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IMAGE ENHANCEMENT BASED ON NONLINEAR AND ANISOTROPIC DIFFUSION

[1]MOHAMMED AIT OUSSOUS and [2]YOUSSEF AIT KHOUYA

[1] Software Engineering and Information Systems Engineering team, Faculty of Sciences and Techniques Errachidia, Moulay Ismail University, Morocco

[2] Software Engineering and Information Systems Engineering team, Faculty of Sciences and Techniques Errachidia, Moulay Ismail University, Morocco

E-mail: [1]aitoussous.med@gmail.com, [2]youssefaitkhouya@gmailcom

ABSTRACT

In this paper, we present evidence of the existence and the uniqueness of the solution of the proposed model in [1], where is applied to image enhancement and filtering. The main idea of the model is to apply a Gaussian filter to the image gradient when computing the diffusion coefficient. The gradient threshold parameter is also calculated from the image gradient at each iteration. The numerical experiments are presented along with a discussion of results.

Keywords: *Nonlinear Reaction- Diffusion, Image Processing, Gaussian Filter, Image Gradient, Existence and and Uniqueness.*

1. INTRODUCTION

The PDE approach is very used to solve many problems in image processing and computer vision such as: image restoration [2-9], edge detection [10-12], object tracking [13], [14] and deformable contours [15-19]. PDE-based methods are one of the mathematically best-founded techniques [20-21]. The oldest and most investigated equation in image processing is probably the parabolic linear heat equation [22-23].

We note that Koenderink in [22] was the first to notice that the evolution of this equation is equivalent to a Gaussian smoothing. The heat equation is an isotropic operation in the sense that the diffusion is carried out in the same manner on all the pixels of the image. It gradually reduces all variations of the image starting with changes in low intensities corresponding to the noise in the image. But it ends by smoothing the edges, which makes the image blurred. To overcome this problem Perona and Malik [24] suggest a nonlinear diffusion method for avoiding the blurring and localization problems of linear diffusion filtering. This equation has recently stimulated a great deal of interest in image processing community. It is commonly believed that the Perona-Malik equation provides a potential algorithm for image processing. The birth of nonlinear and anisotropic methods, which are robust tools in image processing, combine the advantages of nonlinearity and anisotropy.

Morfu [25], gave an algorithm for noise filtering based on nonlinear and anisotropic diffusion processes which also produces contrast enhancement while the image edges are preserved.

He analyses how nonlinearity and anisotropy can be coupled to yield an adequate image processing tool. His model is inspired by the nonlinear Fisher-equation which was proposed by Fisher (1937) as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population [26-27]. But this algorithm has two drawbacks: First, during processing, some details of the image are lost; second, is not resistant to noise, when we increase the noise variance slightly, Morfu's algorithm gives a poor result. To overcome these problems, in [28] we are suggested a new processing algorithm based on anisotropic diffusion and non-linear process for noise filtering and contrast enhancement of an image while preserving edges. In this paper, we establish the uniqueness, the existence and the regularity of the model and give a comparative study with that of Morfu.

This paper is organized as follows: In Section 2, we describe the Morfu model. In Section 3, we review the fundamental concepts of the proposed algorithm. The proof of existence, uniqueness and regularity of the model and the numerical consideration are shown in Sections 4 and 5 respectively. The experimental results are presented in section 6. This paper ends with a brief conclusion section.

2. MORFU MODEL

The nonlinear and anisotropic methods offer many treatments in the image processing field. They couple the advantages of nonlinearity and anisotropy. Morfu [25] produced a processing tool were applied to image filtering and contrast enhancement. This model is inspired from the Fischer model that combines the advantages of nonlinearity and anisotropy. Let us consider a rectangular image domain $\Omega \subset IR^2$ with boundary $\partial\Omega$ and let an image be represented by a mapping I0. He considers the following continuous equation represented by the initial boundary value problem

$$\begin{cases} \frac{\partial I(t,x)}{\partial t} = div(g(|\nabla I|)\nabla I, \lambda) \\ \qquad + f(t,x,I) \quad (t,x) \in (0,T) \times \Omega \\ I(0,x) = I_0 \qquad x \in \Omega \\ \frac{\partial I}{\partial n} = 0 \qquad (t,x) \in (0,T) \times \partial\Omega \end{cases} \quad (2.1)$$

where f is a function which reveals the nonlinearity, usually chosen cubic [29-30], defined as:

$$f(u) = -\beta u(u - \alpha)(u - 1) \quad (2.2)$$

β adjusts the weight of the nonlinearity, so if β tends to 0, equation (2.1) is equivalent to that of Perona and Malik [24]. Moreover, α is called the threshold of the nonlinearity. To ensure the symmetry of the nonlinearity, Morfu [31] choose the coefficient α equal to 1/2. The diffusivity function is decreasing and smooth function defined under conditions: $g : [0,+\infty[\times[0,+\infty[$ where $g(0) = 1$ and $\lim_{s\to+\infty} g(s) = 0$

The function g is defined as:

$$g(||\nabla I||, \lambda) = \frac{d}{1+(\frac{||\nabla I||}{\lambda})^2} \quad (2.3)$$

according to [32], the parameter λ is chosen as:

$$\lambda = 1.4826 MAD(\nabla I)/\sqrt{2} \quad (2.4)$$

where MAD denotes the median absolute deviation and can be calculated as

$$MAD(\nabla I) = median_I[||\nabla I - median_I(||\nabla I||)||] \quad (2.5)$$

where $median_I$ $(||\nabla I||)$ represents the median of the gradient amplitude of the image I.

3. PROPOSED MODEL

The main key of this model is to improve the Morfu model by applying a Gaussian filter on the gradient of the noisy image when computing the anisotropic diffusion coefficient and taking the gradient threshold parameter depending on the gradient of the image at each iteration. The proposed model is defined as [1]:

$$\begin{cases} \frac{\partial I(t,x)}{\partial t} = div(g(|G_\sigma * \nabla I|)\nabla I, \lambda) \\ \qquad + f(t,x,I) \quad (t,x) \in (0,T) \times \Omega \\ I(0,x) = I_0 \qquad x \in \Omega \\ \frac{\partial I}{\partial n} = 0 \qquad (t,x) \in (0,T) \times \partial\Omega \end{cases} \quad (3.1)$$

The nonlinearity function f defined by the equation (2.2). It's easy to observe that, if β tends to 0, equation (2.1) is equivalent to [33] model. The function of diffusivity g is defined by equation (2.3) because it is the higher stable function.
The parameter λ can identify regions of the lowest and the highest image gradient. G_σ is the Gaussian filter where:

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{|x|^2}{4\sigma}\right)}, x \in IR^2 \quad (3.2)$$

Areas of the image where $||G_\sigma * \nabla I|| < \lambda$ are supposed to be uniform regions and the noise need to be reduced.
The d is the higher values of the diffusion function.
Conversely, parts of the image where $||G_\sigma * I|| \geq \lambda$ are supposed brutal variations of intensity produced by the contours necessity to be preserved. The function of diffusivity takes values near 0 blocking the diffusion. The values of parameter λ must be calculated at each iteration of the diffusion process because the diffusion smooths the image see [1] for more details.

$$\lambda(t) = \frac{1.4826 MAD(\nabla I^t)}{\sqrt{2}}$$

The MAD is computed with equation (2.5) and the threshold λ is calculated at each iteration.

