$

$$= n(n - 1) - \frac{1}{2} \times 2 |E(E_n)|$$

$$= (n - 2)^2 + 2$$

Lemma 4. Let $C_n$ be a simple map with $n$ vertices ($n \geq 2$) and let $u$ a vertex of map $C_n$, then

$$W(u, C_n) \geq 2n - \deg(u) - 2$$

Proof:

$$W(u, C_n) = \sum_{v \in V(C_n)} d(u, v)$$

$$= \sum_{v \in V(C_n)} d(u, v) + \sum_{v \in V(C_n)} d(u, v)$$

$$\geq \deg(u) + 2(n - \deg(u) - 1)$$

$$\geq 2n - \deg(u) - 2$$
Theorem 2. Let $C_n$ be a simple map with $n$ vertices, then

$$W(E_n) \leq W(C_n) \leq W(P_n)$$

Proof: By Remark 1, in each deleted edge of $C_n$, we expand the Wiener index. The connected map obtained after deleting all possible edges is a spanning tree of $C_n$. On the other hand, in the trees the Wiener index is maximized by the path $P_n$ with $n$ vertices, hence

$$W(C_n) \leq W(P_n), \quad W(E_n) \leq W(C_n)$$

hence the result.

Remains to show that

- for $n = 2, 3$ and $4$:
  $$W(E_n) = \frac{1}{2} n(n - 1)$$
  and as:
  $$W(C_n) \geq \frac{1}{2} n(n - 1) = W(E_n)$$

hence the result.

- for $n \geq 5$:
  Since $W(u, C_n) \geq 2n - \deg(u) - 2$, we have:

$$W(C_n) = \frac{1}{2} \sum_{u \in V(C_n)} W(u, E_n)$$
$$\geq \frac{1}{2} \sum_{u \in V(E_n)} (2n - \deg(u) - 2)$$
$$\geq n(n - 1) - \frac{1}{2} \sum_{u \in V(E_n)} \deg(u)$$
$$\geq W(E_n)$$

3. CONCLUSION:

For a graph $G$ with $n$ vertices, $W(C_n) \leq W(P_n)$ ($P_n$ is the path with $n$ vertices). The lower bound of $W(C_n)$ is not yet known [5], [8]. But in the case of maps we have given in this paper the upper bound that is $W(E_n)$.

REFERENCES:


