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GLOBAL OUTPUT TRACKING CONTROL FOR HIGH-ORDER NON-LINEAR SYSTEMS WITH TIME-VARYING DELAYS

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ABSTRACT

This paper studies the problem of global practical output tracking for a class of high-order non-linear systems with time-varying delays under the weaker conditions on the system nonlinearities. With the help of an appropriate Lyapunov-Krasovskii functionals and by using the method of adding a power integrator, a continuous state-feedback controller is successfully designed such that all the states of the resulting close loop system are bounded while the output tracking error converges to an arbitrarily small residual set. A numerical example demonstrates the effectiveness of the result.

Keywords: *Practical Output Tracking, State Feedback Control, Nonlinear Systems, Time-Varying Delays, Lyapunov-Krasovskii Functionals*

1. INTRODUCTION

In this paper, we address the global output tracking problem for a class of uncertain nonlinear systems with time-varying delay which is described by

$$\begin{aligned}
&\dot{x}_i(t) = g_i(t)x_{i+1}^{p_i} + f_i(\bar{x}_i(t), x_1(t-d_1(t)),\ldots, x_i(t-d_i(t))),\\
&\qquad\qquad i=1,\ldots,n-1\\
&\dot{x}_n(t) = g_n(t)u^{p_n}(t) + f_n(\bar{x}_n(t), x_1(t-d_1(t)),\ldots, x_n(t-d_n(t))),\\
&y(t) = x_1(t)
\end{aligned} \tag{1}$$

where $x=(x_1,\ldots,x_n)^{\mathrm{T}} \in R^n$ and $u \in R$ are the system state and the control input, respectively; $\bar{x}_i(t) = (x_1(t),\ldots,x_i(t))^T$, $\bar{x}_n(t) = x(t)$, $d_i(t)$, $i=1,\ldots,n, \geq 0$ are time-varying delays satisfying $0 \leq d_i(t) \leq d_i$, $d_i^{'}(t) \leq \lambda_i < 1$ for constants d_i and λ_i. $x(\theta) = \varphi_0(\theta)$, $\theta \in [-d, 0]$ with $d \geq \max_{1\leq i \leq n}\{d_i\}$ and $\varphi_0(\theta)$ being specified continuous initial function; The terms $g_i(\cdot)$ are disturbed virtual control coefficients and $f_i(\cdot)$ represent nonlinear perturbations that are continuous functions; $p_i \in R_{odd}^{\geq 1} = \{\frac{p}{q} \in [0, \infty) : p$ and q are odd integers, $p \geq q$ }; ——— $g_i(t)$, $i = 1,...n$, are disturbed virtual control coefficients; $f_i(x,u)$ $i = 1,...n$, are unknown continuous nonlinear functions.

It has been known that the problem of global output tracking control of nonlinear systems is very challenging and important problems in the field of nonlinear control. During the past two decades, the global output tracking control design for non-linear systems has been extensively investigated. A number of interesting results have been achieved over the past years, see [1-11], as well as the references therein. However, the aforementioned results do not consider the effect of time delay. Time-delay phenomenon exists universally in many practical models, such as mechanical, chemical systems, biological systems and electrical systems. The existence of time-delay may bring about the performance instability, or make the system crashed. Therefore, the study of the problems of global control design of time-delay nonlinear systems has important practical significance. However, due to there being no unified method being applicable to nonlinear control design, this problem has not been fully investigated and there are many significant problems, which remain unsolved. In recent years, by using the Lyapunov-Krasovsky method to deal with the time-delay, control theory, and techniques for stabilization problem of time-delay nonlinear systems were greatly developed and advanced methods have been made; see, for instance, [12-19] and reference therein. In the case when the nonlinearities contain time-delay, for the output tracking problems, some interesting results also have been obtained [20-24]. However, the contributions only considered special cases such as p_i equal one or constant time-delay for the system (1) when the case p_i greater one. When the system under consideration is time-varying delays non-linear, the problem becomes more complicated and remain unsolved. This motivates the research in this paper.

In this paper, under mild conditions on the system nonlinearities involving time-varying delay, we will be to solve the specified problem using of the homogeneous domination technique [15], [25-28] and a homogeneous Lyapunov-Krassovsky functional. The main contributions of the paper can be summarized as follows: First, by comparison with the case in works [21-23], it is difficult to construct the Lyapunov-Krassovsky functional for higher-order nonlinear system (1). Therefore, we solve a number of problems that arose during design and analysis by creating a new Lyapunov-Krassovsky functional for higher-order nonlinear systems and adding the power integrator technique. Second, we extend the result obtained in the work [24] to the case where there is by a time-varying delay [29].

It should be noted that the proposed controller can only work well when the entire state vector can be measured. Therefore, a more interesting problem is how to design a state feedback controller for the systems studied in the paper to make the tracking controller arbitrarily small after a finite time, while keeping all closed-loop signals bounded, if only the state vector is partially measurable, which is currently under our further investigation.

Throughout this study, we use the following notations.

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Notations. R^+ denotes the set of all the nonnegative real numbers and R^n denotes the real n-dimensional space. A function $f: R^n \to R$ is said to be C^k-function, if its partial derivatives exist and are continuous up to order k, $1 \le k < \infty$. A C^0 function means it is continuous. A C^∞ function means it is *smooth*, that is, it has continuous partial derivatives of any order. The arguments of functions (or functional) are sometimes omitted or simplified; for instance, we sometimes denote a function $f(x(t))$ by $f(x)$, $f(\cdot)$ or f.

2. PROBLEM STATEMENT AND PRELIMINARIES

The objective of the paper is to construct an appropriate controller such that the output of system (1) practically tracks a reference signal $y_r(t)$. That is, for any pre-given tolerance $\varepsilon > 0$ to design a state feedback controller of the form

$$u(t) = g(x(t), y_r(t)) \tag{2}$$

such that for the all initial condition

(i) All the trajectories of the closed-loop system (1) with state controller (2) are well-defined and globally bounded on $[0, +\infty)$.

(ii) There exists a finite time $T > 0$, such that

$$|y(t) - y_r(t)| < \varepsilon, \quad \forall t \ge T > 0 \tag{3}$$

To construct a global practical output tracking controller for non-linear system (1), we introduce the following assumptions.

Assumption 1.

For $i = 1, \ldots, n$, there are $a_{ij}(\bar{x}_i)$, $j = 1,2$ non-negative smooth functions and decreasing constants $\tau_1 \ge \tau_2 \ge \cdots \ge \tau_n \ge 0$ ($\tau_i = \frac{p_i}{q_i}$, with an even integer p_i and q_i an odd integer), such that

$$\begin{aligned} &\left| f_i(\bar{x}_i(t), x_1(t-d_1(t)), \ldots, x_i(t-d_i(t))) \right| \\ &\le a_{i1}(\bar{x}_i) \sum_{j=1}^{i} \left(|x_j(t)|^{\frac{r_i+\tau_i}{r_j}} + |x_j(t-d_j(t))|^{\frac{r_i+\tau_i}{r_j}} \right) + a_{i2}(\bar{x}_i) \end{aligned}$$

where r_i are defined as $r_1 = 1$, $r_{i+1} p_i = r_i + \tau_i > 0$, $i = 1, \ldots, n$.

Assumption 2.

For $i = 1, \ldots, n$, there are positive constants b_{i1} and b_{i2} such that

$$b_{i1} \le g_i(t) \le b_{i2}$$

Assumption 3.