4. EXISTENCE AND UNIQUENESS OF THE MODEL

The proposed model is as follows:

$$\begin{cases} \frac{\partial v}{\partial t} - div(g(|\nabla v_\sigma|)\nabla u) = f(t,x,v) \quad in\ Q_T \\ v(0,x) = v_0 \qquad in\ \Omega \\ \frac{\partial v}{\partial n} = 0 \qquad on\ \Sigma_T \end{cases} \quad (4.1)$$

$\Omega =]0,1[\times]0,1[$ is the image domain with bord $\partial\Omega$ and Neumann boundary conditions. Where v_0 is the initial image to be filtered and enhanced, n is the unit vector Normal to domain Ω, $Q_T =]0,T[\times\Omega$, where T is a fixed reel number $(T > 0)$, ∇ is the gradient operator, the gradient norm defined as:

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$$|\nabla w| = \sqrt{\sum_{j=1}^{j=2}\left(\frac{\partial w}{\partial x_j}\right)^2},$$

∇w_σ is the smoothed version of gradient norm where $\nabla w_\sigma = \nabla(w * G_\sigma) = w * \nabla G_\sigma$.

And G_σ is defined as equation (3.2). We propose that the initial image satisfy $0 \le v_0(x)$, and for nonlinearity function f we propose the following hypotheses:

$f: Q_T \times IR \to IR$ is measurable and $f(t,x,.) : IR \to IR$ is continuous (4.2)

we present the principal properties of f :
1- the positivity of the solution v of (4.1) is preserved over time, which is ensured by:
For almost $(t,x) \in Q_T, f(t,x,0) \ge 0$ (4.3)

2- the total mass is controlled in function of time:
$\forall v \in IR$ and for almost $(t,x) \in Q_T$,
$vf(t,x,v) \le 0$ (4.4)
If $f = 0$ this case was discussed by Catté, in [33], where give the proof of the existence, uniqueness and regularity of a solution for $\sigma > 0$ and $v_0 \in L^2(\Omega)$.
Here we defined some functional spaces that will be needed for the proof. For all $k \in IN$ $H^k(\Omega)$ is the set of functions v defined in Ω such as v and its order $D^s v$ derivatives where $|s| = \sum_{j=1}^n s_j \le k$ are in $L^2(\Omega)$: $H^k(\Omega)$ is a Hilbert space for the norm

$$\|v\|_{H^k(\Omega)} = \left(\sum_{|s|\le k}\int_\Omega |D^s v|^2\, dx\right)^{\frac{1}{2}} \quad (4.5)$$

We denote by
$(H^1(\Omega))'$ the dual of $H^1(\Omega)$.

$L^p(0,T;H^k(\Omega))$ is the set of functions v such that, for all every $t \in (0,T)$, $v(t)$ belongs to $H^k(\Omega)$ with the norm

$$\|v\|_{L^p(0,T;H^k(\Omega))} = \left(\int_0^T \|v(t)\|^p_{H^k(\Omega)}\, dt\right)^{\frac{1}{p}}, 1 < p < \infty,\ k \in IN \quad (4.6)$$

$L^\infty(0,T;L^2(\Omega))$ is the set of functions v such that, for all every $t \in (0,T)$, $v(t)$ belongs to $L^2(\Omega)$ with the norm:

$$\|v\|_{L^\infty(0,T;L^2(\Omega))} = \left(\sup_{0<t<T}\|v(t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \quad (4.7)$$

$L^\infty(0,T;C^\infty(\Omega))$ is the set of functions v such that, for all every $t \in (0,T)$, $v(t)$ belongs to $C^\infty(\Omega)$ with the norm

$$\|v\|_{L^\infty(0,T;\,C^\infty(\Omega))} = \inf\{c,\ \|v(t)\|_{C^\infty(\Omega)} \le c \ \ in\ (0,T)\}$$

4.1 Hypotheses. Firstly, to solve the problem (4.1). It's necessary to specify the direction.
Definition 4.1. A function v is a weak solution of (4.1) if

$$\begin{cases} v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), f(t,x,v) \in L^1(Q_T) \\ \text{for all } \ \varphi \in C^1(Q_T) \ \text{ such that } \varphi(T,.) = 0 \\ \int_{Q_T} -v\,\partial\frac{\partial\varphi}{\partial t} + g(|\nabla v_\sigma|)\nabla v\nabla\varphi = \int_{Q_T} f(t,x,v(t))\varphi + \int_\Omega v_0\varphi(0,x) \end{cases}$$
(4.8)
If moreover $v \in C^1(Q_T)$ then we say that v is a classical solution of (4.1).

4.2. Main result.
We have adopted the theorem proposed in [34], defined as:
Theorem 4.1. Assume that (4.2)-(4.4) and that
$\forall R \ge 0 \ \sup_{|v|\le R}(f(t,x,v)) \in L^1(Q_T)$ (4.9)
Then for all fixed $T > 0$ and $\sigma > 0$ and for any $v_0 \in L^2(\Omega)$ such as $v_0 \ge 0$, problem (4.8) admits a weak positive solution. If moreover $\forall r \ge 1$ $f(t,x,r) \le 0$ and $v_0(x) \le 1$ we have $0 \le v(t,x) \le 1$ in Q_T
With $f(t,x,v) = -\beta v(v-a)^{2\alpha}(1-v)$ (4.10)
where $\alpha, \beta > 0$ and $0 < a < 1$.
The proof of Theorem 4.1 is explained in four steps, the reader can consult [34] for more details:
Step 1: Positivity of the solutions: Consider the function
Suppose the function $sign^-$ defined as:

$$sign^-(r) = \begin{cases} -1 & if \quad r < 0 \\ \ \ 0 & if \quad r \ge 0 \end{cases} \quad (4.11)$$

The sequence of convex functions $j_\epsilon(r)$ such as $j'_\epsilon(r)$ is bounded and $\forall r \in IR$
$j'_\epsilon(r) \to sign^-(r)$ when $\epsilon \to 0$
Let v be a solution of (4.8), we multiply both sides of the _first equation by $j'_\epsilon(v)$ and by integrating on $Q_t =]0,t[\times \Omega$ for $t \in [0,T[$, we obtain

$$\int_{Q_t} j'_\epsilon(v)\partial\frac{\partial v}{\partial t}dxdt + \int_{Q_t} A\nabla v.\nabla j'_\epsilon(v)dxdt = \int_{Q_t} f(s,x,v)j'_\epsilon(v)dxds \quad (4.12)$$

Where $A(t,x) = g(|\nabla v_\sigma|) \in L^\infty(0,T;\ C^\infty(\Omega))$ because $v \in L^\infty(0,T;L^2(\Omega))$ and g, G_σ are C^∞ and we can show the existence of a C_0 depends only on G_σ, $\|v_0\|_{L^2(\Omega)}$ such as:
$\|\nabla v_\sigma\|_{L^\infty(Q_T)} \le C_0$ (4.13)
Moreover, as g is decreasing, then there $a = g(C_0) > 0$ which depends only on σ and on $\|v_0\|_{L^2(\Omega)}$ such as: $A(t,x) \ge a \ \ \forall (t,x) \in Q_T$ (4.14)
Consequently,

$$\int_\Omega [j_\epsilon(v)(t) - j_\epsilon(v)(0)]dx + a\int_{Q_t} |\nabla v|^2 j''_\epsilon(v)dxds \leq \int_{Q_t} f(s,x,v)j'_\epsilon(v)dxds \quad (4.15)$$

Since $\int_\Omega j_\epsilon(v)(0)\, dx = 0$ and $\int_{Q_t} |\nabla v|^2 j''_\epsilon(v)dxds \geq 0$, then we have:

$$\int_\Omega j_\epsilon(v)(t)\, dx \leq \int_{Q_T} f(s,x,v)j'_\epsilon(v)dxds$$

$$\leq \int_{[v<0]} f(s,x,v)j'_\epsilon(v)\, dxds + \int_{[v\geq 0]} f(s,x,v)j'_\epsilon(v)dxds$$

On the set where $v \geq 0$ we have $j'_\epsilon(v) = 0$ and $\int_{v\geq 0} f(s,x,v)j'_\epsilon(v)dxds = 0$, therefore

$$\int_\Omega j_\epsilon(v)(t)\, dx \leq \int_{[v<0]} f(s,x,v)j'_\epsilon(v)\, dxds \quad (4.16)$$

When $\epsilon \to 0$, we obtain

$$\int_\Omega (v)^-(t)dx \leq -\int_{v<0} f(s,x,v)\, dxds \quad (4.17)$$

Using (4.4) and the fact that $(v)^-(t) \geq 0$, we obtain $(v)^-(t) = 0$ on Ω therefore, $v \geq 0$ in Q_T.

Step 2: An existence result when f is bounded:

Theorem 4.2. Assume (4.2)-(4.3), and that there $\exists\ M \geq 0$ such as for almost $(t,x) \in Q_T, \forall\ r \in IR$,

$$|f(t,x,r)| \leq M \quad (4.18)$$

Then for all $v_0 \in L^2(\Omega)$, problem (2.1) admits a weak solution. Moreover, there exists $C = C(M, a, T, \|v_0\|_{L^2(\Omega)})$ such that

$$\sup_{0<t<T} \|v(t)\|_{L^2(\Omega)} + \|v\|_{L^2(0,T;H^1(\Omega))} \leq C \quad (4.19)$$

Proof. The Schauder fixed point theorem is used to demonstrate the existence of a weak solution.

Firstly, we defined the space

$$W(0,T) = \left\{u \in L^2(0,T;H^1(\Omega)), \frac{du}{dt} \in L^2(0,T;\ (H^1(\Omega))')\right\} \quad (4.20)$$

which is a Hilbert space for the graph norm. Let $u \in W(0,T) \cap L^\infty(0,T;\ L^2(\Omega))$ and we consider $v(u)$ the solution of the linear problem:

$$\begin{cases} v(u) \in C([0,T]; L^2(\Omega)) \cap L^2(0,T;\ H^1(\Omega)), \\ \text{for all} \quad \varphi \in C^1(Q_T) \quad \text{such that } \varphi(T,.) = 0 \\ \int_{Q_T} -v(u)\frac{\partial\varphi}{\partial t} + g(|\nabla u_\sigma|)\nabla v(u)\nabla\varphi = \\ \qquad \int_{Q_T} -f(t,x,u(t))\varphi + \int_\Omega v_0\varphi(0,x) \end{cases} \quad (4.21)$$

Based on the classical theory defined in [35, 36], equation (4.21) admits a unique solution $v(u) \in W(0,T)$, and by applying a classic bootstrap argument, we have $v(u)(t) \in H^1(\Omega)$ for all $t > 0$ since $f(t,x,u(t)) \in L^\infty(Q_T)$ then $v(u)(t) \in H^1(\Omega)$ for all $t > 0$. By application the general classical theory defined in [37] and according to [34] the $v(u)$ is a classical solution and $v(u) \in C^\infty(]0,T[\times\ \Omega)$

We take $\varphi = v(u)$ in (4.21), and deduce that for all $0 < t < T$:

$$\frac{1}{2}\int_\Omega [v(u)^2(t) + \int_{Q_t} g(|\nabla u_\sigma|)|\nabla v(u)|^2 = \int_{Q_t} f(t,x,u(t))v(u) + \frac{1}{2}\int_\Omega v_0^2\ dx \quad (4.22)$$

Using (4.14) and the assumption (4.18) on f, we obtain

$$\frac{1}{2}\int_\Omega [v(u)^2(t) + a\int_{Q_t} |\nabla v(u)|^2 \leq M(1 + \int_{Q_t} v(u)^2) + \frac{1}{2}\int_\Omega v_0^2\ dx \quad (4.23)$$

so, by Gronwall's lemma, we obtain the estimation of (4.19) which guide to introduce the space

$$W_0(0,T) = \left\{ \begin{array}{l} u \in W(0,T) \cap L^\infty(0,T;\ L^2(\Omega)), u(0) = v_0 \ \ and \\ \sup_{0<t<T} \|v(t)\|_{L^2(\Omega)} + \|v\|_{L^2(0,T;H^1(\Omega))} \leq C \end{array} \right\}$$

where $C = C(M, a, T, \|v_0\|_{L^2(\Omega)})$ is the constant obtained in (4.19).

We can easily show that $W_0(0,T)$ is a nonempty closed convex in $W(0,T)$ moreover it injects with a compact way in $L^2(0,T;L^2(\Omega))$ Then we define the application:

$$F\colon W_0(0,T) \to W_0(0,T)$$
$$u \to v(u) \text{ where } v \text{ is a solution of (4.21)} \quad (4.24)$$

Estimate (4.18) shows that F is well defined. According to Schauder fixed point theorem, we show that F is weakly continuous from $W_0(0,T)$ in $W_0(0,T)$.

