Journal of Theoretical and Applied Information Technology
15th December 2021. Vol.99. No 23

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

DEVELOPMENT OF PATH ANALYSIS BASED ON NONPARAMETRIC REGRESSION

FATHIYATUL LAILI NUR RASYIDAH[1] , ADJI ACHMAD RINALDO FERNANDES[2] , ATIEK IRIANY[3] , SOLIMUN[4], NI WAYAN SURYA WARDHANI[5]

[1,2,3,4,5]Brawijaya University, Faculty of Mathematics and Natural Sciences, Department of Statistics, Indonesia

E-mail: [1]fathiyatullaili@student.ub.ac.id, [2]fernandes@staff.ub.ac.id, [3] atiekiriany@ub.ac.id, [4]solimun@ub.ac.id, [5]wswardhani@ub.ac.id

ABSTRACT

This study aims to develop a robust nonparametric regression-based path analysis with the assumption of linearity. This study used a multivariate approach, namely nonparametric path analysis. The conclusion that can be obtained is that the properties of the spline estimator in Nonparametric Regression-Based Path Analysis using the PWLS approach, hypothesis testing on each relationship between variables in Nonparametric Regression-Based Path Analysis using the PWLS approach, as well as several findings regarding confidence intervals. The novelty of this research is to describe the estimation of nonparametric path analysis parameters through lemmas and theorems.

Keywords: *Path Analysis, Spline, Nonparametric, Regression, Penalized Weighted Least Square*

1. INTRODUCTION

Path analysis is the development of a parametric regression analysis that has more than one equation and between equations is presented structurally, where the structural equation is characterized by at least one exogenous variable (X), at least one endogenous intervening variable (Y), and one pure endogenous variable (Z). The main assumption in path analysis is that the relationship between variables is linearity [1]. The assumption of linearity sometimes cannot be fulfilled in several studies, such as the results found by Arisoesilaningsih et al. [2] stated that the relationship between growth and production of porang tubers is influenced by variations in plant age, vegetation conditions, soil conditions, and agroforestry climatic conditions that do not meet the linearity assumption. In the application, it is very difficult to get these functions precisely, and even symptoms often show that the data obtained does not or does not show a relationship pattern that is easy to describe. If the assumption of linearity is not fulfilled and the form of nonlinearity is unknown, then one alternative that can be used is a nonparametric regression model.

The pattern of the relationship between responses and unknown predictors can be estimated using the Spline function approach [3]-[5], Local Polynomials [6], Kernel [7], Wavelets [8], and the Fourier Series [9]. This study developed a nonparametric regression-based path analysis using a spline path estimator. The regression curve used is assumed to be smooth, in the sense that it is contained in a certain function space, especially the Sobolev space, where m is the order of the spline polynomial. To get the regression curve estimation, the optimization is used Weighted Least Square (WLS) or Penalized Weighted Least Square (PWLS) [10].

Based on the above phenomena, this study aims to develop a nonparametric regression-based path analysis that is robust in the assumption of linearity. This research is expected to obtain the properties of the spline estimator in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach, to obtain hypothesis testing on each relationship between variables in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach, and to obtain the optimal confidence interval for each relationship between variables in Path Analysis. based on Nonparametric Regression using the Penalized Weighted Least Square approach.

2. LITERATURE REVIEW

2.1 Path Analysis

Path analysis was developed by Sewall Wright in 1934. Sewall Wright developed this method as a means of studying the direct effects and indirect effects of several variables where some variables are seen as causes and other variables are seen as effects [11]. Path analysis is also known as causing modeling because path analysis makes it possible to test the theoretical proportions of the cause and effect relationships of certain variables.

The following are the assumptions underlying path analysis, namely: The relationship between variables must be linear and additive [1]. The path diagram is the basis of path analysis, which is a procedure for empirical estimation of the strength of each relationship depicted in the path diagram [12]. Path diagrams are used to graphically display both measured and unmeasured causal relationships

2.2 Nonparametric Regression Analysis

This study developed a nonparametric regression-based path analysis, specifically spline. Therefore, this research requires several supporting theories that can later be used to complete this research. Some of these theories are as follows:

1. Kernel Functions and Spline Functions and their properties. Kernel functions include Gaussian Kernel, Ephanicov Kernel and Uniform Kernel, and the Spline function will be applied to investigate the behavior of the Kernel and Spline mixture estimators for estimating multivariable semiparametric regression curves.
2. Nonparametric regression analysis. Nonparametric regression analysis, is used as the basic theory in developing a mixture estimator of Spline and Kernel in multivariable semiparametric regression.
3. Optimization of Penalized Weighted Least Square (PWLS). The Penalized Weighted Least Square optimization concept is used to find the estimator form of a mixture of Kernel and Spline to estimate the regression curve, both in multivariable nonparametric regression and multivariable semiparametric regression.
4. The Hilbert Space Reproducing Kernel Method (HSRK). The Space Reproducing Kernel Method method was used to solve the PLS optimization in multivariable and semiparametric multivariable regression models.
5. The basic concept of the Sobolev space. The Sobolev space is used as a basic function space in multivariable and semiparametric nonparametric regression analysis, where the regression curve is assumed to fit in the Sobolev space.
6. The concept of the Generalized Cross Validation (GCV) method. The Generalized Cross Validation method, which has been developed by researchers on cross section data, will be generalized into a function that will be used as a method for selecting optimal knot points and bandwidth parameters in the Spline and Kernel mixture estimators in multivariable semiparametric regression.

3. METHOD

This research was conducted at the Statistics Computer Laboratory, Statistics Department, Universitas Brawijaya for the development of statistical modeling theory. This study describes the properties of the spline estimator in Nonparametric Regression-Based Path Analysis using the Penalized Weighted Least Square approach, and constructs the lemma theorem.

4. RESULTS AND DISCUSSION

Properties of Estimates

The first objective of the study is to obtain the properties of the spline estimator in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach.

Asymptotic properties

Several goodness estimator criteria have been developed by many authors. In nonparametric regression, it is often noted that the estimator's asymptotic behavior is based on a certain measure of goodness. If squared risk criteria are used as the goodness of the regression curve estimator, The estimator will converge in terms of velocity, $n^{-\delta}, 0<\delta<1$, in contrast to the parametric case which can achieve velocity n^{-1} under regular terms [13].

In this paper, we investigate the asymptotic nature of a weighted spline estimator $\hat{f}_\lambda$ based on the Integrated Mean Square Error (IMSE) criteria. Previously, the following assumptions were given.