The reference signal $y_r(t)$ is continuously differentiable. Moreover, there is a known constant $M > 0$, such that

$$|y_r(t)| + |\dot{y}_r(t)| \le M, \qquad \forall t \in [0, \infty)$$

This section cites some definitions and technical lemmas, which are used in the main body of this investigation.

Next, we will present several useful Lemmas borrowed from [10], [12], [13] and [18], which will play an important role in our later controller design.

Lemma 1.

For all $x, y \in R$ and a constant $p \geq 1$, the following inequalities hold:

(i) $$|x+y|^p \leq 2^{p-1}|x^p + y^p|,$$

$$(|x|+|y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(|x|+|y|)^{\frac{1}{p}}$$

if $p \in \mathrm{R}_{odd}^{\geq 1}$ then

(ii) $$|x-y|^p \leq 2^{p-1}|x^p - y^p|,$$

$$\left|x^{\frac{1}{p}} - y^{\frac{1}{p}}\right| \leq 2^{\frac{p-1}{p}}|x-y|^{\frac{1}{p}}.$$

Lemma 2.

For given positive real numbers *m*, *n* and a positive function *a* (*x*, *y*), there exists a positive function *c* (*x*, *y*), such that

$$a(x,y)|x|^m|y|^n \leq c(x,y)|x|^{m+n} + \frac{n}{n+m}\left(\frac{m}{(m+n)c(x,y)}\right)^{\frac{m}{n}} a(x,y)^{\frac{m+n}{n}}|y|^{m+n}$$

Lemma 3.

For any positive real numbers x, y and $m \geq 1$, the following inequality holds

$$x \leq y + \left(\frac{x}{m}\right)^m \left(\frac{m-1}{y}\right)^{m-1}.$$

3. CONTINUOUSLY DIFFERENTIABLE STATE FEEDBACK CONTROLLER DESIGN

In this section, we shall construct a *continuously differentiable* state feedback tracking controller which is addressed in a step-by-step manner for system (1).

Theorem 1.

Under Assumptions 1-3, the global practical output tracking problem of system (1) can be solved by a continuously state feedback controller of the form (2).

Proof:

Let $z_1(t) = x_1(t) - y_r(t)$ and given $z_i(t) = x_i(t), i = 2, \ldots, n.$ Then we have

$$\begin{aligned}
\dot{z}_1(t) &= g_1(t)z_2^{p_1}(t) + f_1(z_1(t)+y_r(t),\\
&\qquad z_1(t-d_1(t)) + y_r(t-d_1(t))) - \dot{y}_r(t), \sqrt{b^2-4ac}\\
\dot{z}_j(t) &= g_j(t)z_{j+1}^{p_j} + f_j(\overline{z}_j(t), z_1(t-d_1(t)) + y_r(t-d_1(t)),\\
&\qquad z_2(t-d_2(t)), \ldots, z_j(t-d_j(t))),\ j = 2, \ldots, n-1,\\
\dot{z}_n(t) &= g_n(t)u^{p_n}(t) + f_n(\overline{z}_j(t), z_1(t-d_1(t)) + y_r(t-d_1(t)),\\
&\qquad z_2(t-d_2(t)), \ldots, z_j(t-d_j(t)))\\
y(t) &= z_1(t) + y_r(t)
\end{aligned} \tag{4}$$

Let $\rho \in R_{odd}^{+}$ be a constant satisfying

$\rho \geq \max_{1 \leq i \leq n}\{r_i + \tau_i\}$, where τ_i and r_i are defined by Assumption 1.

Initial step. Choose the Lyapunov–Krasovskii functional

$$V_1(z_1) = U_1(z_1) + W_1(z_1) \tag{5}$$

where

$$U_1(z_1) = \frac{1}{2\rho - \tau_1} z_1^{2\rho-\tau_1} = \int_{z_1^*}^{z_1}\left(s^{\frac{\rho}{r_1}} - z_1^{*\frac{\rho}{r_1}}\right)^{\frac{2\rho-\tau_1-r_1}{\rho}} ds,$$

$z_1^* = 0$ and

$$W_1(z_1) = \frac{n}{1-\lambda_1}\int_{t-d_1(t)}^{t} e^{s-t} z_1^{2\rho}(s)ds + $$
$$+\frac{n-1}{1-\lambda_2}\int_{t-d_2(t)}^{t} e^{s-t} z_1^{2\rho}(s)ds.$$

With the help of Assumptions 1 and 3 and Lemma 2, we have (which is positive definite, proper, and C^1 due to the fact that $2\rho-\tau_1 \geq 2r_1+\tau_1$. Then, the time derivative of $V_1(z_1)$ along the trajectory of (1) is)

$$\dot{V}_1(z_1) = z_1^{(2\rho-\tau_1-r_1)/r_1}[g_1(t)z_2^{p_1} + f_1(z_1(t)+$$
$$+y_r(t), z_1(t-d_1(t)) + y_r(t-d_1(t))) -$$
$$-\dot{y}_r(t)] - \left(\frac{n}{1-\lambda_1}\int_{t-d_1(t)}^{t} e^{s-t} z_1^{2\rho}(s)ds + \right.$$
$$\left.+\frac{n-1}{1-\lambda_2}\int_{t-d_2(t)}^{t} e^{s-t} z_1^{2\rho}(s)ds\right) + \frac{n}{1-\lambda_1}\times$$
$$\times\left(z_1^{2\rho}(t) - e^{-d_1(t)} z_1^{2\rho}(t-d_1(t))(1-\dot{d}_1(t))\right) +$$
$$+\frac{n-1}{1-\lambda_2}\left(z_1^{2\rho}(t) - e^{-d_2(t)} z_1^{2\rho}(t-d_1(t))(1-\dot{d}_2(t))\right)$$

Further, it follows from Assumptions 1 (ii), 2 and Lemmas 1-3 that

$$\dot{V}_1(z_1) \leq g_1 z_1^{\frac{2\rho-\tau_1-r_1}{r_1}} z_2^{*p_1} + g_1 z_1^{\frac{2\rho-\tau_1-r_1}{r_1}}\left(z_2^{p_1} - z_2^{*p_1}\right) +$$
$$+a_{11}(z_1+y_r) z_1^{\frac{2\rho-\tau_1-r_1}{r_1}}\left(|z_1|^{\frac{\tau_1+r_1}{r_1}} + |z_1(t-d_1(t))|^{\frac{\tau_1+r_1}{r_1}}\right) +$$
$$+\left(2^{\frac{\tau_1}{r_1}} a_{11}(z_1+y_r) M^{\frac{\tau_1+r_1}{r_1}} + M + a_{12}(z_1+y_r)\right)$$
$$z_1^{\frac{2\rho-\tau_1-r_1}{r_1}} - W_1(z_1) + \frac{n}{1-\lambda_1} z_1^{2\rho} +$$
$$+\frac{n-1}{1-\lambda_2} z_1^{2\rho} - e^{-d_1(t)}\frac{n(1-\dot{d}_1(t))}{1-\lambda_1} z_1^{2\rho}(t-d_1(t))$$
$$-e^{-d_2(t)}\frac{(n-1)(1-\dot{d}_2(t))}{1-\lambda_2} z_1^{2\rho}(t-d_2(t)) \leq$$
$$\leq g_1 z_1^{\frac{2\rho-\tau_1-r_1}{r_1}} z_2^{*p_1} - W_1(z_1) - e^{-d}(n-1)*$$
$$*\left(z_1^{2\rho}(t-d_1(t)) + z_1^{2\rho}(t-d_2(t))\right) +$$
$$+z_1^{2\rho}\left(\frac{n}{1-\lambda_1} + \frac{n-1}{1-\lambda_2} + \tilde{\alpha}_1(z_1)\right) + g_1 z_1^{\frac{2\rho-\tau_1-r_1}{r_1}} \times$$
$$\times\left(z_2^{p_1} - z_2^{*p_1}\right) + \delta$$