Then consider a sequence (u_n) in $W_0(0,T)$, such as $v_n \to v$ in $W_0(0,T)$, and let $v_n = F(u_n)$ According to the classical results of compactness, we can extract from the sequence (v_n) a subsequence yet denoted (v_n) such that

- $v_n \to v$ weakly in $L^2(0,T;H^1(\Omega))$
- $v_n \to v$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T
- $\nabla v_n \to \nabla v$ weakly in $L^2(0,T;L^2(\Omega))$
- $u_n \to u$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T
- $\nabla G_\sigma * u_n \to \nabla G_\sigma * u$ strongly in $L^2(0,T,L^2(\Omega))$ and almost everywhere in Q_T
- $g(|\nabla G_\sigma * u_n|) \to g(|\nabla G_\sigma * u|)$ strongly in dans $L^2(0,T;L^2(\Omega))$
- $f(t,x,u_n) \to f(t,x,u)$ strongly in $L^1(Q_T)$

The latter is obtained by applying the dominated convergence theorem. We can then pass to the limit in (4.21), with $v = F(u)$ By uniqueness of the solution of (4.21), then the sequence $v_n = F(u_n)$ converges weakly to $v = F(u)$ in $W_0(0,T)$. Converges weakly to $v \in W_0(0,T)$ such as $v = F(v)$. and thus the existence of $v \in W(0,T)$ such us $v = U$:

Step 3: Approximate problem and a priori estimates Consider the truncation

Function $\Psi_n \in C_0^\infty(IR)$ defined by

$$\Psi_n(r) = \begin{cases} 1 & if \quad |r| \leq n \\ 0 & if \quad |r| \geq n+1 \end{cases} \quad (4.25)$$

We truncate the nonlinearity f by Ψ_n

$$f_n(t,x,v) = \Psi_n(|v|)f(t,x,v) \quad (4.26)$$

Thus, we can easily check that f_n satisfies (4.2), (4.3), (4.4) with $M = M(n)$ and for almost $(t,x) \in Q_T, \forall r \in IR$ $f_n(t,x,r) \to f(t,x,r)$
Since $v_0 \in L^2(\Omega)$ and $|f_n(t,x,r)| \le M_n$, theorem equation (4.18) is applied, then we can deduce the existence of a weak solution of the problem

$$\begin{cases} \frac{\partial v_n}{\partial t} - div(g(|\nabla(v_n)_\sigma|)\nabla v_n) = f_n(t,x,v_n) & in\ Q_T \\ v_n(0,x) = v_0 & on\ \Omega \\ \frac{\partial v_n}{\partial n} = 0 & on\ \Sigma_T \end{cases} \quad (4.27)$$

Remark 4.1. Since $v_0 \ge 0$ on Ω the (i) assures that $v_n \ge 0$ is in Q_T. Moreover, under the assumption (1.7) we have also $f_n(t,x,v_n) \le 0$ in Q_T.

Now we will show that a subsequences v_n converges to the weak solution v of problem (4.1). For this we need to prove the following result:

Lemma 4.1. Let (v_n) the sequence of weak solutions defined by (4.19), then we have:

(i) $\int_{Q_T} |f_n(t,x,v_n)| \le \int_\Omega |v_0|\ dx$

(ii) (v_n) is bounded in $L^2(0,T;H^1(\Omega))$ *and*

$$\int_{Q_T} |v_n f_n(t,x,v_n)|dxdt \le \frac{1}{2}\int_\Omega {v_0}^2\ dx$$

(iii) (v_n) is relatively compact in $L^2(Q_T)$.

Proof. (i) By Remark 4.1, $|f_n(t,x,v_n)| = -f_n(t,x,v_n)$, Thus by integrating the equation satisfied by v_n in Q_T we obtain

$$\int_\Omega v_n(T)dx - \int_{Q_T} f_n(t,x,v_n)dxdt = \int_\Omega v_0\ dx \quad (4.28)$$

Therefore

$$\int_{Q_T} |f_n(t,x,v_n)|dxdt \le \int_\Omega |v_0|\ dx \quad (4.29)$$

(ii) Firstly we show that v_n is bounded in $L^2(Q_T)$. for this we consider $\varphi = v_n$ as a function test in (4.27), we then deduce that

$$\frac{1}{2}\int_\Omega {v_n}^2(t) + \int_{Q_t} g(|\nabla(v_n)_\sigma|)|\nabla v_n|^2 = \int_{Q_t} f(t,x;v_n)v_n + \frac{1}{2}\int_\Omega v_0^2\ dx \quad (4.30)$$

Then we use (4.14) and the hypothesis (4.15) on f, to obtain

$$\frac{1}{2}\int_\Omega {v_n}^2(t) + a\int_{Q_t} |\nabla v_n|^2 \le \frac{1}{2}\int_\Omega v_0^2\ dx \quad (4.31)$$

We have also

$$\int_{Q_T} v_n|f_n(t,x,v_n)|dxdt \le \frac{1}{2}\int_\Omega v_0^2\ dx \quad (4.32)$$

where we have

$$\sup_{0<t<T} \|v_n(t)\|_{L^2(\Omega)} \le \|v_0\|_{L^2(\Omega)}$$

$$\|v_n\|_{L^2(0,T;H^1(\Omega))} \le \left(1 + \frac{1}{2a}\right)\|v_0\|_{L^2(\Omega)}$$

(iii) Since $\frac{\partial v_n}{\partial t} - div(A_n \nabla v_n) + f_n(t,x,v_n)$ is bounded in $L^1(0,T;(H^1(\Omega))') + L^1(\Omega)$.

Since v_n is also bounded in $L^2(0,T;H^1(\Omega))$ and that the injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact, it follows that (v_n) is relatively compact in $L^2(Q_T)$ [38].