Assumption:

$$(A0).t_j = \frac{2j-1}{2n}, j = 1,2,\ldots,n \tag{1}$$

Then, Integrated Mean Square Error *(λ)* is composed of two terms, bias and variance.

$$
\begin{aligned}
IMSE(\lambda) &= E\int_0^1 \left(f(t) - f_\lambda(t)\right)^2 w(t)\,dt \\
&= \int_0^1 E\left(f(t) - Ef_\lambda(t)\right)^2 w(t)\,dt \\
&\quad + \int_0^1 E\left(Ef(t) - f_\lambda(t)\right)^2 w(t)\,dt \\
&\quad + 2\int_0^1 \left(f(t) - Ef_\lambda(t)\right) \\
&\quad \left(Ef(t) - Ef_\lambda(t)\right) w(t)\,dt \\
&= b^2(\lambda) + V(\lambda) \qquad (2)
\end{aligned}
$$

With $b^2(\lambda) = \int_0^1 E\left(f(t) - Ef_\lambda(t)\right)^2 w(t)\,dt$

and $V(t) = \int_0^1 E\left(Ef(t) - f_\lambda(t)\right)^2 w(t)\,dt$

The first step, investigated the asymptotic properties of the quadratic bias component $b^2(\lambda)$

Theorem 1

If $f_\lambda(t)$ completion of minimizing the PLST:

$$n^{-1}\sum_{j=1}^{n} w_j \left(y_j - g(t_j)\right)^2 + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \qquad (3)$$

then the solution is to minimize the PLST:

$$is\ g_\lambda{*}(t) = Ef_\lambda(t). \qquad (4)$$

Proof:

Theorem 1 provides a solution to minimize the Penalized Weighted Least Square:

$$n^{-1}\sum_{j=1}^{n} w_j \left(y_j - g(t_j)\right)^2 + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \qquad (5)$$

can be written as:

$$
\begin{aligned}
f_\lambda(t) &= \Big(\varphi'\left(\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\boldsymbol{T}\right)^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W} + \qquad (6) \\
&\psi'\boldsymbol{S}^{-1}\boldsymbol{W}\left(\boldsymbol{I} - \boldsymbol{T}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\boldsymbol{T}\right)^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\right)\Big)\tilde{y}
\end{aligned}
$$

By pairing $f(t_j) = y_j, j = 1, 2, ..., n$ the minimum completion:

$$n^{-1}\sum_{j=1}^{n} w_j \left(f(t_j) - g(t_j)\right)^2 + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \qquad (7)$$

can be written as:

$$
\begin{aligned}
g_\lambda{*}(t) &= \left(\varphi'\left(\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\boldsymbol{T}\right)^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\right) \\
&\quad \left(+\psi'\boldsymbol{S}^{-1}\boldsymbol{W}\left(\boldsymbol{I} - \boldsymbol{T}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\boldsymbol{T}\right)^{-1}\right)\right) \\
&\quad \left(\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}\right) f(t) \\
&= Ef_\lambda(t), \text{ with } f(t) = \left(f(t_1), ..., f(t_n)\right)' \qquad (8)
\end{aligned}
$$

The following theorem, presents the asymptotic properties of the quadratic bias component.

Theorem 2

If (A0) applies then $\tilde{b}^2(\lambda) \le \boldsymbol{O}(\lambda)$, $n \to \infty$.

Proof:

Suppose that $g_\lambda{*}(t)$ the minimum completion:

$$\int_0^1 \left(f(t) - g(t)\right)^2 w(t)\,dt + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \qquad (9)$$

Because (A0), then for $n \to \infty$.

$$
\begin{aligned}
& n^{-1}\sum_{j=1}^{n} w_j \left(f(t_j) - g(t_j)\right)^2 \\
& \approx \int_0^1 \left(f(t) - g(t)\right)^2 w(t)\,dt \qquad (10)
\end{aligned}
$$

as a result, so $g_\lambda(t) = Ef_\lambda(t) \approx g_\lambda(t)$

So for every $g \in W_2^m[0,1]$,

$$
\begin{aligned}
\underset{\sim}{b}^2(\lambda) &= \int_0^1 E\left(f(t) - Ef_\lambda(t)\right)^2 w(t)\,dt \\
&\le \int_0^1 E\left(f(t) - Ef_\lambda(t)\right)^2 w(t)\,dt \qquad (11) \\
&\quad + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt
\end{aligned}
$$

Remember that $Ef_\lambda(t) \approx g_\lambda(t)$, so obtained:

$$
\begin{aligned}
\tilde{b}^2(\lambda) &\le \int_0^1 E\left(f(t) - g_\lambda(t)\right)^2 w(t)\,dt \\
&\quad + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \qquad (12)
\end{aligned}
$$

to $\tilde{g} \in \boldsymbol{W}_2^m[0,1]$, so we can conclude that:

$$
\begin{aligned}
\tilde{b}^2(\lambda) &\le \int_0^1 E\left(f(t) - g(t)\right)^2 w(t)\,dt \\
&\quad + \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \qquad (13)
\end{aligned}
$$

Because it applies to every $\tilde{g} \in \boldsymbol{W}_2^m[0,1]$, then by taking $g(t) = f(t)$, can be obtained that:

$$
\begin{aligned}
\underset{\sim}{b}^2(\lambda) &\le \lambda\int_0^1 \left[g^{(m)}(t)\right]^2 dt \\
&= \mathbf{O}(\lambda) \qquad (14)
\end{aligned}
$$

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

The following is derived asymptotic properties $V(\lambda)$.

Define:

$\Phi(\tilde{f}_\lambda, \tilde{h}; \gamma) = R(\tilde{f}_\lambda + \gamma\tilde{h}) + \lambda J(\tilde{f}_\lambda + \gamma\tilde{h})$, and $\tilde{h} \in W_2^m[0,1]$,

With $(g) = n^{-1}\sum_{j=1}^{n} w_j (y_j - g(t_j))^2$ and

$$J(g) = \int_0^1 \left[g^{(m)}(t)\right]^2 dt$$

For any value $\underset{\sim}{f}, \underset{\sim}{g} \in W_2^m[0,1]$,

$$\Phi(\tilde{f}, \tilde{g}, \gamma) = n^{-1}\sum_{j=1}^{n} w_j (y_j - f(t_j) - \gamma g(t_j))^2 + \lambda\int_0^1 (f^{(m)}(t) + \gamma g^{(m)}(t))^2 dt \quad (15)$$

$\dfrac{d\Phi(f,g,\gamma)}{d\gamma} = 0$ and $\gamma = 0$ will give:

$$n^{-1}\sum_{j=1}^{n} w_j g(t_j)(y_j - f(t_j)) = \lambda\int_0^1 f^{(m)}(t) g^{(m)}(t) dt$$

If $\phi_1,...,\phi_n$ base for natural splines and $f(t) = \sum_{k=1}^{n}\beta_k\phi_k(t)$, then Lemma 1, gives:

$$\sum_{j=1}^{n} w_j g(t_j)\left(y_j - \sum_{k=1}^{n}\beta_k\phi_k(t_j)\right) = n\lambda(-1)^m(2m-1)!\sum_{j=1}^{n} g(t_j)\sum_{k=1}^{n}\beta_k d_{jk} \quad (16)$$

Because it applies to every $\tilde{g} \in W_2^m[0,1]$, the last equation is equivalent to finding β_k which fulfills:

$$y_j = \sum_{k=1}^{n}\left(n\lambda(-1)^m(2m-1)!w_j^{-1}d_{jk} + \phi_k(t_j)\right)\beta_k \quad (17)$$

$j = 1, 2, ..., n$

By presenting the matrix, it can be obtained:

$$\underset{\sim}{y} = \left(n\lambda(-1)^m(2m-1)!W^{-1}K + \phi\right)\beta \quad (18)$$

with $K = \{d_{jk}\}, k, j = 1, 2, ..., n$,and

$\phi = \{\phi_k(t_j)\}, k, j = 1, 2, ..., n$

Then, can be obtained that:

$$\underset{\sim}{y} = \left(n\lambda W^{-1}FB^{-1}F'W^{-1}\phi + \phi\right)\beta, \quad (19)$$

with $B = F'VF$.