where $\tilde{\alpha}_1(z_1) = \dfrac{[\alpha_1(z_1)]^{\frac{2\rho}{2\rho-\tau_1-r_1}}}{\delta^{\frac{\tau_1+r_1}{2\rho-\tau_1-r_1}}}$,

$$\alpha_1(z_1) \geq \left(2^{\frac{\tau_1}{r_1}} a_{11}(z_1+y_r) M^{\frac{\tau_1+r_1}{r_1}} + M + a_{12}(z_1+y_r)\right)$$

and $\delta > 0$ is any real constant. Since $a_{11}(z_1+y_r)$ and $a_{12}(z_1+y_r)$ are smooth functions and $y_r(t)$ is bounded, so we can choose $\tilde{\alpha}_1(z_1) \geq a_{11}(z_1+y_r)$. Design the virtual controller $z_2^{*p_1}$ as

$$z_2^{*p_1} = -\frac{1}{b_{11}}\left(n + \frac{n}{1-\lambda_1} + \frac{n-1}{1-\lambda_2} + \tilde{\alpha}_1(z_1)\right) z_1^{\frac{\tau_1+r_1}{r_1}} =$$
$$= -\beta_1(z_1)^{\frac{\tau_1+r_1}{\rho}}\left(z_1^{\frac{\rho}{\tau_1}}\right)^{\frac{\tau_1+r_1}{\rho}} = -\beta_1(z_1)^{\frac{r_2 p_1}{\rho}} \xi_1^{\frac{r_2 p_1}{\rho}}$$

with a smooth function

$$\beta_1(z_1)=\left(\frac{1}{b_{11}}\left(n+\frac{n}{1-\lambda_1}+\frac{n-1}{1-\lambda_2}+\tilde{\alpha}_1(z_1)\right)\right)^{\frac{\rho}{r_2 p_1}},$$ and

then we have

$$V_1(z_1)\le -nz_1^{2\rho}-e^{-d}(n-1)\left(z_1^{2\rho}(t-d_1(t))+z_1^{2\rho}(t-d_2(t))\right)- \\ -W_1(z_1)+g_1 z_1^{\frac{2\rho-\tau_1-r_1}{r_1}}(z_2^{p_1}-z_2^{*p_1})+\delta \tag{6}$$

Inductive step. Suppose at step, there exist a series of smooth functions $\beta_i(z_1,\ldots,z_i)>0$, $i=1,\ldots,k-1$, with the following virtual controllers $k-1$

$$\begin{aligned} &z_1^*=0, && \xi_1=z_1^{\frac{\rho}{r_1}}-z_1^{*\frac{\rho}{r_1}} \\ &z_2^*=-\xi_1^{\frac{r_2}{\rho}}\beta_1(z_1), && \xi_2=z_2^{\frac{\rho}{r_2}}-z_2^{*\frac{\rho}{r_2}} \\ &\vdots && \vdots \\ &z_i^*=-\xi_{i-1}^{\frac{r_i}{\rho}}\beta_{i-1}(\bar{z}_i), && \xi_i=z_i^{\frac{\rho}{r_i}}-z_i^{*\frac{\rho}{r_i}}, \end{aligned} \tag{7}$$

such that

$$\dot{V}_{i-1}(\bar{z}_i)\le -(n-i+2)\sum_{j=1}^{i-1}\xi_j^2 \\ -(n-i+1)\sum_{j=1}^{i-1}\left(\xi_j^2(t-d_j(t))+\xi_j^2(t-d_{j+1}(t))\right) \\ +g_{i-1}\xi_{i-1}^{\frac{2\rho-r_{i-1}-\tau_{i-1}}{\rho}}\left(z_i^{p_{i-1}}-z_i^{*p_{i-1}}\right)+(i-1)\delta. \tag{8}$$

We claim that (8) also holds at *Step k*. To prove this claim, consider the Lyapunov function

$$V_i(\bar{z}_i)=V_{i-1}(\bar{z}_{i-1})+U_i(\bar{z}_i)+W_i \tag{9}$$

where

$$V_{i-1}(\bar{z}_{i-1})=\sum_{j=1}^{i-1}U_j(\bar{z}_j)+\sum_{j=1}^{i-1}W_i,$$

$$U_i(\bar{z}_i)=\int_{z_i^*}^{z_i}\left(s^{\frac{\rho}{r_i}}-z_i^{*\frac{\rho}{r_i}}\right)^{\frac{2\rho-r_i-\tau_i}{\rho}}ds$$

and

$$W_i=\frac{n}{1-\lambda_i}\int_{t-d_i(t)}^{t}e^{s-t}\xi_i^2(s)ds+ \\ +\frac{n-1}{1-\lambda_{i+1}}\int_{t-d_{i+1}(t)}^{t}e^{s-t}\xi_i^2(s)ds \tag{10}$$

The function $U_i(\bar{z}_i)$ can be shown to be C^1, proper and positive definite with the following property: for $j=1,\ldots,i-1$,

$$\frac{\partial U_i}{\partial z_j}=-\frac{2\rho-r_i-\tau_i}{\rho}\int_{z_i^*}^{z_i}\left(s^{\frac{\rho}{r_i}}-z_i^{*\frac{\rho}{r_i}}\right)^{\frac{2\rho-r_i-\tau_i}{\rho}-1}ds\,\frac{\partial\left(z_i^{*\frac{\rho}{r_i}}\right)}{\partial z_j}, \tag{11}$$

$$\frac{\partial U_i}{\partial z_i}=\left(z_i^{\frac{\rho}{r_i}}-z_i^{*\frac{\rho}{r_i}}\right)^{\frac{2\rho-r_i-\tau_i}{\rho}}=\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}} \tag{12}$$

and there is a known constant $L>0$, such that

$$U_i\ge L\left(z_i-z_i^*\right)^{\frac{2\rho-r_i-\tau_i}{r_i}} \tag{13}$$

Proofs of these properties proceed just in the same way as in the proofs for [20, propositions 1 and 2] and [21], where the set of positive odd integers is considered instead of R_{odd} which is used in this paper.