Step 4: Convergence According to (iii), the sequence (v_n) is relatively compact in $L^2(Q_T)$, so we can extract a subsequence still denoted (v_n) such that:

- $v_n \to v$ strongly in $L^2(Q_T)$ and almost everywhere in Q_T
- $\nabla G_\sigma * v_n \to \nabla G_\sigma * v$ strongly in $L^2(Q_T)$ and almost everywhere in Q_T
- $g(|\nabla G_\sigma * v_n|) \to g(|\nabla G_\sigma * v|)$ strongly in $L^2(Q_T)$
- $f_n(t,x,v_n) \to f(t,x,v)$ for almost everywhere in Q_T

To prove that u is a weak solution of (1.1), it suffices to prove that $f_n(t,x,v_n) \to f(t,x,v)$ in $L^1(Q_T)$. Since $f_n(t,x,v_n) \to f(t,x,v)$ almost everywhere in Q_T, we will demonstrate that $(f_n(t,x,v_n))$ is uniformly integrable in $L^1(Q_T)$. For this we show that:

$\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that $\forall\ E \subset Q_T$ measurable with $|E| < \delta$ we have $\int_E |f_n(t,x,v_n)| \le \varepsilon$ (4.33)

Then for all $k \ge 0$,

$$\int_E |f_n(t,x,v_n)|dx \le \int_{E\cap[v_n \le k]} |f_n(t,x,v_n)|dx + \int_{E\cap[v_n>k]} |f_n(t,x,v_n)|dx \quad (4.34)$$

For the first term on the right-hand side, we have

$$\int_{E\cap[v_n\le k]} |f_n(t,x,v_n)|dx \le \int_E \sup_{|r|\le k} |f(t,x,r)|dx \quad (4.35)$$

According to (4.9), we have $\sup_{|v|\le k} f(t,x,v) \in L^1(Q_T)$ is uniformly integrable in $L^1(Q_T)$, therefore $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$. such that if $|E| < \delta$ then $\int_E \sup_{|v|<=k} |f(t,x,v)|dx \le \frac{\varepsilon}{2}$ (4.36)

For the second term we have

$$\int_{E\cap[v_n>k]} |f_n(t,x,v_n)|dx \le \frac{1}{k}\int_{Q_T} v_n|f_n(t,x,v_n)|dx \quad (4.37)$$

Then, using (4.32) we obtain

$$\int_{E\cap[v_n>k]} |v_n f_n(t,x,v_n)|dx \le \frac{1}{2k}\|v_0\|^2_{L^2(\Omega)}$$

Now if we choose $k \ge \frac{\|v_0\|^2_{L^2(\Omega)}}{\varepsilon}$ then we have

$$\int_{E\cap[v_n>k]} |f_n(t,x,v_n)|dx \le \frac{\varepsilon}{2} \quad (4.38)$$

consequently, (4.33) follows from (4.36) and (2.38).

Using the following lemma, we complete the proof of Theorem 4.1

Lemma 4.2. Let v be a weak solution of (4.8), and assume that $0 \le v_0 \le 1$ in Ω. Then, $0 \le v \le 1$ in Q_T.

Proof. We have already obtained the positivity of weak solutions if the initial data is positive. So, we assume that $v_0 \le 1$ and proof that $v \le 1$.

For this, we take $\bar{v} = 1 - v$, then we have $\nabla\bar{v} = \nabla v$ we can verify that $\bar{v}$ satisfies

$$\begin{cases} \bar{v} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), f(t,x,1-\bar{v}) \in L^1(Q_T) \\ \text{for all } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0 \\ \int_{Q_T} -\bar{v}\,\partial\frac{\partial\varphi}{\partial t} + g(|\nabla\bar{v}_\sigma|)\nabla\bar{v}\nabla\varphi = \int_{Q_T} f(t,x,1-\bar{v}(t))\varphi - \int_\Omega v_0\varphi(0,x) \end{cases}$$

(4.39)

Then we consider the sequence of convex functions $j_\epsilon(r)$ such as $j'_\epsilon(r)$ is bounded and $\forall r \in IR$, $j'_\epsilon(r) \to sign^-(r)$ when $\varepsilon \to 0$. We take $\varphi = j'_\epsilon(\bar{v})$ as a test function in (4.39) and integrating with respect to $t \in]0,T[$, we obtain

$$-\int_\Omega j_\epsilon(\bar{v})(t,x)dx \le \int_0^t \int_\Omega f(t,x,1-\bar{v}) j'_\epsilon(\bar{v})dxdt \quad (4.40)$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$-\int_\Omega (\bar{v})^-(t,x)dx \le \int_0^t \int_{[v\ge 1]} f(t,x,v)dxdt \quad (4.41)$$

The fact f satisfies $\forall r \ge 1, f(t,x,r) \le 0$, we deduce

$$\int_\Omega (\bar{v})^-(t,x)dx \ge 0 \quad (4.42)$$

Therefore $\bar{v}(t) \ge 0$ which implies $v = 1 - \bar{v} \le 1$

5. NUMERICAL CONSEDIRATION

The proposed model is discretised as:

$$I^{t+dt}(i,j) = I^t(i,j) + dt.div\big(g(\|\nabla G_\sigma * I^t(i,j)\|,\lambda)\nabla I^t(i,j)\big) + dt.f\big(I^t(i,j)\big)$$

(5.1)

The gradient operator ∇ is discretised by finite difference on the right

$$\nabla I^t(i,j) = \begin{pmatrix} \nabla_{x_1} I^t(i,j) \\ \nabla_{x_2} I^t(i,j) \end{pmatrix} = \begin{pmatrix} I^t(i+1,j) - I^t(i,j) \\ I^t(i,j+1) - I^t(i,j) \end{pmatrix}$$

Then the discretization of the divergence is given by finite difference left

$$div(P(i,j) = P_1(i,j) - P_1(i-1,j) + P_2(i,j) - P_2(i,j-1)$$

Where

$$P(i,j) = \begin{pmatrix} P_1(i,j) \\ P_2(i,j) \end{pmatrix} = \begin{pmatrix} g(\|h(i,j)\|,\lambda)\nabla_{x_1} I(i,j) \\ g(\|h(i,j)\|,\lambda)\nabla_{x_2} I(i,j) \end{pmatrix}$$

With

$$h(i,j) = \begin{pmatrix} h_1(i,j) \\ h_2(i,j) \end{pmatrix} = \begin{pmatrix} (G_\sigma * \nabla_{x_1} I)(i,j) \\ (G_\sigma * \nabla_{x_2} I)(i,j) \end{pmatrix}$$

The convolution product with the Gaussian kernel is discretised as:

$$(G_\sigma * X)(i,j) = \sum_{k_1=-1}^{k_1=1} \sum_{k_2=-1}^{k_2=1} G_\sigma(k_1,k_2) X(i-k_1, j-k_2)$$

$$X(i,j) = \{\nabla_{x_1} I(i,j) \text{ or } \nabla_{x_2} I(i,j)\}$$

6. EXPERIMENTAL RESULTS

We make a comparative study between the proposed model and the Morfu model for removing noise and contrast enhancement. We fixed the parameters of the nonlinearity β to 1 and α to 0.5 to conserve the symmetry of the function f. The parameter λ is calculated at each time with equation (2.4). The processing times T is set to 5 and dt to 0.01. The parameter σ is fixed to 0.7.

We used the EME criteria to compare the results obtained by two models, it's defined as:

Let an image $I(N,M)$ be split into $k_1 k_2$ blocks $w_{k,l}(i,j)$ of sizes $l_1 l_2$then we define

$$EME = \frac{1}{k_1 k_2} \sum_{l=1}^{k_2} \sum_{k=1}^{k_1} 20log\left(\frac{I^w_{max;k,l}}{I^w_{min;k,l}}\right) \quad (5.2)$$

Where $I^w_{max;k,l}$ and $I^w_{min;k,l}$ are respectively maximum and minimum values of the image $I(N,M)$ inside the block $w_{k,l}$. When the parameter EME is high the image becomes clear.