If the last equation is multiplied from the left with ϕ' can be obtained that:

$$\phi'\tilde{y} = \left(n\lambda\phi'W^{-1}FB^{-1}F'W^{-1}\phi + \phi'\phi\right)\beta \quad (20)$$

So, the estimator β_λ obtained from the equation:

$$\beta_\lambda = \left(n\lambda\phi'\mathbf{W}^{-1}\mathbf{F}\mathbf{B}^{-1}\mathbf{F}'\mathbf{W}^{-1}\phi + \phi'\phi\right)^{-1}\phi'\underset{\sim}{y} = diag\left(\frac{1}{1+n\lambda\theta_1}, ..., \frac{1}{1+n\lambda\theta_n}\right)\phi'\underset{\sim}{y} \quad (21)$$

Estimator $f_\lambda(t)$ can presented in the equation:

$$f_\lambda(t) = \sum_{k=1}^{n}\beta_{\lambda k}\phi_k(t) = \sum_{k=1}^{n}\frac{1}{1+n\lambda\theta_k}\phi_k' y\phi_k \quad (22)$$

The asymptotic property of the variance component is given by the theorem below.

Theorem 3

If (A0) applies then $V(\lambda) \le O\left(\dfrac{1}{1+n\lambda^{1/2m}}\right)$, $n \to \infty$

Proof:

$$f_\lambda(t) = \sum_{k=1}^{n}\frac{1}{1+n\lambda\theta_k}\phi_k' y\phi_k(t) = \sum_{k=1}^{n}\frac{1}{1+n\lambda\theta_k}\sum_{r=1}^{n}\phi_k(t_r)y_r\phi_k(t) \quad (23)$$

$$\mathrm{Var}(f_\lambda(t)) = \sigma^2\sum_{k=1}^{n}\frac{\phi_k^2(t)}{(1+n\lambda\theta_k)^2}\sum_{r=1}^{n}\phi_k(t_r)\phi_k(t_r)w_r^{-1} \le \left(\underset{i\le j\le n}{\mathrm{Max}}\{w_j^{-1}\}\right)\sigma^2\sum_{k=1}^{n}\frac{\phi_k^2(t)}{(1+n\lambda\theta_k)^2} \quad (24)$$

So that:

$$V(\lambda) = \int_0^1 E(Ef2(t) - f2(t))^2 w(t)dt \quad (25)$$

$$\le (\underset{1\le j\le n}{Max}\{w_j^{-1}\}\sigma^2\sum_{k=1}^{n}\frac{1}{(1+n\lambda\theta k)^2}\int_0^1\phi_k^2(t)w(t)dt$$

For n ->∞, [13] and [14], provide an approach that:

$$n^{-1} = n^{-1}\sum_{r=1}^{n} w_r\phi_k(t_r) \approx \int_0^1\phi_k(t)w(t)dt \quad (26)$$

$$V(\lambda) \le \left(\underset{1\le j\le n}{Max}\{w_j^{-1}\}\right)\sigma^2 n^{-1}\sum_{k=1}^{n}\frac{1}{(1+\lambda\gamma k)^2} \quad (27)$$

Furthermore [13]:

$$V(\lambda) \le \left(\underset{1\le j\le n}{Max}\{w_j^{-1}\}\right)\sigma^2 n^{-1}\sum_{k=1}^{n}\frac{1}{\left[1+\lambda(\pi k)^{2m}\right]^2} \quad (28)$$

With the integral approach obtained that:

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$$V(\lambda) \le \left(\underset{1\le j\le n}{Max}\left\{w_j^{-1}\right\}\right)\frac{\sigma^2}{\pi n\lambda^{1/2m}}\int_0^\infty \frac{dx}{(1+x^{2m})^2}$$

$$= \frac{1}{n\lambda^{1/2m}}\boldsymbol{K}(m,\sigma)$$

$$= \boldsymbol{O}\left(\frac{1}{n\lambda^{1/2m}}\right) \quad (29)$$

With $\boldsymbol{K}(m,\sigma) = \frac{\sigma^2}{\pi}\left(\underset{1\le j\le n}{Max}\left\{w_j^{-1}\right\}\right)\int_0^\infty \frac{dx}{(1+x^{2m})^2}$

By combining the quadratic bias and variance components, the asymptotic behavior of Integrated Mean Square Error is obtained, which is fully provided by the following results.

Consequence

If (A0) applies then

$$IMSE(\lambda) \le \boldsymbol{O}(\lambda) + \left(\frac{1}{n\lambda^{1/2m}}\right), n \to \infty \quad (30)$$

Proof:

This result is proofed by combining Theorem 2 and Theorem 3

The following theorem, presents the convergence speed of the weighted spline estimator which can reach the level $n^{-2m/(2m+1)}$, based on Integrated Mean Square Error criteria.

Theorem 4

if $\tilde{f} \in \boldsymbol{W}_2^m[0,1]$ dan $(\mathrm{A0})$ applies then

to $n \to \infty, \lambda \to 0, n\lambda^{1/2m} \to \infty$

(i). $\lambda_{opi} = \boldsymbol{O}\left(n^{-2m/(2m+1)}\right)$ and

(ii). $IMSE\left(\lambda_{opt}\right) = \boldsymbol{O}\left(n^{-2m/(2m+1)}\right)$

Proof:

$IMSE(\lambda)$ can be written as:

$$IMSE(\lambda)) = \boldsymbol{K}(m)\lambda + \boldsymbol{K}(\sigma,m)\left(\frac{1}{n\lambda^{1/2m}}\right), \quad (31)$$

With K (m) and K (σ, m) constants that do not contain λ. by lowering the Integrated Mean Square Error (λ) against λ, then the result is equalized to zero, the first part of Theorem is proofed. The second part of theorem is obtained by substituting λ_{opt}. Theorem of the first part into the $IMSE(\lambda)$.

Consistent Properties of Parametic Component Estimators

In this section we will investigate the consistent nature of the $\hat{\underset{\sim}{\beta}}$ in the nonparametric regression model. For this purpose, the following assumptions are given

A1. $\{X_n; n \ge 1\}$ is a sequence of random variables that are distributed identically and independently with zero mean and with variation $v_j < v, 0 < v < \infty, j = 1,2,\dots,rp$

A2. $E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0}$

A3. $E\left(x_{jk}^2 e_k^2\right) < \vartheta < \infty, j = 1,2,\dots,rp; k = 1,2,\dots,rn$

A4. $E\left(x_{jk}^2 s_k^2\right) < \omega < \infty, j = 1,2,\dots,rp; k = 1,2,\dots,rn$

Based on these assumptions, the following items are compiled to demonstrate the consistent nature of the parametric component estimators $\hat{\underset{\sim}{\beta}}$.