With these properties, we obtain

$$\dot{V}_i(\bar{z}_i) \le -(n-i+2)\sum_{j=1}^{i-1}\xi_j^2 - -e^{-d}(n-i+1)\sum_{j=1}^{i-1}\left(\xi_j^2(t-d_j(t))+\xi_j^2(t-d_{j+1}(t))\right) - -\sum_{j=1}^{i-1}W_j + g_{i-1}\xi_{i-1}^{\frac{2\rho-r_{i-1}-\tau_{i-1}}{\rho}}\left(z_i^{p_{i-1}} - z_i^{*p_{i-1}}\right) + +(i-1)\delta - W_i + \frac{n-i+1}{1-\lambda_i}\xi_i^2 + +\frac{n-i}{1-\lambda_{i+1}}\xi_i^2 - e^{-d}\frac{(n-i+1)(1-\dot{d}_i(t))}{1-\lambda_i}\xi_i^2(t-d_i(t)) - -e^{-d}\frac{(n-i)(1-\dot{d}_{i+1}(t))}{1-\lambda_{i+1}}\xi_i^2(t-d_{i+1}(t)) + +g_i\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}}z_{i+1}^{*p_i} + g_i\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}}(z_{i+1}^{p_i} - z_{i+1}^{*p_i}) + +\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}}f_i(z_1+y_r, z_2, \ldots, z_i) + \sum_{j=1}^{i-1}\frac{\partial U_i}{\partial z_j}\dot{z}_j \tag{14}$$

for a virtual controller $z_{i+1}^{*p_i}$ to be determined later. In order to proceed further, a bounding estimate for each term in the right-hand side of (14) is needed. The terms in (14) can be estimated using the Propositions 1-3 in the Appendix.

Substituting the results of the Propositions 1-3 into (14), we arrive at

$$\dot{V}_i(\bar{z}_i) \le -(n-i+1)\sum_{j=1}^{i}\xi_j^2 -e^{-d}(n-i)\sum_{j=1}^{i}\left(\xi_j^2(t-d_j(t))+\xi_j^2(t-d_{j+1}(t))\right) -\sum_{j=1}^{i}W_j + g_i\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}}z_{i+1}^{*p_i} + g_i\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}}\left(z_{i+1}^{p_i} - z_{i+1}^{*p_i}\right) +\left(\frac{n-i+1}{1-\lambda_i} + \frac{n-i}{1-\lambda_{i+1}} + \tilde{\alpha}_i(\bar{z}_i)\right)\xi_i^2 + i\delta \tag{15}$$

where

$$\tilde{\alpha}_i(\bar{z}_i)\ (\ge h_{i1}(\bar{z}_i) + h_{i2}(\bar{z}_i) + h_{i3}(\bar{z}_i)\ )$$

is a smooth positive function.

Therefore, if we take the virtual control z_{i+1}^* as

$$z_{i+1}^{*p_i} = -\xi_i^{\frac{r_{i+1}p_i}{\rho}}\left\{\frac{1}{b_{i1}}\left(n-i+1+\frac{n-i+1}{1-\lambda_i}+\frac{n-i}{1-\lambda_{i+1}}+ +\tilde{\alpha}_i(\bar{z}_i)\right)\right\} =: -\xi_i^{\frac{r_{i+1}p_i}{\rho}}\beta_i(\bar{z}_i)^{\frac{r_{i+1}p_i}{\rho}}, \tag{16}$$

then, we obtain

$$\dot{V}_i(\bar{z}_i) \le -(n-i+1)\sum_{j=1}^{i}\xi_j^2 -e^{-d}(n-i)\sum_{j=1}^{i}\left(\xi_j^2(t-d_j(t))+\xi_j^2(t-d_{j+1}(t))\right) -\sum_{j=1}^{i}W_j + g_i\xi_i^{\frac{2\rho-r_i-\tau_i}{\rho}}\left(z_{i+1}^{p_i} - z_{i+1}^{*p_i}\right) + i\delta$$

which proves the inductive argument.

At the nth step, by applying the feedback control

$$u^{p_n} = -\xi_n^{\frac{r_{n+1}p_n}{\rho}}\beta_n^{\frac{r_{n+1}p_n}{\rho}}(z) \tag{17}$$

with the C^1, proper and positive definite Lyapunov–Krasovskii functional $V_n(z)$ constructed via the inductive procedure, we arrive at

$$\dot{V}_n(z) \le -\sum_{i=1}^{n} \xi_i^2 - \sum_{i=1}^{n} W_i + n\delta \tag{18}$$

where

$$V_n(z) = \sum_{i=1}^{n} U_i(\bar{z}_i) + \sum_{i=1}^{n} W_i \quad \text{and}$$

$$\begin{aligned}\sum_{i=1}^{n} W_i &= \sum_{i=1}^{n} \frac{n-i+1}{1-\lambda_i} \int_{t-d_i(t)}^{t} e^{s-t} \xi_i^2(s) ds \\ &+ \sum_{i=1}^{n} \frac{n-i}{1-\lambda_{i+1}} \int_{t-d_{i+1}(t)}^{t} e^{s-t} \xi_i^2(s) ds\end{aligned} \tag{19}$$

Moreover, we have

$$\begin{aligned}U_i(\bar{z}_i) &\le \left|z_i - z_i^*\right| \left| z_i^{\frac{\rho}{r_i}} - z_i^{*\frac{\rho}{r_i}} \right|^{\frac{2\rho - r_i - \tau_i}{\rho}} \\ &\le 2^{1-\frac{r_i}{\rho}} \left| z_i^{\frac{\rho}{r_i}} - z_i^{*\frac{\rho}{r_i}} \right|^{\frac{r_i}{\rho}} \left| z_i^{\frac{\rho}{r_i}} - z_i^{*\frac{\rho}{r_i}} \right|^{\frac{2\rho - r_i - \tau_i}{\rho}} \\ &\le \left(2^{\rho/\tau_i}\right)^{\frac{\tau_i}{\rho}} \left|\xi_i\right|^{2-\frac{\tau_i}{\rho}} \\ &\le (1-\frac{\tau_i}{2\rho}) \left|\xi_i\right|^2 + \frac{2^{\frac{2\rho-1}{\tau_i}-1} \tau_i}{\rho} \\ &\le \left|\xi_i\right|^2 + \frac{2^{\frac{2\rho-1}{\tau_i}-1} \tau_i}{\rho}\end{aligned} \tag{20}$$

So,

$$\sum_{i=1}^{n} U_i(\bar{z}_i) \le \sum_{i=1}^{n} \left|\xi_i\right|^2 + n \frac{2^{\frac{2\rho}{\tau_i}-1} \tau_i}{\rho},$$

Therefore,

$$\begin{aligned}\dot{V}_n(z) &\le -\sum_{i=1}^{n} \xi_i^2 - \sum_{i=1}^{n} W_i + n\delta \\ &\le -\sum_{i=1}^{n} U_i - \sum_{i=1}^{n} W_i + n \frac{2^{\frac{2\rho}{\tau_i}-1} \tau_i}{\rho} + n\delta \\ &= -V_n(z) + n \frac{2^{\frac{2\rho}{\tau_i}-1} \tau_i}{\rho} + n\delta\end{aligned} \tag{21}$$

Inequality (21) will be shown that the state $z(t)$ of closed-loop system (4)-(17) is well-defined on $[0,+\infty)$ and globally bounded. To prove this, first introduce the following set

$$\Omega = \left\{ z(t) \in R^n \;\middle|\; V_n(z) \ge n \frac{2^{\frac{2\rho}{\tau_i}-1} \tau_i}{\rho} + 2n\delta \right\}, \tag{22}$$

and let $z(t)$ be the trajectory of (4) with an initial state $z(0)$. If $z(t) \in \Omega$, then it follows from (21) that

$$\dot{V}_n(z(t)) \le -V_n(z(t)) + n \frac{2^{\frac{2\rho}{\tau_i}-1} \tau_i}{\rho} + n\delta \le -n\delta < 0. \tag{23}$$