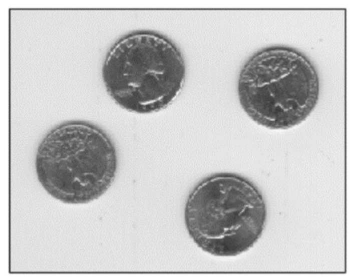

Figure 1: Original image without noise

We used another criterion providing an idea of the quality of the image filtered and enhanced compared with that reference image called the PSNR defined as:

$$SNR = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} [I(i,j) - I_{ref}(i,j)]^2$$

$$PSNR = 10 \log_{10}\left(\frac{255^2}{SNR}\right)$$

where I is the filtered image with enhancement and I_{ref} is the reference image, the image enhanced without noise with the proposed method .

To show the robustness of the proposed method, we used different images and noise levels. The original image is degraded with Gaussian noise for different value of noise estimator λ_0, as shown in the Figures below.

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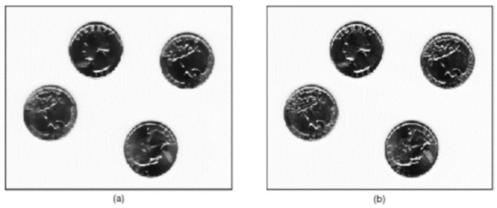

Figure 2: Original image enhanced: (a) S. Morfu method, $EME = 5.64$ *and (b) proposed method,* $EME = 7.62$

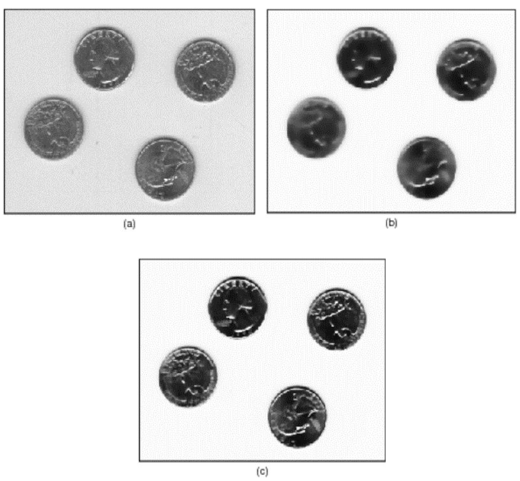

Figure 3: (a) Noisy image $\lambda_0 = 0.0362$*; (b) S. Morfu method, EME = 3.66 and (c) Proposed method,* $EME = 6.85. d = 1, \beta = 1$

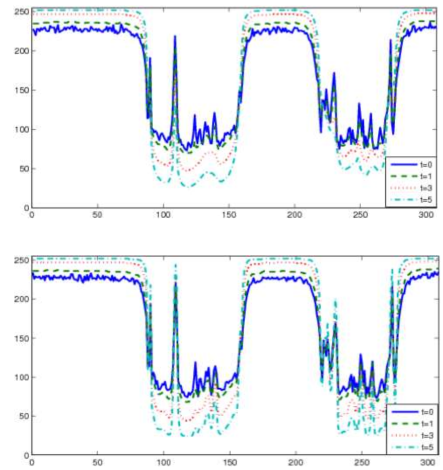

Figure 4 Profile of the line number 50 for different processing times d = 1, β = 1 and $\lambda_0 = 0.0362$ *(a) S. Morfu method. (b) Proposed method (see online version for colors)*

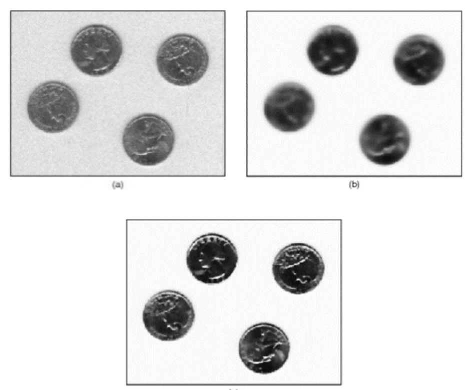

Figure 5 (a) Noisy image $\lambda_0 = 0.0965$*. (b) S. Morfu method,* $EME = 2.92$*. (c) Proposed method,* $EME = 6.02. d = 1, \beta = 1$

Table 1 PSNR(dB) between the reference image and filtered, enhanced image by the two methods

Image	*Noise estimator* λ_0	*Morfu*	*Proposed*
Eight	$\lambda 0$ =0.0362	23.9878	35.7587
	$\lambda 0$ =0.0965	21.2595	28.9170
	$\lambda 0$ =0.2515	20.5291	24.5635

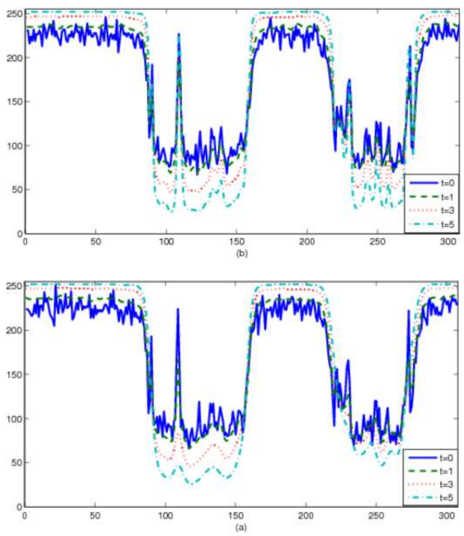

Figure 6 Profile of the line number 50 for different processing times $d = 1$, $\beta = 1$ *and* $\lambda 0 = 0.0965$ *(a) S. Morfu method. (b) Proposed method (see online version for colors)*

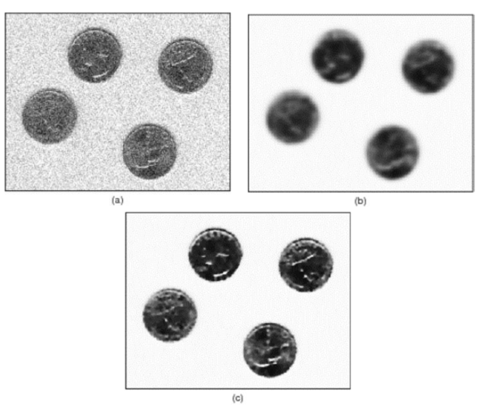

Figure 7 (a) Noisy image $\lambda 0 = 0.2515$. *(b) S. Morfu method,* $EME = 2.84$. *(c) Proposed method,* $EME = 5.54$. $d = 1, \beta = 1$