Lemma 1:

If $\hat{\underset{\sim}{\beta}}_n$ is a sequence of parametic component estimators given by Theorem 1, then:

$$\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta} = \left[\boldsymbol{X}^T\boldsymbol{B}\boldsymbol{X} - \boldsymbol{X}^T\boldsymbol{A}^T(\underset{\sim}{\lambda})\boldsymbol{B}\boldsymbol{X}\right]^{-1}$$
$$\left[\left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\underset{\sim}{f}\right] + \quad (32)$$
$$\left[\boldsymbol{X}^T\boldsymbol{B}\underset{\sim}{\varepsilon} - \boldsymbol{X}^T\boldsymbol{A}^T(\underset{\sim}{\lambda})\boldsymbol{B}\underset{\sim}{\varepsilon}\right]\right]$$

Proof:

Based on Theorem 1, can be obtained that:

$$\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta} = \left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\boldsymbol{X}\right]^{-1}\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T$$
$$\boldsymbol{B}\left(\boldsymbol{X}\underset{\sim}{\beta} + \underset{\sim}{f} + \underset{\sim}{\varepsilon}\right) - \underset{\sim}{\beta}$$
$$= \left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\boldsymbol{X}\right]^{-1}\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T$$
$$\boldsymbol{B}\left(\underset{\sim}{f} + \underset{\sim}{\varepsilon}\right)$$
$$= \left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\boldsymbol{X}\right]^{-1}\left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\right.$$
$$\left.\boldsymbol{B}\left(\underset{\sim}{f} + \underset{\sim}{\varepsilon}\right)\right]$$
$$= \left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\boldsymbol{X}\right]^{-1}\left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\right.$$
$$\left.\boldsymbol{B}\underset{\sim}{f} + \boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\underset{\sim}{\varepsilon}\right]$$
$$= \left[\boldsymbol{X}^T\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\boldsymbol{X}\right]^{-1}$$
$$\left[\boldsymbol{X}^T\left(\left[\boldsymbol{I} - \boldsymbol{A}(\underset{\sim}{\lambda})\right]^T\boldsymbol{B}\underset{\sim}{f} + \boldsymbol{B}\underset{\sim}{\varepsilon}\right)\right.$$
$$\left.-\boldsymbol{X}^T\boldsymbol{A}^T(\underset{\sim}{\lambda})\boldsymbol{B}\underset{\sim}{\varepsilon}\right]$$

$$=\left[X^T\left[I-A(\underset{\sim}{\lambda})\right]^T BX\right]^{-1}\left[X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}+X^T B\underset{\sim}{\varepsilon}-X^T A^T(\underset{\sim}{\lambda})B\underset{\sim}{\varepsilon}\right] \quad (33)$$

$$=\left[X^T BX-X^T A^T(\underset{\sim}{\lambda})BX\right]^{-1}\left[X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}+X^T B\underset{\sim}{\varepsilon}-X^T A^T(\underset{\sim}{\lambda})B\underset{\sim}{\varepsilon}\right]$$

Furthermore, to show that the sequence of the estimator $\underset{\sim}{\hat{\beta}_n}$ is a consistent estimator, the following lemma is needed which states the convergence property of the right-hand product term on the right side of Lemma 1.

Lemma 2.

If the assumptions of A1-A3 are met, then

i. $\dfrac{X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn} \xrightarrow{p} 0, n\to\infty$

ii. $\dfrac{X^T B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n\to\infty$

iii. $\dfrac{X^T A^T(\underset{\sim}{\lambda})B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n\to\infty$

Proof:

i. To show that

$\dfrac{X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn} \xrightarrow{p} 0, n\to\infty$, will be used Chebychev's inequality. It will be shown that $Var\left(\dfrac{X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn}\right)_j \to 0, n\to\infty$. Then, $, j=1,2,...,rp$

take

$$\underset{\sim}{z}=\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}, \underset{\sim}{z}=(z_1,z_2,...,z_{rn})$$

and $u=\underset{1\le i\le rn}{Max}\{z_i\}$

So,

$$Var\left(\mathbf{X}^T\left[\mathbf{I}-\mathbf{A}(\underset{\sim}{\lambda})\right]^T \mathbf{B}\underset{\sim}{f}\right)_j$$

$$=Var\left(\mathbf{X}^T \underset{\sim}{z}\right)_j$$

$$=Var\left(\sum_{i=1}^{rn} x_{ji}z_i\right)$$

$$=\sum_{i=1}^{rn} Var\left(x_{ji}z_i\right)$$

$$=\sum_{i=1}^{rn} z_i^2 Var\left(x_{ji}\right)\le\left(\underset{1\le i\le rn}{Max}\{z_i\}\right)^2\sum_{i=1}^{rn} Var\left(x_{ji}\right)$$

$$=u^2\sum_{i=1}^{rn} v_j$$

$$=u^2 rn\left(v_j\right) \quad (34)$$

$$\le rnu^2 v$$

And the consequences will lead to:

$$Var\left(\frac{X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn}\right)_j$$

$$=\frac{1}{(rn)^2}Var\left(X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}\right)_j$$

$$=\frac{1}{(rn)^2}rnu^2 v_j \quad (35)$$

$$=\frac{1}{rn}u^2 v_j$$

$$=o(1), n\to\infty$$

So,

$$Var\left(\frac{X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn}\right)_j \to 0, n\to\infty$$

So, $\dfrac{X^T\left[I-A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn} \xrightarrow{p} 0, n\to\infty$

ii. To show that $\dfrac{X^T B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n\to\infty$, and it will show that:

$$Var\left(\frac{X^T B\underset{\sim}{\varepsilon}}{rn}\right)_j \to 0, n\to\infty, j=1,2,...,rp$$

Take $\underset{\sim}{e}=B\underset{\sim}{\varepsilon},\ \underset{\sim}{e}=\{e_1,e_2,...,e_{rn}\}$

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$$
\begin{aligned}
Var\left(X^T B\underset{\sim}{\varepsilon}\right)_j &= Var\left(X^T \underset{\sim}{e}\right) \\
&= ar\left(\sum_{i=1}^{rn} x_{ji} e_i\right) \\
&= \sum_{i=1}^{rn} Var\left(x_{ji} e_i\right) \\
&= \sum_{i=1}^{rn} E\left(x_{ji}^2 e_i^2\right) \\
&\le rn\vartheta
\end{aligned} \tag{36}
$$

So,

$$
\begin{aligned}
Var\left(\frac{X^T B\underset{\sim}{\varepsilon}}{rn}\right) &= \frac{1}{(rn)^2} Var\left(X^T B\underset{\sim}{\varepsilon}\right) \\
&= \frac{rn\vartheta}{(rn)^2} \\
&= \frac{\vartheta}{rn} \\
&= o(1), \quad n \to \infty
\end{aligned} \tag{37}
$$

The result is obtained,

$$
Var\left(\frac{X^T B\underset{\sim}{\varepsilon}}{rn}\right)_j \to 0, \quad n \to \infty \tag{38}
$$

So, it has been proven that

$$
\frac{X^T B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n \to \infty
$$

iii. $\frac{X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n \to \infty$ applies if

$$
Var\left(\frac{X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}}{rn}\right)_j \to 0, n \to \infty
$$

