SHIFTED LEGENDRE POLYNOMIAL BASED GALERKIN AND COLLOCATION METHODS FOR SOLVING FRACTIONAL ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this article, effective numerical methods for the solution of fractional order delay differential equations (FODDEs) are presented. The fractional derivative (FD) is defined in Caputo sense. Shifted Legendre polynomials are used in the Collocation and Galerkin methods to convert FDDEs to the linear and/or nonlinear system in algebraic form of equations. Example problems are addressed to show the powerfulness and efficacy of the methods.

Keywords: *Residual, Galerkin Method, Fractional Delay Differential Equation, Legendre Polynomial, Caputo Fractional Derivatives*

1. INTRODUCTION

Fractional Calculus (FC) is a vital branch in applied mathematics. It is a generalization of the classical ordinary calculus (differentiation and integration) to an arbitrary order. FC arises in electromagnetism, rheology, electrochemistry,viscoelasticity, and so on. For points of interest, one may see the references [1-5]. In current years, physicists likewise mathematicians have dedicated considerable efforts to discover robust and stable semi-approximate, numerical and analytical methods aimed at solving the fractional differential equation of substantial interest. Some of the numerical techniques may be recorded as a generalized differential transform method (See [6] to [7]), Sumudu transform method [8], Adomian decomposition method (See [9] to [10]), homotopy perturbation technique (See [11] to [13]), Residual power series method (See [14] to [15]), differential transform method (See [16] to [18]) and Homotopy analysis method [19]. One may see the detailed study on fractional calculus in [20]. Also, a new solution method in analytical form has been presented in [21] to solve "The Time-Fractional Coupled-Korteweg-deVries Equations" through homotopy decomposition method by the same researchers. The sinc methods have been illustrated in [22] and extended in [23] by Frank Stenger. The sinc functions have been firstly examined in Ref. [24-25]. In Ref. [26-27], the sinc-Galerkin method has been applied to the nonlinear differential equations processing homogeneous and or nonhomogeneous boundary conditions.

Differential Equations (DEs) are of various forms. They appear in several arms of disciplines ranging from social sciences (as in the theory of economics), sciences, and engineering with basic structures in modeling. DEs can be ordinary or partial in nature. In pure and applied mathematics, Delay Differential Equations, shortly written as DDEs represent a type of DE where the unknown derivative functions at a specific time is defined in terms of values of the concerned function at previous times. DDEs can take systematic forms. The delay term or function can be proportion or constant. substantial skill, softwares and methods are required to solve a realistic system of DDEs. Therefore, obtatining the solutions of DDEs (if they exist) need reliable and effective solution approaches.

In this paper, two methods (Shifted Legendre polynomials and Galerkin) are proposed for handling the FODDEs. However, to the best of our knowledge, Legendre Galerkin and Legendre Collocation methods have not been used for the fractional order delay differential equations. It may be worth mentioning that these well-known methods turn out to be simpler for handling the titled problems by the use of the shifted Legendre polynomials. Fractional order delay differential equation may be written as [28],

$$\begin{cases} w^{(\alpha)}(x) = f\left(w(h(x)), w(x), x\right) \\ w(a)=\gamma_0, w(b)=\gamma_1 \text{ (BD)} \\ a \le x \le b,\ 1<\alpha \le 2 \end{cases}, \tag{1}$$

where α denotes the fractional-order derivatives, BD boundary conditions, and h represents the supposed continuous function in delay form defined on the interval $[a,b]$.

The remaining parts of the article are organized as follows. Some essential definitions in line with titled problems are introduced are contained in Section 2. Similarly, in Section 3, the mathematical expression of the Legendre series is included. Legendre Collocation and Legendre Galerkin methods w.r.t fractional order delay differential equation are proposed in Section 4 and 5. In Section 6, four cases of examples are included to demonstrate the effectiveness and accuracy of the present method. Also, a conclusion is given in Section 7.