This implies that, as long as $z(t) \in \Omega$, $V_n(z(t))$ is strictly decreasing with time *t*, and hence $z(t)$ must enter the complement set $R^n - \Omega$ in a finite time $T \ge 0$ and stay there forever. Therefore, (23) leads to

$$V_n(z(t))-V_n(z(0))=\int_0^t \dot{V}_n(z(t))dt \quad <0, \quad t\in[0,\ T)$$
$$V_n(z(t)) < nn\frac{2^{\frac{2\rho}{\tau_i}-1}\tau_i}{\rho}+2n\delta, \quad t\in[T,\infty) \tag{24}$$

which shows $V_n \in L^\infty$, and so do z_1 and W_k. By $z_1 = x_1 + y_r$ and $y_r \in L^\infty$, we conclude $x_1 \in L^\infty$ as well. Noting

$$x_2^{*p_1} = -x_1^{\frac{r_1+\tau_1}{r_1}}\left(n+\tilde{\kappa}_1(x_1)\right) = -x_1^{\frac{r_1+\tau_1}{r_1}}\beta_1(x_1)$$

and $\tilde{\kappa}_1(x_1)$ is smooth function of x_1, we have $x_2^{*p_1} \in L^\infty$.

Since $W_2 \in L^\infty$ and (13), we have $(x_2 - x_2^*) \in L^\infty$ and $x_2 \in L^\infty$. Inductively, we can prove $x_i \in L^\infty$, $i = 3,4,...,n$ and so do $x(t)$.

Thus, the solution $x(t)$ of the system (4) is well-defined and globally bounded on $[0,+\infty)$.

Next, it will be shown that

$$|y(t)-y_r(t)| = |z_1(t)-y_r(t)| < \varepsilon, \quad \forall t \ge T > 0. \tag{25}$$

This is easily shown from (13), (24) and by tuning the parameter δ as follows

$$|y(t)-y_r(t)| = |x_1(t)| \le V_n(x(t)) \le n\frac{r_1+\tau_1}{r_1}+2n\delta < \varepsilon.$$

Therefore, for any $\varepsilon > 0$, there is globally practical output-tracking such that (25) holds.

This completes the proof of Theorem 1.

4. AN ILLUSTRATIVE EXAMPLE

$$\begin{cases} \dot{x}_1(t) = x_2^{\frac{11}{7}}(t) + \frac{1}{7}x_1^{\frac{1}{3}}(t-d_1(t)), \\ \dot{x}_2(t) = u^{\frac{7}{5}}(t) + \frac{1}{6}x_2(t)e^{x_2(t)} \\ y(t) = x_1(t) \end{cases} \tag{26}$$

where: $d(t) = \frac{1}{6}(1+\sin(t))$; $g_1(t) = g_2(t) = 1$ - disturbed virtual control coefficients. $f_1 = \frac{1}{7}x_1^{\frac{1}{3}}(t-d(t))$, $f_2 = \frac{1}{6}x_2(t)e^{x_2(t)}$. We choose $\tau_1 = \tau_2 = \frac{4}{7}$ that together with $r_1 = 1$ and $p_1 = \frac{11}{7}$, $p_2 = \frac{7}{5}$, implies that

$$r_2 = \frac{r_1+\tau_1}{p_1} = \frac{1+2}{3} = 1, \quad r_3 = \frac{r_2+\tau_2}{p_2} = \frac{1+\frac{4}{7}}{\frac{7}{5}} = \frac{55}{49},$$

it is easy to obtain

$$|f_1| \le \frac{1}{2}(1+x_1^2)\left(|x_1|^{\frac{11}{7}} + |x_1(t-d(t))|^{\frac{11}{7}}\right) + a_{12}(\bar{x}_1),$$
$$|f_2| \le \frac{1}{5}(1+x_1^2+e^{x_2})\left(|x_1|^{\frac{11}{7}} + |x_1(t-d(t))|^{\frac{11}{7}} + \left|x_2^{\frac{11}{7}}\right|\right) + \\ +a_{22}(\bar{x}_2)$$

Clearly, Assumption 1 and 2 are satisfied with $a_{11} = \frac{1}{2}$, $a_{21} = \frac{1}{5}$. Moreover, noting that $\dot{d}(t) = \frac{1}{6}\cos t \le \frac{1}{6} < 1$ the controller proposed in this paper is applicable.

By choosing

$$\rho = 2 \ \left(\rho \geq \max\{r_1+\tau_1; r_2+\tau_2; r_3+\tau_3\} = = \max\left\{\frac{11}{7}; \frac{11}{7}; \frac{83}{49}\right\} = \frac{83}{49}\right)$$

$$V_1(z_1) = U_1(z_1) + W_1(z_1)$$

$$U_1(z_1) = \int_0^{z_1} \left(s^2\right)^{\frac{4-\frac{4}{7}-1}{2}} ds = \left.\frac{7s^{\frac{24}{7}}}{24}\right|_0^{z_1} = \frac{7z_1^{\frac{24}{7}}}{24};$$

$$W_1(z_1) = \frac{2}{1-\lambda_1} \int_{t-d_1(t)}^{t} e^{s-t} z_1^4(s) ds$$

$$V_1(z_1) = \frac{7z_1^{\frac{24}{7}}}{24} + W_1(z_1)$$

$$\dot{V}_1(z_1) = z_1^{\frac{17}{7}} + \dot{W}_1(z_1)$$
$$= z_1^{\frac{17}{7}} \Bigg[z_2^{\frac{11}{7}} + f_1\left(z_1(t) + y_r(t), z_1\left(t - d_1(t)\right) + y_r(t-d_1(t))\right) - \dot{y}_r(t) \Bigg] - \frac{2}{1-\lambda_1} \int_{t-d_1(t)}^{t} e^{s-t} z_1^4(s) ds$$
$$+ \frac{2}{1-\lambda_1}\left(z_1^4(t) - e^{-d(t)} z_1^4\left(t - \frac{1}{6} - \frac{1}{6}\sin t\right)\left(1 - \frac{1}{6}\cos t\right)\right)$$

$$\dot{V}_1(z_1) \leq z_1^{\frac{17}{7}} z_2^{*\frac{11}{7}} - W_1(z_1) - e^{-d}(z_1^4(t-d_1(t)))$$
$$+ z_1^4\left(\frac{2}{1-\lambda_1} + \frac{1}{1-\lambda_2} + \tilde{\alpha}_1(z_1)\right) + z_1^{\frac{17}{7}}(z_2^{\frac{11}{7}} - z_2^{*\frac{11}{7}}) + \delta$$

where $\tilde{\alpha}_1(z_1) = \dfrac{[\alpha_1(z_1)]^{\frac{28}{17}}}{\delta^{\frac{11}{17}}}$,

$$\alpha_1(z_1) \geq \left(2^{\frac{4}{7}} a_{11}(z_1+y_r) M^{\frac{11}{7}} + M + a_{12}(z_1+y_r)\right)$$

and $\delta > 0$ $\forall$ real const.