$, j = 1, 2, \ldots, rn$

Then, given that $\underset{\sim}{s} = A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}$,

$\underset{\sim}{s} = (s_1, s_2, \ldots, s_{rn})$

$$
\begin{aligned}
Var\left(X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}\right)_j &= Var\left(X^T \underset{\sim}{s}\right)_j \\
&= Var\left(\sum_{i=1}^{rn} x_{ji} s_i\right) \\
&= \sum_{i=1}^{rn} Var\left(x_{ji} s_i\right) \\
&= \sum_{i=1}^{rn} E\left(x_{ji}^2 s_i^2\right) \\
&\le rn\omega
\end{aligned} \tag{39}
$$

As a result,

$$
\begin{aligned}
Var\left(\frac{X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}}{rn}\right)_j &= \frac{1}{(rn)^2} Var\left(X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}\right)_j \\
&= \frac{rn\omega}{(rn)^2} \\
&= \frac{\omega}{rn} \\
&= o(1), \quad n \to \infty
\end{aligned} \tag{40}
$$

So,

$$
Var\left(\frac{X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}}{rn}\right)_j \to 0, n \to \infty \tag{41}
$$

So, it has been proven that $\frac{X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n \to \infty$

Furthermore, based on Lemma 1 and 2, the consistency of the estimator $\hat{\underset{\sim}{\beta}}$ is shown in the following Theorem.

Theorem 5

If $\hat{\underline{\beta}}_n$ sequence of parametric component estimators given by Theorem 1, then $\hat{\underset{\sim}{\beta}}_n \xrightarrow{p} \underset{\sim}{\beta}, n \to \infty$

Proof:

Based on Lemma 1, it is obtained,

$$
\begin{aligned}
\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta} = {} & \left[X^T B X - X^T A^T(\underset{\sim}{\lambda}) B X\right]^{-1} \\
& \left[X^T \left[I - A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f} \right. \\
& \left. + X^T B\underset{\sim}{\varepsilon} - X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}\right]
\end{aligned} \tag{42}
$$

Based on Lemma 2, it has been promoted that

i. $\frac{X^T \left[I - A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f}}{rn} \xrightarrow{p} 0, n \to \infty$

ii. $\frac{X^T B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n \to \infty$

iii. $\frac{X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}}{rn} \xrightarrow{p} 0, n \to \infty$

Furthermore, with Theorem 3 about convergent properties in probability, will be obtained

$$
\left[X^T \left[I - A(\underset{\sim}{\lambda})\right]^T B\underset{\sim}{f} + X^T B\underset{\sim}{\varepsilon} - X^T A^T(\underset{\sim}{\lambda}) B\underset{\sim}{\varepsilon}\right] \tag{43}
$$

$\underset{\sim}{\varepsilon} \xrightarrow{p} \underline{0}, \quad n \to \infty$

As a result, with Theorem 3 obtained

$$\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta} \xrightarrow{p} 0 \text{ atau } \hat{\underset{\sim}{\beta}}_n \xrightarrow{p} \underset{\sim}{\beta}, \quad n \to 0$$

Hyphotesis Testing

In the second objective of the study, namely to obtain hypothesis testing on each relationship between variables in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square (PWLS) approach. Hypothesis testing was carried out to determine the significance of the path analysis formed [15]. There are procedures that can be used in writing hypotheses in general and in specifics [16]. The parameter function hypothesis is:

$H : k'\beta = m$

Bayes Estimator

The observed sampling is given $(t_1, y_1), \ldots, (t_n, y_n)$ obtained from a stochastic process $\{Y(t); t \in [0,1]\}$ and follow the model:

$$Y(t) = f(t) + \varepsilon(t), t \in [0,1]. \tag{44}$$

$\{f(t); t \in [0,1]\}$ has a prior improper distribution:

$$\sum_{t=1}^{m} \alpha_1 \varphi_1(t) + b^{1/2} Z(t), \tag{45}$$

b> 0, polynomial coefficient $\alpha = (\alpha_1, \ldots, \alpha_m)'$ has the distribution $N(0, al), a \to \infty$

$\{Z(t); t \in [0,1]\}$, with:

$$Z(t) = \int_0^1 \frac{(t-u)^{m-1}}{(m-1)!} dw(u) \tag{46}$$

Is a Wiener process with zero mean and covariance of Reproducing Kernel:

$$R^1(s,t) = \int_0^1 \frac{(s-u)_+^{m-1}(t-u)_+^{m-1}}{[(m-1)!]^2} du \tag{47}$$

Polynomial Coefficient $\alpha_1, i = 1, \ldots, m$ and $Z(t)$ is uncorrelated [4] and $\{\varepsilon(t)\}$ is a normal process with mean zero and

$$\operatorname{cov}(\varepsilon(t)), \varepsilon(s)) = \begin{cases} \sigma^2 / w(t), & \text{if } t = s \\ 0 & \text{, if } t \neq s \end{cases}$$

Theorem 2 provides a weighted spline estimator $f_\lambda(t)$ can be written as follows

$$\begin{aligned} & f_\lambda(t) \\ & = (\varphi_1(t), \ldots, \varphi_m(t))(\boldsymbol{T}'\boldsymbol{S}^{-1}(\tilde{\lambda})\boldsymbol{W}\boldsymbol{T})^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}(\lambda)\boldsymbol{W}\tilde{y} + \\ & (\psi_1(t), \ldots, \psi_n(t))\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W} \\ & (\boldsymbol{I} - \boldsymbol{T}(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W}\boldsymbol{T})^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W})\tilde{y}. \end{aligned} \tag{48}$$

The following is given a Theorem which is used to derive the Bayes estimator in a weighted spline.

Theorem 6

Given $\underset{\sim}{y}, \underset{\sim}{f}, \underset{\sim}{\varepsilon}$ Gaussian random vector with zero mean and following the model:

$$\underset{\sim}{y} = \underset{\sim}{f} + \underset{\sim}{\varepsilon}$$

$E(\underset{\sim}{f}\underset{\sim}{f}') = \underset{\sim}{b}V_f, E(\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}') = \sigma^2 \boldsymbol{W}^{-1}$ and $E(\underset{\sim}{f}\underset{\sim}{\varepsilon}') = 0$ if h is normally distributed with

$E(\underset{\sim}{h}) = 0, E(\underset{\sim}{h}\underset{\sim}{h}') = \underset{\sim}{b}\boldsymbol{V}_h, E(\underset{\sim}{h}\underset{\sim}{f}') = \underset{\sim}{b}\boldsymbol{V}_{hf}$

and $E(\underset{\sim}{h}\underset{\sim}{\varepsilon}') = 0$ so:

(i). $E(\underset{\sim}{h} \mid \underset{\sim}{y}) = \boldsymbol{V}_{hf}(\boldsymbol{V}_f + n\lambda \boldsymbol{W}^{-1})^{-1}\underset{\sim}{y}$, and