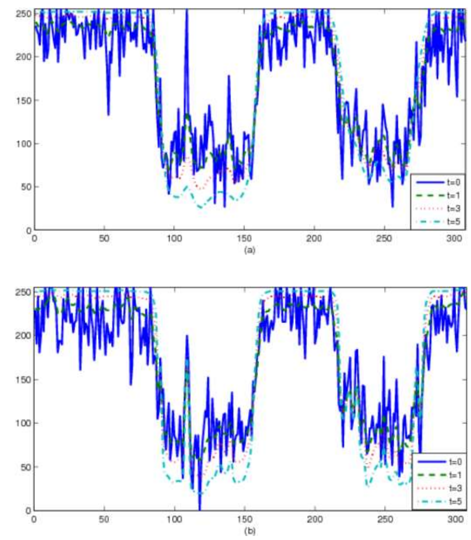

Figure 8 Profile of the line number 50 for different processing times $d = 1, \beta = 1$ *and* $\lambda 0 = 0.2515$ *(a) S. Morfu method. (b) Proposed method (see online version for colours)*

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7. CONCLUSION

In this paper, we are giving proof of the existence and the uniqueness of the solution of the proposed model in [1], to remove noise and contrast enhancement while preserving edges. This model is purely nonlinear and anisotropic diffusion. The focus of the proposed model is to apply a Gaussian filter to the image gradient when computing the diffusion coefficient in the model proposed by S. Morfu. The Numerical experiments presented in this paper applied to different noisy images show that the proposed model is more robust noise-removing and simultaneously contrast enhancement and preserving edges.

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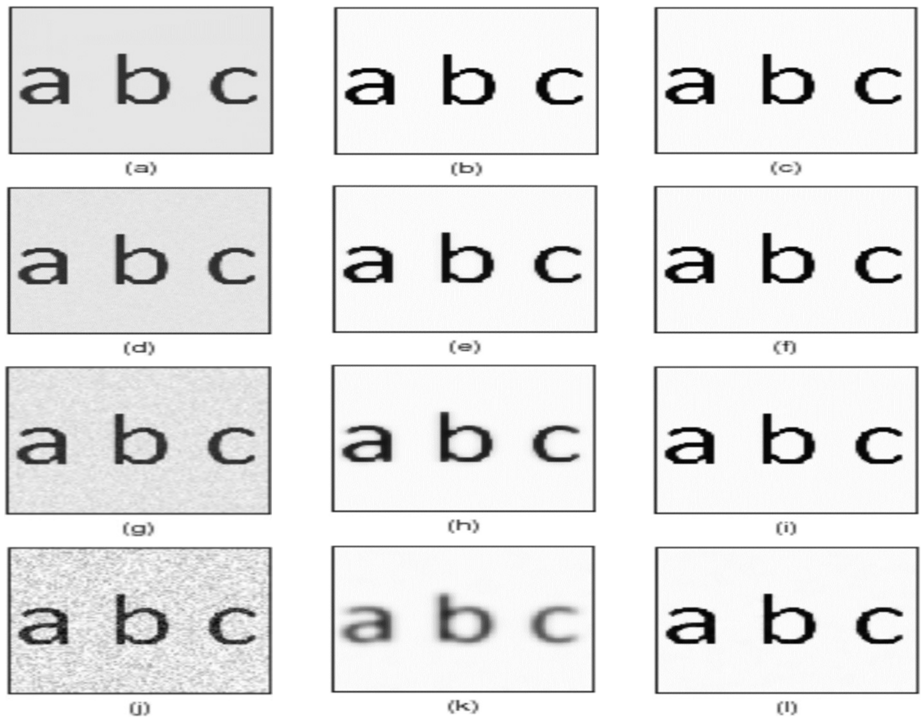

Figure 9: (a) Original image; (b) Original image enhanced by S. Morfu, $EME = 9.58$; (c) Original image enhanced by the proposed method, $EME = 9.69$; (d), (g) and (j) Noisy images for different noise levels; (e), (h) and (k) Noisy images enhanced by S. Morfu; (f), (i) and (l) Noisy images enhanced by the proposed method

Table 2 Comparison between the two methods, S. Morfu and proposed method for different images and noise levels

Image	Noise estimator λ_0	EME		PSNR	
		Morfu	Proposed	Morfu	Proposed
Alphabet	0.0270	7.7621	9.6833	36.8678	52.7056
	0.0836	5.5206	9.6689	25.8879	42.4668
	0.2378	2.8535	9.6144	18.0362	32.7579
Cameramen	0.0437	11.0950	16.1537	27.0422	39.5306
	0.1008	9.4410	15.2790	23.4242	36.8779
	0.2591	8.3809	13.0980	19.5716	23.9754
Bird	0.0359	6.8864	9.0091	34.9041	46.0855
	0.0926	6.1723	8.6152	27.1813	37.2098
	0.2609	6.1483	8.4477	24.5506	29.2444