$$z_2^{*\frac{11}{7}} = -\beta_1^{\frac{11}{14}}(z_1) \cdot z_1^{\frac{11}{7}} = -\beta_1^{\frac{11}{14}}(z_1) \cdot \xi_1^{\frac{11}{14}},$$

where $\beta_1(z_1) = \dfrac{1}{b_{11}}\left(2 + \dfrac{2}{1-\lambda_1} + \tilde{\alpha}_1(z_1)\right)$. Then we have

$$\dot{V}_1(z_1) \leq -2z_1^4 - e^{-d}(z_1^4(t-d_1(t))) - W_1(z_1) +$$
$$+ z_1^{\frac{17}{7}}(z_2^{\frac{11}{7}} - z_2^{*\frac{11}{7}}) + \delta$$

$$z_1^* = 0 \qquad \xi_1 = z_1^2 - z_1^{*2}$$
$$z_2^* = -\xi_1^{\frac{1}{2}} \beta_1(z_1) \qquad \xi_2 = z_2^2 - z_2^{*2}$$

$$\dot{V}_1(\bar{z}_2) \leq -2\xi_1^2 - \xi_1^2\left(t - d_1(t)\right) + \xi_1^{\frac{17}{14}}\left(z_2^{\frac{11}{7}} - z_2^{*\frac{11}{7}}\right) + \delta$$

$$\dot{V}_2(\bar{z}_3) \leq -\left(\xi_1^2 + \xi_2^2\right) + \xi_2^{\frac{17}{14}}(z_3^{\frac{7}{5}} - z_3^{*\frac{7}{5}}) + 2\delta$$

$$V_2(\bar{z}_2) = V_1(\bar{z}_1) + U_2(\bar{z}_2) + W_2$$

$$\dot{V}_2(\bar{z}_2) \leq -2\xi_1^2 - e^{-d}\xi_1^2(t-d_1(t))$$
$$+ W_1 + \xi_1^{\frac{17}{14}}(z_2^{\frac{11}{7}} - z_2^{*\frac{11}{7}}) + \delta - W_2 + \xi_2^{\frac{17}{14}} z_3^{*\frac{7}{5}} +$$
$$+ \xi_2^{\frac{17}{14}}\left(z_3^{\frac{7}{5}} - z_3^{*\frac{7}{5}}\right) + \xi_2^{\frac{17}{14}} f_2(z_1 + y_r, z_2) + \frac{\partial U_2}{\partial z_1}\dot{z}_1$$

Where,

$$\frac{\partial U_2}{\partial z_1} = -\frac{2*2-1-\frac{4}{7}}{2} \int_{z_2^*}^{z_2} \left(s^2 - z_2^{*2}\right)^{\frac{4-1-\frac{4}{7}}{2}} ds \frac{\partial\left(z_2^{*2}\right)}{\partial z_1}$$
$$= -\frac{17}{14} \int_{z_2^*}^{z_2} \left(s^2 - z_2^{*2}\right)^{\frac{17}{14}} ds \frac{\partial\left(z_2^{*2}\right)}{\partial z_1}$$

$$\frac{\partial U_2}{\partial z_2} = \left(z_2^2 - z_2^{*2}\right)^{\frac{17}{14}} = \xi_2^{\frac{17}{14}};$$

$$U_2 \geq L(z_2 - z_2^*)^{\frac{17}{14}}$$

$$\dot{V}_1(\bar{z}_1) \le -2\xi_1^2 - e^{-d}\xi_1^2(t-d_1(t)) - W_1 + \xi_1^{\frac{17}{14}} z_2^{*\frac{11}{7}} + \xi_1^{\frac{17}{14}}(z_2^{\frac{11}{7}} - z_2^{*\frac{11}{7}}) + \left(\frac{2}{1-\lambda_1} + \tilde{\alpha}_1(\bar{z}_1)\right)\xi_1^2 + \delta$$

$$\dot{V}_2(\bar{z}_2) \le -\left(\xi_1^2 + \xi_2^2\right) - (W_1 + W_2) + \xi_2^{\frac{17}{14}}(z_3^{\frac{7}{5}} - z_3^{*\frac{7}{5}}) + 2\delta$$

$$z_2^{*\frac{11}{7}} = -\xi_1^{\frac{11}{14}}\left\{\frac{1}{b_{11}}\left(2 + \frac{2}{1-\lambda_1} + \frac{1}{1-\lambda_2}\right) + \tilde{\alpha}_1(z_1)\right\} =: \;=: \xi_1^{\frac{11}{14}}\beta_1^{\frac{11}{14}}(\bar{z}_1)$$

$$u^{\frac{11}{7}} = -\xi_1^{\frac{11}{14}}\beta_1^{\frac{11}{14}}(z)$$

$$u^{\frac{7}{5}} = -\xi_2^{\frac{11}{14}}\beta_2^{\frac{11}{14}}(z)$$

$$\dot{V}_2(z) \le -(\xi_1^2 + \xi_2^2) - (W_1 + W_2) + 2\delta$$

where $V_2(z) = (U_1(\bar{z}_1) + U_2(\bar{z}_2)) + W_1 + W_2$

$$W_1 + W_2 = \frac{2}{1-\lambda_1}\int_{t-d_1(t)}^{t} e^{s-t}\xi_1^2(s)ds$$

$$U_1(\bar{z}_1) + U_2(\bar{z}_2) \le |\xi_1|^2 + |\xi_2|^2 + 2 * 2^{\frac{28}{-3}} * \frac{4}{14} \le$$

$$\le |\xi_1|^2 + |\xi_2|^2 + \frac{1}{896\sqrt[3]{2}}$$

$$V_2(z(t)) < 2\cdot 2^{\frac{4}{\frac{4}{7}-1}}\frac{4}{14} + 4\delta \le \frac{1}{1129} + 4\delta$$

$$x_2^{*\frac{11}{7}} = -x_1^{\frac{11}{7}}(2 + \tilde{\kappa}_1(x_1)) = -x_1^{\frac{11}{7}}\beta_1(x_1)$$

$$|y(t) - y_r(t)| = |x_1(t)| \le V_2(x(t)) \le \frac{1}{1129} + 4\delta < \varepsilon,$$

for any $\varepsilon > 0$

$\delta = 0,0023$ the tracking error obtained is about 0.01; $\delta = 0.025$ then the tracking error reduces to about 0.025.

5. CONCLUSION

This paper studied the problem of global practical output tracking for a class of high-order non-linear systems with time-varying delays under the weaker conditions on the system nonlinearities. With the help of an appropriate Lyapunov-Krasovskii functionals and by using the method of adding a power integrator, a continuous state-feedback controller is successfully designed such that all the states of the resulting close loop system are bounded while the output tracking error converges to an arbitrarily small residual set. A numerical example demonstrates the effectiveness of the result.