(ii). $\begin{aligned} & Var(\underset{\sim}{h} \mid \underset{\sim}{y}) = \underset{\sim}{b}(\boldsymbol{V}_h - \boldsymbol{V}_{hf}\boldsymbol{V}_f^{-1}\boldsymbol{V}_{fh}) \\ & + \sigma^2 \boldsymbol{V}_{hf}\boldsymbol{V}_f^{-1}\boldsymbol{A}(\underset{\sim}{\lambda})\boldsymbol{W}^{-1}\boldsymbol{V}_f^{-1}\boldsymbol{V}_{fh} \end{aligned}$

with $\boldsymbol{A}(\underset{\sim}{\lambda}) = \boldsymbol{V}_f(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1})^{-1}$ and $n\underset{\sim}{\lambda} = \sigma^2 / \underset{\sim}{b}$

Proof:

If $\boldsymbol{X}_1 = \underset{\sim}{h}, \boldsymbol{X}_2 = \underset{\sim}{y}$, Lemma 2 will give:

$$\sum_{12} = \underset{\sim}{b}\boldsymbol{V}_{hf} \text{ and} \sum_{22} = \underset{\sim}{b}(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1}) \tag{49}$$

As the results:

(i). $E(\underset{\sim}{h} \mid \underset{\sim}{y}) = \boldsymbol{V}_{hf}(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1})^{-1}\underset{\sim}{y}$ and

(ii). $Var(\underset{\sim}{h} \mid \underset{\sim}{y}) = \underset{\sim}{b}\{\boldsymbol{V}_h - \boldsymbol{V}_{hf}(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1})^{-1}\boldsymbol{V}_{fh}\}$

On the other hand, the Sherman-Morisson-Woodbury formula for matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C},$ dan $\boldsymbol{D}$ will give:

$$(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{D})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C} + \boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{D}\boldsymbol{A}^{-1} \tag{50}$$

If the similarities are taken:

$$\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{C} = \boldsymbol{V}_f \text{ dan } \mathbf{D} = n\underset{\sim}{\lambda}\boldsymbol{W}^{-1} \tag{51}$$

Will be given:

$$(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1})^{-1} = \boldsymbol{V}_f^{-1} - n\underset{\sim}{\lambda}(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1})^{-1}\boldsymbol{W}^{-1}\boldsymbol{V}_f^{-1} \tag{52}$$

As the results:

$$\begin{aligned} Var(\underset{\sim}{h} \mid \underset{\sim}{y}) & = \underset{\sim}{b}(\boldsymbol{V}_h - \boldsymbol{V}_{hf}\boldsymbol{V}_f^{-1}\boldsymbol{V}_{fh}) \\ & + n\underset{\sim}{\lambda}\underset{\sim}{b}\boldsymbol{V}_{hf}(\boldsymbol{V}_f + n\underset{\sim}{\lambda}\boldsymbol{W}^{-1})^{-1}\boldsymbol{W}^{-1}\boldsymbol{V}_f^{-1}\boldsymbol{V}_{fh} \\ & = \underset{\sim}{b}(\boldsymbol{V}_h - \boldsymbol{V}_{hf}\boldsymbol{V}_f^{-1}\boldsymbol{V}_{fh}) \\ & + \sigma^2 \boldsymbol{V}_{hf}\boldsymbol{V}_f^{-1}\boldsymbol{A}(\underset{\sim}{\lambda})\boldsymbol{W}^{-1}\boldsymbol{V}_f^{-1}\boldsymbol{V}_{fh} \end{aligned} \tag{53}$$

With prior and weighted quadratic loss function, the estimator $f_\lambda(t)$ is the posterior mean f, which is fully presented in the following Theorem.

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

Theorem 7

Bayes estimation $f_{\lambda,\phi}(t)$ *from* $f(t)$ *given by :*

$$\begin{aligned} &f_{\lambda,\phi}(t) = E\left(f(t)\middle| \underset{\sim}{y}\right) \\ &= \left(\varphi_1(t),...,\varphi_m(t)\right). \\ &\phi\mathbf{T}'\left(\phi\mathbf{TT}'+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})^{-1}\underset{\sim}{y}+\psi_1(t),...,\psi_m(t)\right) \\ &\left(\phi\mathbf{TT}'+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})^{-1}\underset{\sim}{y}\right) \end{aligned} \quad (54)$$

with $\varphi = a/b$ *and* $\boldsymbol{S}(\underset{\sim}{\lambda}) = \boldsymbol{WV} + n\underset{\sim}{\lambda}\boldsymbol{I}$

Proof:

Rewrite $\underset{\sim}{f} = \left(f(t_1),...,f(t_n)\right)', \underset{\sim}{\varepsilon} = (\varepsilon_1,...,\varepsilon_m)'$

$$\boldsymbol{T} = \left\{t_j^{-1}/(i-1)!\right\} \quad (55)$$

$j = 1,2,...,n, i = 1,2,...,m$

and

$$\boldsymbol{V} = \left\{\boldsymbol{R}^1(t_i,t_j)\right\}, i,j = 1,2,...,n \quad (56)$$

When paired $\underset{\sim}{h} = f(t)$ so that

$$\begin{aligned} &V_{hf} = E\left(f(t)\underset{\sim}{f}'\right)/\underset{\sim}{b} \\ &= \left(\varphi_1(t),...,\varphi_m(t)\right).\phi\boldsymbol{T}'+\left(\phi_1(t),...,\phi_m(t)\right) \end{aligned} \quad (57)$$

From Theorem 7 obtained that:

$$\begin{aligned} \mathbf{V}_f &= E\left(\underset{\sim}{f}\,\underset{\sim}{f}'\right)/\underset{\sim}{b} \\ &= \phi\mathbf{TT}'+\mathbf{V} \end{aligned} \quad (58)$$

With a few elaborations obtained:

$$\begin{aligned} \mathbf{V}_h &= E\left(f(t)f(t)\right)/\underset{\sim}{b} \\ &= \phi\underset{\sim}{\varphi}'\underset{\sim}{\varphi}+\psi(t) \end{aligned} \quad (59)$$

Finally, Theorem 7 and Theorem 1 will give;

$$\begin{aligned} E\left(f(t)\middle|\underset{\sim}{y}\right) &= \left(\varphi_1(t),...,\varphi_m(t)\right) \\ &.\phi\mathbf{T}'\left(\phi\mathbf{TT}'+\mathbf{V}+n\underset{\sim}{\lambda}\mathbf{W}^{-1}\right)^{-1}\underset{\sim}{y}+ \\ &\left(\psi_1(t),...,\psi_m(t)\right) \\ &\left(\phi\mathbf{TT}'+\mathbf{V}+n\underset{\sim}{\lambda}\mathbf{W}^{-1}\right)^{-1}\underset{\sim}{y} \\ &= \left(\varphi_1(t),...,\varphi_m(t)\right).\phi\mathbf{T}' \\ &\left(\phi\mathbf{TT}'+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})\right)^{-1}\underset{\sim}{y}+ \\ &\left(\psi_1(t),...,\psi_m(t)\right) \\ &\left(\phi\mathbf{TT}'+\mathbf{V}+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})\right)^{-1}\underset{\sim}{y} \end{aligned} \quad (60)$$

The weighted spline estimator given by Theorem 2 is a Bayes estimator with the prior improper estimator, which is given in full by Theorem below.