2. PRELIMINARIES

Definition 2.1 (See [4] and [5]):

According to Riemann-Liouville, the fractional differential (FD) operator $\boldsymbol{D^\alpha}$ of order $\boldsymbol{\alpha}$ is defined as:

$$D^\alpha w(x) = \begin{cases} \dfrac{d^n}{dx^n} w(x),\ \alpha = n \\ \left\{\Gamma(n-\alpha)\right\}^{-1} \dfrac{d}{dx^n}\displaystyle\int_0^x \frac{w(t)}{h(x,t)}dt, \\ h(x,t) = (x-t)^{\alpha-n+1}, \\ n-1<\alpha<n \end{cases} \tag{2}$$

where $n \in Z^+$, $\alpha \in R^+$ and

$$\begin{cases} D^{-\alpha} w(x) = \eta \displaystyle\int_0^x (x-t)^{\alpha-1} w(t)dt,\ \alpha \in (0,1] \\ \eta = \dfrac{1}{\Gamma(\alpha)}. \end{cases} \tag{3}$$

Definition 2.2 (See [1], [2] and [5]):

In fractional order form, the Riemann-Liouville integral operator J^α is described as

$$\begin{cases} J^\alpha w(x) = \eta \displaystyle\int_0^x (x-t)^{\alpha-1} w(t)dt, \\ t>0, \alpha>0,\ \eta = \dfrac{1}{\Gamma(\alpha)}. \end{cases} \tag{4}$$

Following Podlubny [5] we may have

$$\left.\begin{aligned} J^\alpha t^n &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} t^{n+\alpha}, \\ D^\alpha t^n &= \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}. \end{aligned}\right\} \tag{5}$$

Definition 2.3 (See [2], [4] and [5]):

The Caputo FD operator D^α of order α is given as

$${}^C D^\alpha w(x) = \begin{cases} \left\{\Gamma(m-\alpha)\right\}^{-1} \\ \times \displaystyle\int_0^x \frac{w^m(t)}{(x-t)^{\alpha-m+1}} dt,\ \alpha \in (m-1, m), \\ \dfrac{d^m}{dt^m} w(x), \qquad \alpha = m. \end{cases} \tag{6}$$

3. MATHEMATICAL EXPRESSION FOR LEGENDRE SERIES

The Legendre polynomials are well-defined on a given interval $I = [-1,1]$. The following recurrence formula yields these polynomials [29]

$$\begin{cases}(k+1)\,p_{k+1}(x)=(2k+1)\,xp_k(x)\\ \qquad\qquad -kp_{k-1}(x),\\ k=1(1)\cdots,\end{cases} \tag{7}$$

where $p_0(x)=1$ and $p_1(x)=x$. The analytic form of the Legendre polynomial of order n is given by

$$p_n(x)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k}. \tag{8}$$

In order to handle the title problem on the defined interval, $[0,1]$, $\phi_n(x)$ is defined as the shifted Legendre polynomials. These are defined in term of Legendre polynomials $p_n(x)$ by the following relation

$\phi_n(x)=p_n(2x-1)$, and the recurrence formula for this is

$$\varphi_{n+1}(x)=\frac{2n+1}{n+1}(2x-1)\varphi_n(x) -\frac{n}{n+1}\varphi_{n-1}(x),\ n=1,2,\cdots \tag{9}$$

where $\phi_0(x)=1$ and $\phi_1(x)=2x-1$. The shifted Legendre polynomial $\phi_k(x)$ of degree k in analytic form is given by

$$\phi_k(x)=\sum_{i=0}^{k}(-1)^{k+i}\frac{(k+i)!x^i}{(k-i)!(i!)^2} \tag{10}$$

where $\{\varphi_k(1)=1, \text{ and } \varphi_k(0)=(-1)^k$.