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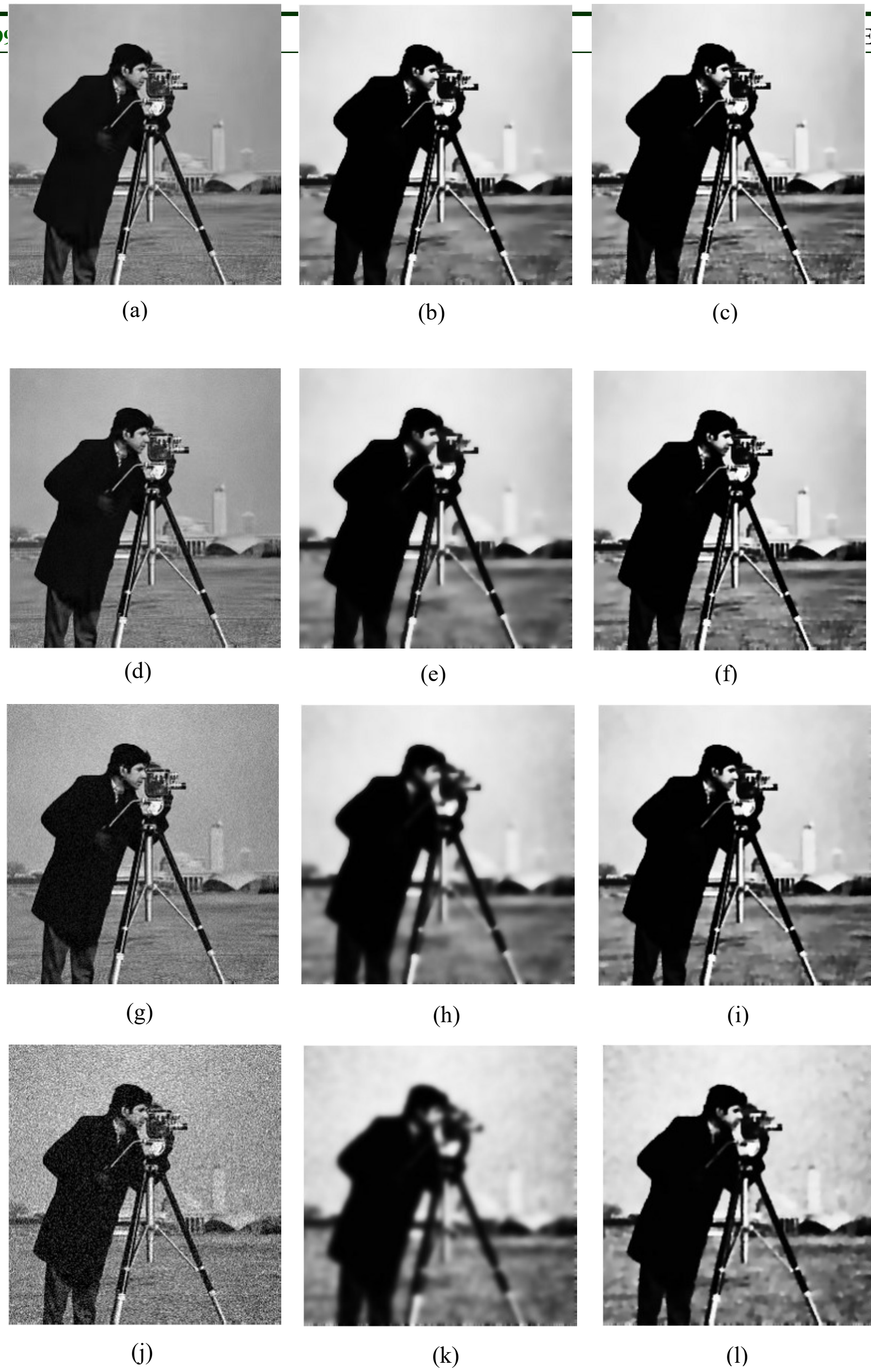

Figure 10: (a) Original image; (b) Original image enhanced by S. Morfu, $EME = 13.78$*; (c) Original image enhanced by the proposed method,* $EME = 16.75$*; (d), (g) and (j) Noisy images for different noise levels; (e), (h) and (k) Noisy images enhanced by S. Morfu; (f), (i) and (l) Noisy images enhanced by the proposed method*

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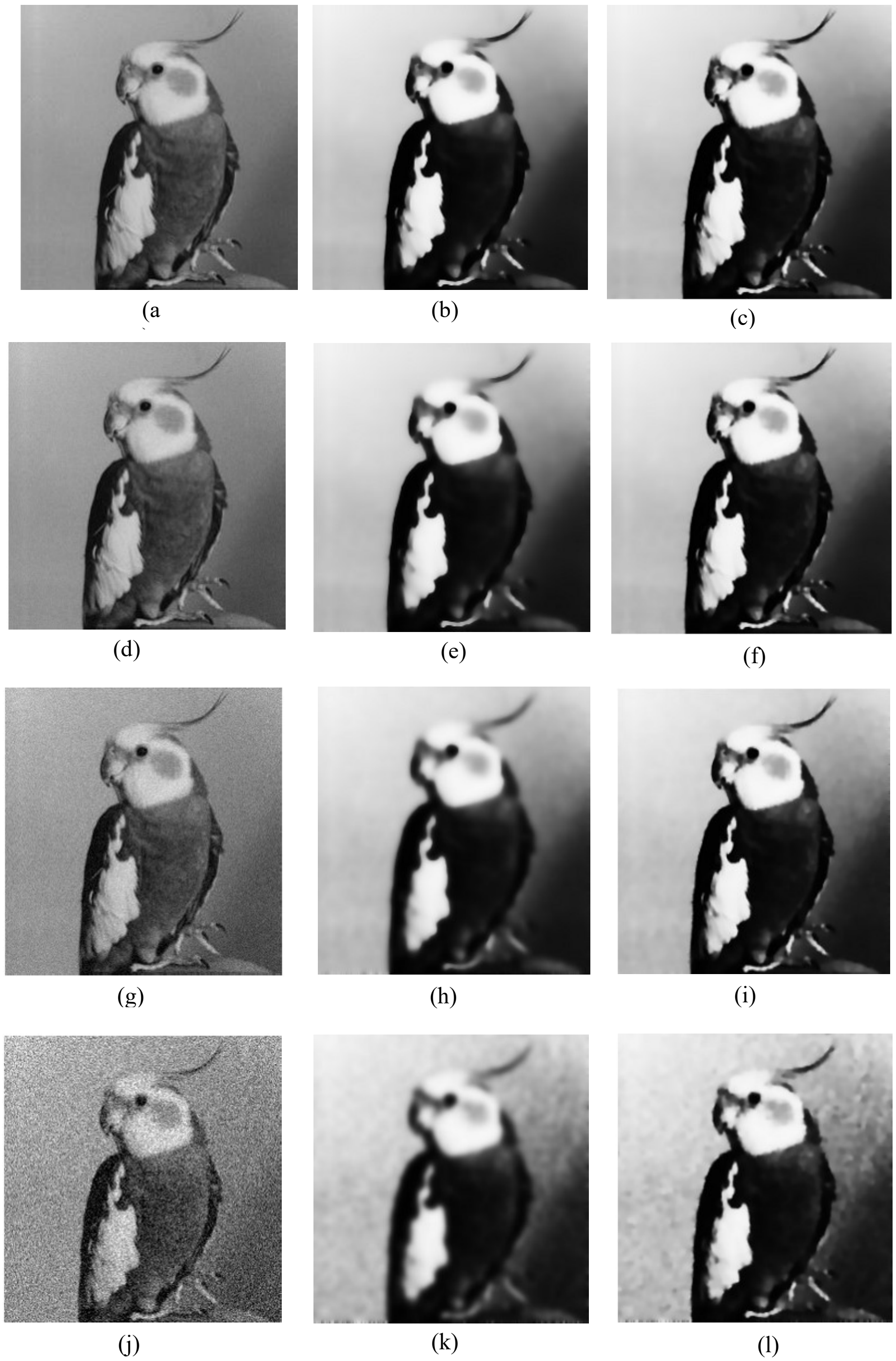

Figure 11: (a) Original image; (b) Original image enhanced by S. Morfu, EME = 7.79; (c) Original image enhanced by the proposed method, EME = 9.18; (d), (g) and (j) Noisy images for different noise levels; (e), (h) and (k) Noisy images enhanced by S. Morfu; (f), (i) and (l) Noisy images enhanced by the proposed method