Theorem 8

If $f(t), t \in [0,1]$ *has a prior improper distribution so that:*

$$\lim_{a\to\infty} E_a\left(f(t)\middle|y\right) = f_\lambda(t), \quad (61)$$

With $f_\lambda(t)$ *weighted natural polynomial spline is obtained by minimizing:*

$$n^{-1}\sum_{j=1}^{n} w_j\left(y_j - f(t_j)\right)^2 + \lambda\int_0^1\left[f^{(m)}(t)\right]^2 dt \quad (62)$$

Proof:

Sherman-Morisson-Woodbury formula for equation matrix:

$$\boldsymbol{A} = \boldsymbol{W}^{-1}\boldsymbol{S}(\underset{\sim}{\lambda}), \boldsymbol{B} = \boldsymbol{T}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1} \quad (63)$$

$$\boldsymbol{C}^{-1} = \phi\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right) \text{ dan } \boldsymbol{D} = \boldsymbol{T}' \quad (64)$$

Will give:

$$\begin{aligned} &\left(\phi\boldsymbol{TT}'+\boldsymbol{W}^{-1}\boldsymbol{S}(\underset{\sim}{\lambda})\right)^{-1} = \boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W} + \\ &-\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1} \\ &\left[\phi^{-1}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1}+\boldsymbol{I}\right]\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W} \end{aligned}$$

Lemma 3 give that:

$$\begin{aligned} &\left(\boldsymbol{I}+\phi^{-1}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1}\right)^{-1} \\ &= \boldsymbol{I}-\phi^{-1}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1} \\ &+\phi^{-2}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-2}-... \end{aligned} \quad (65)$$

As the results:

$$\begin{aligned} &\left(\phi\boldsymbol{TT}'+\boldsymbol{W}^{-1}\boldsymbol{S}(\underset{\sim}{\lambda})\right)^{-1} = \boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W}-\boldsymbol{S}^{-1}(\underset{\sim}{\lambda}) \\ &\boldsymbol{WT}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W} + \\ &\phi^{-1}\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\lambda)\boldsymbol{WT}\right)^{-2}\boldsymbol{T}'\boldsymbol{S}^{-1}\boldsymbol{W}+\boldsymbol{O}\left(\phi^{-2}\right) \\ &\lim_{\phi\to\infty}\phi\boldsymbol{T}'\left(\phi\boldsymbol{TT}'+\boldsymbol{W}^{-1}\boldsymbol{S}(\underset{\sim}{\lambda})\right)^{-1} \\ &= \left(\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{WT}\right)^{-1}\boldsymbol{T}'\boldsymbol{S}^{-1}(\underset{\sim}{\lambda})\boldsymbol{W} \\ &\lim_{\phi\to\infty}\left(\phi TT'+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})\right)^{-1} \\ &= \mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{W}-\mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{WT}\left(\mathbf{T'S}^{-1}(\underset{\sim}{\lambda})\mathbf{WT}\right)^{-1} \\ &\mathbf{T'S}^{-1}(\underset{\sim}{\lambda})\mathbf{W} \\ &= \mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{W}\left(\mathbf{I}-\mathbf{T}\left(\mathbf{T'S}^{-1}(\underset{\sim}{\lambda})\mathbf{WT}\right)^{-1}\right. \\ &\left.\mathbf{T'S}^{-1}(\underset{\sim}{\lambda})\mathbf{W}\right) \end{aligned} \quad (66)$$

Finally, Theorem 2 will give:

$$\lim_{\phi\to\infty} E\left(f(t)\middle|y\right)$$
$$=\left(\varphi_1(t),\ldots,\varphi_m(t)\right)\lim_{\phi\to\infty}\phi\mathbf{T}'\left(\phi\mathbf{TT}'+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})\right)^{-1}+$$
$$\left(\psi_1(t),\ldots,\psi_m(t)\right)\lim_{\phi\to\infty}\left(\phi\mathbf{TT}'+\mathbf{W}^{-1}\mathbf{S}(\underset{\sim}{\lambda})\right)^{-1}\underset{\sim}{y}$$
$$=\left(\varphi_1(t),\ldots,\varphi_m(t)\right)\left(\mathbf{T}'\mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{WT}\right)^{-1}\mathbf{T}'\mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{W}\underset{\sim}{y}+$$
$$\left(\psi_1(t),\ldots,\psi_m(t)\right)\mathbf{S}^{-1}(\underset{\sim}{\lambda}) \tag{67}$$
$$\mathbf{W}\left(\mathbf{I}-\mathbf{T}\left(\mathbf{T}'\mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{WT}\right)^{-1}\mathbf{T}'\mathbf{S}^{-1}(\underset{\sim}{\lambda})\mathbf{W}\right)\underset{\sim}{y}$$

The following shows the method of selecting the smoothing parameter λ for a weighted spline estimator based on the Bayes approach, namely Generalized Maximum Likelihood (GML). The basic idea of using the Generalized Maximum Likelihood method in the original nonparametric spline regression was first given by Wahba [4]. Then developed by Wang [17] for correlated data.

Given that ϑ and ω with the decomposition:

$$\begin{pmatrix}\vartheta\\ \ldots\\ \omega\end{pmatrix}=\begin{pmatrix}F'\\ \ldots\\ \dfrac{T'}{\sqrt{\phi}}\end{pmatrix}y,$$ and fulfill $F'T=0$.

First, the following Theorem is given:

(i) ϑ *distributed* $N\left(0,\underset{\sim}{b}\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)$

(ii) ω *distributed* $N\left(0,\underset{\sim}{b}\left(\boldsymbol{T}'\boldsymbol{T}\right)\left(\boldsymbol{T}'\boldsymbol{T}\right)\right)$

Proof:

Given the decomposition and the model

$$\underset{\sim}{y}=\underset{\sim}{f}+\tilde{\varepsilon}$$

Will give:

$$\operatorname{Var}(y)=\underset{\sim}{a}\boldsymbol{T}'\boldsymbol{T}+\underset{\sim}{b}\boldsymbol{V}+\sigma^2\boldsymbol{W}^{-1} \tag{68}$$

With another elaborations, obtained that:

$$\operatorname{Var}(y)=\underset{\sim}{b}\left(\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)+\phi\boldsymbol{TT}'\right) \tag{69}$$

On the other hand, will give:

$$\begin{aligned}\operatorname{Cov}(\vartheta,\omega)&=\underset{\sim}{b}\phi^{-1/2}\mathbf{F}'\left(\mathbf{W}^{-1}\mathbf{S}(\lambda)+\phi\mathbf{TT}'\right)\mathbf{T}\\&=\underset{\sim}{b}\phi^{-1/2}\mathbf{FS}(\lambda)\mathbf{T}\end{aligned} \tag{70}$$

So, can be obtained that:

$$\operatorname{Var}(\omega)=\underset{\sim}{b}\phi^{-1}\boldsymbol{T}'\left(\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)+\phi\boldsymbol{TT}'\right)\boldsymbol{T} \tag{71}$$

and

$$\begin{aligned}\operatorname{Var}(\vartheta)&=\underset{\sim}{b}\mathbf{F}'\left(\mathbf{W}^{-1}\mathbf{S}(\lambda)+\phi\mathbf{TT}'\right)\mathbf{F}\\&=\underset{\sim}{b}\mathbf{F}'\mathbf{W}^{-1}\mathbf{S}(\lambda)\mathbf{F}\end{aligned} \tag{72}$$

For $a\to\infty$, obtained:

$$\begin{aligned}\lim_{a\to\infty}\operatorname{Cov}(\vartheta,\omega_j)&=\underset{\sim}{b}\lim_{\phi\to\infty}\phi^{-1/2}\mathbf{FS}(\lambda)\mathbf{T}\\&=0,\\ \lim_{a\to\infty}\operatorname{Var}(\omega)&=\underset{\sim}{b}\lim_{a\to\infty}\phi^{-1}\mathbf{T}'\left(\mathbf{W}^{-1}\mathbf{S}(\lambda)+\phi\mathbf{TT}'\right)\mathbf{T}\\&=\underset{\sim}{b}\left(\mathbf{T}'\mathbf{T}\right)\left(\mathbf{T}'\mathbf{T}\right)\end{aligned} \tag{73}$$

For $a\to\infty$, remember that y normally distributed with a mean of zero, then Theorem is proved.