Appendix

Proposition 1: There exists a positive constant h_{k1} such that

$$\xi_{k-1}^{\frac{2\rho - r_{k-1} - \tau_{k-1}}{r_n + \tau_n}}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \le \frac{1}{3}\xi_{k-1}^{\frac{2\rho}{r_n+\tau_n}} + h_{k1}\xi_k^{\frac{2\rho}{r_n+\tau_n}}.$$

Proof: Due to the fact that $\rho \ge \max_{1\le i\le n}\{r_i + \tau_i\}$ and $r_k p_{k-1} = r_{k-1} + \tau_{k-1}$, we have $\frac{r_k p_{k-1}}{\rho} \le 1$. So, it follows from Lemma 1 that

$$\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \le \left|\left(x_k^{\rho/r_k}\right)^{\frac{r_k p_{k-1}}{\rho}} - \left(x_k^{*\rho/r_k}\right)^{\frac{r_k p_{k-1}}{\rho}}\right| \le 2^{1-\frac{r_k p_{k-1}}{\rho}}\left|x_k^{\frac{\rho}{r_k}} - x_k^{*\frac{\rho}{r_k}}\right|^{\frac{r_k p_{k-1}}{\rho}} \le 2^{1-\frac{r_k p_{k-1}}{\rho}}|\xi_k|^{\frac{r_k p_{k-1}}{\rho}} = \tilde{h}_{k1}|\xi_k|^{\frac{r_k p_{k-1}}{\rho}}. \quad \text{(A1)}$$

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By Lemma 2, Assumption 2 and (A1), and noting $r_k p_{k-1} = r_{k-1} + \tau_{k-1}$, it can be seen that

$$\begin{aligned} g_{k-1}(t)\xi_{k-1}^{\frac{2\rho - r_{k-1}-\tau_{k-1}}{\rho}}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) &\le b_{k1}\left|\xi_{k-1}\right|^{\frac{2\rho - r_{k-1}-\tau_{k-1}}{\rho}}\tilde{h}_{k1}\left|\xi_k\right|^{\frac{r_k p_{k-1}}{\rho}} \\ &\le \frac{1}{3}\xi_{k-1}^2 + h_{k1}\xi_k^2 \end{aligned} \tag{A2}$$

for a positive constant h_{k1}. Proposition 1 is proved.

Proposition 2: There exists a positive smooth function $h_{k2}(\bar{x}_k)$ and any real number $\delta > 0$ such that

$$\begin{aligned} \xi_k^{\frac{2\rho - r_{k-1}-\tau_{k-1}}{\rho}}(t) f_k(\cdot) &\le \frac{1}{4}\sum_{j=1}^{k-1}\xi_j^2(t) + \frac{1}{2}\sum_{j=1}^{k}\xi_j^2(t-d_j(t)) + \\ &+ \frac{1}{2}\sum_{j=1}^{k-1}\xi_j^2(t-d_{j+1}(t)) + h_{k2}(\bar{z}_k(t))\xi_k^2(t) + \frac{1}{2}\delta. \end{aligned}$$

where

$$\begin{aligned} f_k(\cdot) := f_k(&z_1(t)+y_r(t), z_2(t), \dots, z_k(t), \\ &z_1(t-d_1(t)) + y_r(t-d_1(t)), z_2(t-d_2(t)), \dots, \\ &z_k(t-d_k(t))) \end{aligned}$$

Proof: Using Lemma 1, Assumptions 1 and (7), it can be rewritten as (for $l = 2,\dots,k$)

$$\begin{aligned} &\left|f_k(\cdot)\right| \le b_{k1}(z_1(t)+y_r(t), z_2(t), \dots, z_k(t)) \\ &\left[\left(\left|z_1(t)+y_r(t)\right|^{\frac{r_1+\tau_1}{r_1}} + \left|z_2(t)\right|^{\frac{r_2+\tau_2}{r_2}} + \dots + \left|z_k(t)\right|^{\frac{r_k+\tau_k}{r_k}}\right)\right. \\ &+ \left|(z_1(t-d_1(t)) + y_r(t-d_1(t))\right|^{\frac{r_1+\tau_1}{r_1}} + \\ &\left. + \left|z_2(t-d_2(t))\right|^{\frac{r_2+\tau_2}{r_2}} + \left|z_k(t-d_k(t))\right|^{\frac{r_k+\tau_k}{r_k}}\,)\right] + \\ &+ b_{k2}(z_1(t)+y_r(t), z_2(t), \dots, z_k(t)) \\ &\le \tilde{b}_{k1}(\bar{z}_k(t))\sum_{j=1}^{k}\left(\left|\xi_j(t) + \beta_{j-1}\xi_{j-1}(t)\right|^{\frac{r_j}{\rho}}\right)^{\frac{r_k+\tau_k}{r_j}} + \\ &+ \tilde{b}_{k1}(\bar{z}_k(t))\sum_{j=1}^{k}\left(\left|\xi_j(t-d_j(t)) + \right.\right. \\ &\left.\left. + \beta_{j-1}\xi_{j-1}(t-d_{j-1}(t))\right|^{\frac{r_j}{\rho}}\right)^{\frac{r_k+\tau_k}{r_j}} + 2\tilde{b}_{k1}(\bar{z}_k(t))M^{\frac{r_k+\tau_k}{\rho}} + \\ &+ \tilde{b}_{k2}(\bar{z}_k(t)) \le \tilde{b}_{k1}(\bar{z}_k(t))\sum_{j=1}^{k}(1+\beta_{j-1}^{\frac{r_k+\tau_k}{\rho}})\left|\xi_j(t)\right|^{\frac{r_k+\tau_k}{\rho}} + \\ &+ \tilde{b}_{k1}(\bar{z}_k(t))\sum_{j=1}^{k}\left|\xi_j(t-d_j(t))\right|^{\frac{r_k+\tau_k}{\rho}} + \\ &+ \tilde{b}_{k1}(\bar{z}_k(t))\sum_{j=1}^{k}\beta_{j-1}^{\frac{r_k+\tau_k}{\rho}}\left|\xi_j(t-d_{j-1}(t))\right|^{\frac{r_k+\tau_k}{\rho}} \\ &+ 2\tilde{b}_{k1}(\bar{z}_k(t))M^{\frac{r_k+\tau_k}{\rho}} + \tilde{b}_{k2}(\bar{z}_k(t)) \end{aligned} \tag{A5}$$

for smooth, positive nonzero functions $\bar{\beta}_i(x_1,\dots,x_i) = \beta_i^{\frac{r_n+\tau_n}{r_{l+1}p_l}}(x_1,\dots,x_i)$, $i = 1,2,\dots,l$ and $\bar{b}_k(x_1,\dots,x_k)$.

By Lemmas 2-3 and (A5), with

$$\frac{2\rho - r_l - \tau_l}{r_n + \tau_n} + \frac{r_{l+1}p_l}{r_n + \tau_n} = \frac{2\rho}{r_n + \tau_n},$$

$$\xi_k^{\frac{2\rho-r_k-\tau_k}{\rho}}(t) f_k(\cdot) \le |\xi_k|^{\frac{2\rho-r_k-\tau_k}{\rho}}$$
$$\tilde{b}_{k1}(\bar{z}_k(t))\sum_{j=1}^{k}(1+\beta_{j-1}^{\frac{r_k+\tau_k}{\rho}})|\xi_j(t)|^{\frac{r_k+\tau_k}{\rho}} + \tilde{b}_{k1}(\bar{z}_k(t))$$
$$\left(\sum_{j=1}^{k}|\xi_j(t-d_j(t))|^{\frac{r_k+\tau_k}{\rho}} + \tilde{b}_{k1}(\bar{z}_k(t))\right.$$
$$\sum_{j=1}^{k}\beta_{j-1}^{\frac{r_k+\tau_k}{\rho}}|\xi_j(t-d_{j-1}(t))|^{\frac{r_k+\tau_k}{\rho}}$$
$$\left.+2\tilde{b}_{k1}(\bar{z}_k(t))M^{\frac{r_k+\tau_k}{\rho}} + \tilde{b}_{k2}(\bar{z}_k(t))\right)$$
$$\le \frac{1}{4}\sum_{j=1}^{k-1}\xi_j^2(t) + \frac{1}{2}\sum_{j=1}^{k}\xi_j^2(t-d_j(t))$$
$$+\frac{1}{2}\sum_{j=1}^{k-1}\xi_j^2(t-d_{j+1}(t)) + h_{k2}(\bar{z}_k(t))\xi_k^2(t) + \frac{1}{2}\delta. \tag{A6}$$

for a smooth function $h_{k2}(\bar{z}_k) > 0$ and any real number $\delta > 0$. Proposition 2 is proved.