Theorem 4 shows that for $a\to\infty$ the distribution of ϑ that depends on λ and $\vartheta\sim N\left(0,\underset{\sim}{b}\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)$. Based on the ϑ distribution, the Likelihood and log Likelihood functions are obtained respectively:

$$L\left(\lambda,b\middle|\vartheta\right)=\frac{1}{(2\pi)^{r/2}\left|\tilde{b}\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right|^{1/2}}$$
$$\exp\left[-\frac{1}{2}\vartheta'\left(\tilde{b}\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\vartheta\right)\right]$$

$$L\left(\lambda,b\middle|\vartheta\right)=\left(\frac{1}{(2\pi)^{r/2}\left|\underset{\sim}{b}\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right|^{1/2}}\right)$$
$$\left(\exp\left[-\frac{1}{2}\vartheta'\left(\underset{\sim}{b}\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\vartheta\right)\right]\right)$$

$$\begin{aligned}\log L\left(\lambda,b\middle|\vartheta\right)&=-\frac{r}{2}\log\underset{\sim}{b}-\frac{1}{2}\log\left|\underset{\sim}{b}\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right|\\&-\frac{1}{2b}\left(\vartheta'\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\vartheta+K\right)\end{aligned} \tag{74}$$

This log Likelihood function provides the Maximum Likelihood estimator:

$$\hat{\underset{\sim}{b}}_{\lambda}=\frac{\vartheta'\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\vartheta}{r} \tag{75}$$

By resubstituting $\hat{b}_{\lambda}$ n the log Likelihood function is obtained:

$$\log L\left(\lambda\middle|\vartheta\right)$$
$$=-\frac{r}{2}\log\left(\frac{\vartheta'\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\vartheta}{\left|\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\right|^{1/r}}\right)+K_2 \tag{76}$$

with K1 and K2 constants that are independent of λ and b. Maximizing $\log L\left(\lambda\middle|\vartheta\right)$ is equivalent to minimizing:

$$GML(\lambda)=\frac{\vartheta'\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\vartheta}{\left|\left(\boldsymbol{F}'\boldsymbol{W}^{-1}\boldsymbol{S}(\lambda)\boldsymbol{F}\right)^{-1}\right|^{1/r}} \tag{77}$$

The optimal λ value is obtained by minimizing the GML (λ).

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

5. CONCLUSION

The conclusion that can be obtained is obtained the properties of the spline estimator in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach, hypothesis testing on each relationship between variables in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach, as well as some findings regarding the confidence interval. optimal on each relationship between variables in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach, complementing the previous research findings, namely a theoretical review of Non-Parametric Path Analysis, as well as a theoretical review regarding the estimation of the variance-covariance error matrix of the Non-Parametric Path Analysis model, as well as selection of smoothing parameters. On the other hand, the third objective is to theoretically test the optimal confidence interval for each relationship between variables in the Nonparametric Regression-based Path Analysis using the Penalized Weighted Least Square approach.

REFERENCES:

[1] Solimun. “Analisis Multivariat Pemodelan Struktural”, Malang: Citra, 2010.

[2] E. Arisoesilaningsih, S. Indriyani, R. Retnowati, A.A.R. Fernandes, “Pemodelan pertumbuhan vegetatif dan produksi umbi porang pada beberapa umur tanaman, kondisi vegetasi, tanah dan iklim agroforestri”, Research Report, Universitas Brawijaya. Malang, 2009.

[3] P. Craven, and G. Wahba, “Smoothing Noisy Data With Spline Function: Estimating the Correct Degree of Smoothing by the Method of Generalized Cross Validation”, *Numerical Mathematics*, vol. 31, no. 1, 1979, pp. 377-403.

[4] G. Wahba, “Spline Models for Observational Data”, Pensylvania: SIAM, 1990.

[5] I.N. Budianara, Subanar, and Z. Soejoeti, “Weighted Spline Estimator”, *Bulletin of the International Statistical Institute*,vol. 51, no. 1, 1997, pp.333-334.

[6] J. Fan, and I. Gijbels, “Local Polynomial Modelling and Its Applications”, New York: Chapman and Hall, 1997.

[7] W. Hardle, “Applied Nonparametric Regression”, New York: Cambridge University Press, 1990.

[8] A. Antoniadis, G. Gregoire, and I.W. McKeague, “Wavelet methods for curve estimation”, *Journal of the American Statistical Association*, vol. 89, no. 428, 1994, pp.1340-1353.

[9] P. Malliavin, and M.E. Mancino, “A Fourier Transform Method for Non-parametric Estimation of Multivariate Volatility”, *The Annals of Statistics,* vol. 37, no. 4, 2009, pp.1983-(2010).

[10] I.N. Budiantara, B. Lestari and A. Islamiyati, “Estimator Spline Terbobot Spline Parsial Terbobot dalam Regresi Nonparametrik dan Semi parametrik Heteroskedastik untuk Data Longitudinal”, Surabaya: Laporan Akhir Program Hibah Kompetensi Tahun 1, Lembaga Penelitian dan Pengabdian Kepada Masyarakat. Institut Teknologi Sepuluh Nopember, 2009.

[11] W.R. Dillon, and M. Goldstein, “Multivariate Analysis Methods and Applicationds”, New York: John Wiley dan Sons, 1984.

[12] J. Supranto, “Statistik : Teori dan Aplikasi Edisi 8”, Jakarta: Erlangga, 2010.

[13] R.L. Eubank, “Nonparametric Regression and Spline Smoothing. Second Edition”, New York: Marcel Dekker, Inc, 1999.

[14] P. Speckman, “Spline smoothing and optimal rates of convergence in nonparametric regression models”, *The Annals of Statistics*, 1985, pp. 970-983.

[15] W.R.J. Mendengall, Beaver, and B.M. Beaver, “). Introduction to Probabilty and Statistics. Edisi 14th”, United States of America: Brooks/Cole Cengage Learning, 2013.

[16] S.R. Searles, “Linear models”, New York: Wiley, 1971.

[17] Y. Wang, “Smoothing spline models with correlated random errors”, *Journal of the American Statistical Association*, vol. 93, no. 441, 1998, pp. 341-348.