Proposition 3: There exists a positive smooth function $h_{k3}(\bar{x}_k)$ and any real number $\delta > 0$ such that

$$\left|\sum_{j=1}^{k-1}\frac{\partial U_k}{\partial z_j}\dot{z}_j\right| \le \frac{1}{4}\sum_{j=1}^{k-1}\xi_j^2 + \frac{1}{2}\sum_{j=1}^{k-1}\xi_j^2(t-d_j(t)) +$$
$$+\frac{1}{2}\sum_{j=1}^{k-2}\xi_j^2(t-d_{j+1}(t)) + h_{k3}(\bar{z}_k)\xi_k^2 + \frac{1}{2}\delta.$$

Proof: Noting that

$$z_i^{*\frac{\rho}{r_i}} = -\beta_{i-1}(\bar{z}_{i-1})\xi_{i-1} = \sum_{j=1}^{i-1}B_j z_j^{\frac{\rho}{r_j}}$$

where $B_j = \beta_{i-1}\cdots\beta_j$, $j = 1,\ldots,i-1$. Then, for $j = 1,\ldots,i-1$, we have

$$\frac{\partial z_i^{*\frac{\rho}{r_i}}}{\partial z_j} = -\left(\sum_{j=1}^{i-1}\frac{\partial B_j}{\partial z_j}z_j + \frac{\rho}{r_j}B_j\right)z_j^{\frac{\rho-r_j}{r_j}} \tag{A7}$$

By (7), (11), (A5), (A7) and Assumption 1-2, we have

$$\sum_{j=1}^{k-1}\frac{\partial U_k}{\partial z_j}\dot{z}_j =$$
$$=\sum_{j=1}^{k-1}\left(-\frac{2\rho-r_k-\tau_k}{\rho}\int_{z_k^*}^{z_k}\left(s^{\frac{\rho}{r_k}} - z_k^{*\frac{\rho}{r_k}}\right)^{\frac{2\rho-r_k-\tau_k}{\rho}-1} ds\right.$$
$$\left.\frac{\partial\left(z_k^{*\frac{\rho}{r_k}}\right)}{\partial z_j}\right)\left(g_j z_{j+1}^{p_j} + f_j(\cdot)\right) \le$$
$$\le b_k\sum_{j=1}^{k-1}|\xi_k|^{\frac{\rho-r_k}{\rho}}\left|\frac{\partial z_k^{*\frac{\rho}{r_k}}}{\partial z_j}\right|\left(b_{j2}\left|z_{j+1}^{p_j}\right| + |f_j(\cdot)|\right)$$
$$\le b_k\sum_{j=1}^{k-1}\left(\sum_{j=1}^{i-1}\frac{\partial B_j}{\partial z_j}z_j + \frac{\rho}{r_j}B_j\right)z^{\frac{\rho-r_j}{r_j}}|\xi_k|^{\frac{\rho-r_k}{\rho}}\times$$
$$\times\left(b_{j2}\left|z_{j+1}^{p_j}\right| + |f_j(\cdot)|\right) \le b_k\sum_{j=1}^{k-1}\left(\sum_{j=1}^{i-1}\frac{\partial B_j}{\partial z_j}z_j + \frac{\rho}{r_j}B_j\right)\times$$
$$\times\left|\xi_j + \beta_{j-1}\xi_{j-1}\right|^{\frac{\rho-r_j}{r_j}}|\xi_k|^{\frac{\rho-r_k}{\rho}}\left[b_{j2}\left(\left|\xi_{j+1} + \beta_j\xi_j\right|^{\frac{r_{j+1}}{\rho}}\right)^{p_j} +\right.$$
$$+\tilde{b}_{k1}(\bar{z}_k(t))\sum_{l=1}^{j}(1+\beta_{l-1}^{\frac{r_k+\tau_k}{\rho}})|\xi_l(t)|^{\frac{r_k+\tau_k}{\rho}} +$$
$$+\tilde{b}_{k1}(\bar{z}_k(t))\sum_{l=1}^{j}|\xi_l(t-d_l(t))|^{\frac{r_k+\tau_k}{\rho}}$$
$$+\tilde{b}_{k1}(\bar{z}_k(t))\sum_{l=1}^{j}\beta_{l-1}^{\frac{r_k+\tau_k}{\rho}}|\xi_l(t-d_{l-1}(t))|^{\frac{r_k+\tau_k}{\rho}}$$
$$\left.+2\tilde{b}_{k1}(\bar{z}_k(t))M^{\frac{r_k+\tau_k}{\rho}} + \tilde{b}_{k2}(\bar{z}_k(t))\right] \tag{A8}$$

Noting that $p_j r_{j+1} = r_j + \tau_j$ and $\tau_j \ge \tau_k$, by using Lemma 2, we have (see (A9))

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$$
\begin{aligned}
&\sum_{j=1}^{k-1}\frac{\partial U_k}{\partial z_j}\dot{z}_j \\
&\le \tilde{b}_k\sum_{j=1}^{k-1}\left|\xi_k\right|^{\frac{\rho-r_k}{\rho}}\left|\xi_j+\beta_{j-1}\xi_{j-1}\right|^{\frac{\rho-r_j}{r_j}}\Bigg[b_{j2}\left|\xi_{j+1}+\beta_j\xi_j\right|^{\frac{r_j+\tau_j}{\rho}} \\
&\quad+\tilde{b}_{k1}(\bar{z}_k(t))\sum_{l=1}^{j}(1+\beta_{l-1}^{\frac{r_k+\tau_k}{\rho}})\left|\xi_l(t)\right|^{\frac{r_k+\tau_k}{\rho}}+\tilde{b}_{k1}(\bar{z}_k(t)) \\
&\quad\sum_{l=1}^{j}\left|\xi_l(t-d_l(t))\right|^{\frac{r_k+\tau_k}{\rho}}+\tilde{b}_{k1}(\bar{z}_k(t))\sum_{l=1}^{j}\beta_{l-1}^{\frac{r_k+\tau_k}{\rho}}\cdot \\
&\cdot\left|\xi_l(t-d_{l-1}(t))\right|^{\frac{r_k+\tau_k}{\rho}}+2\tilde{b}_{k1}(\bar{z}_k(t))M^{\frac{r_k+\tau_k}{\rho}}+\tilde{b}_{k2}(\bar{z}_k(t))\Bigg] \\
&\quad\le\frac{1}{4}\sum_{j=1}^{k-1}\xi_j^2+\frac{1}{2}\sum_{j=1}^{k-1}\xi_j^2(t-d_j(t)) \\
&\quad+\frac{1}{2}\sum_{j=1}^{k-2}\xi_j^2(t-d_{j+1}(t))+h_{k3}(\bar{z}_k)\xi_k^2+\frac{1}{2}\delta.
\end{aligned}
\tag{A9}
$$

where $\tilde{b}_k$ and $h_{k3}(\bar{z}_k)$ are non-negative smooth functions. Proposition 3 is proved.

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