Thus, the condition for orthogonality is:

$$\int_0^1\varphi_i(x)\varphi_j(x)dx=\begin{cases}\dfrac{1}{2i+1} & , j=i\\ 0 & , j\neq i\end{cases}$$

It may be noted that Legendre polynomials can easily approximate the solution of a given differential equation with dependent variable, $u(x)\in C[0,1]$ as follows

$$\begin{cases}w(x)\approx\sum_{i=0}^{\infty}c_i\varphi_i(x)\\ c_i=(2i+1)\int_0^1 w(x)\varphi_i(x)dx,\ i=1,2,3,\cdots\end{cases} \tag{11}$$

4. LEGENDRE GALERKIN METHOD

Consider the FODDE of the form [28]

$$\begin{cases}u^{\alpha}(x)=f(x,u(x),u(g(x))),\\ u(a)=\gamma_0,\ u(b)=\gamma_1\\ a\le x\le b,\ 1<\alpha\le 2.\end{cases} \tag{12}$$

We assume the following as an approximate solution for (12):

$$u(x)=h(x)\left(\sum_{i=0}^{n}c_i\varphi_i(x)\right), \tag{13}$$

where $h(x)$ controls the boundary conditions and φ_i's are the shifted Legendre polynomials.

Substituting Eq. (13) in Eq. (12) one may get the residual R as [30].

$$\left.\begin{aligned}R(x,c_i)&=\sum_{i=0}^{n}c_ih^{\alpha}(x)\varphi_i^{\alpha}(x)\\ &-f\left(x,\sum_{i=0}^{n}c_ih(x)\varphi_i(x),\sum_{i=0}^{n}c_ih(g)\varphi_i(g)\right)\end{aligned}\right\},$$

$R(x,c_i)=R(x,c_0,c_1,....,c_n),g=g(x)$.

Here, orthogonalizing R to the $(n+1)$ functions $\varphi_0,\varphi_1,\cdots,\varphi_n$, gives

$$\begin{cases}\int_a^b R(x,c_0,c_1,c_2,....,c_n)\varphi_j(x)dx=0,\\ j=0(1)n\end{cases}$$

$$\left.\begin{aligned}&\int_a^b\left\{\sum_{i=0}^{n}c_ih^{\alpha}(x)\varphi_i^{\alpha}(x)\varphi_j(x)dx\right\}\\&-\int_a^b\left\{f\left(x,\sum_{i=0}^{n}c_ih(x)\varphi_i(x),w(\varpi)\right)\varphi_j(x)dx\right\}=0,\\&w(\varpi)=\sum_{i=0}^{n}c_ih(g(x))\varphi_i(g(x)),\\&j=0(1)n.\end{aligned}\right\}\tag{14}$$

Eq. (14) gives $(n+1)$ system of equations involving $(n+1)$ unknown variables, that can be resolved by using any known method. Thereafter, substituting the estimated constants $c_0,c_1,\cdots,c_n$ in Eq. (13) one may get the approximate solution for Eq. (12).

5. LEGENDRE COLLOCATION METHOD

Consider the FODDE of the form [28]

$$\begin{cases}u^{\alpha}(x)=f\left(u(g(x)),u(x),x\right),\\u(a)=\gamma_0,u(b)=\gamma_1,\\a\le x\le b,1<\alpha\le 2.\end{cases}\tag{15}$$

Again, an approximate solution of (15) is assumed to satisfy the boundary condition with the unknown constants $c_i:\ i=0(1)n$ as:

$$u(x)=h(x)\left(\sum_{i=0}^{n}c_i\phi_i(x)\right),\tag{16}$$

where $h(x)$ controls the boundary conditions and ϕ_i's are the shifted Legendre polynomials

Substituting Eq. (16) in Eq. (15) one may get the residual R as [30]

$$R(x,c_i)=\begin{cases}\sum_{i=0}^{n}c_ih^{\alpha}(x)\varphi_i^{\alpha}(x)\\-f\left(x,\sum_{i=0}^{n}c_ih(x)\varphi_i(x),\sum_{i=0}^{n}c_ih(g)\varphi_i(g)\right),\\g=g(x),R(x,c_i)=R(x,c_0,c_1,\ldots.,c_n).\end{cases}$$

In this method, we force the residual, R to become zero at $(1+n)$ points: say, $x_0,x_1,\cdots,x_n\in[a,b]$ in that is:

$$\begin{cases}\sum_{i=0}^{n}c_ih^{\alpha}(x_j)\varphi_i^{\alpha}(x_j)\\-f\left(x_j,\sum_{i=0}^{n}c_ih(x_j)\varphi_i(x_j),\sum_{i=0}^{n}c_ih(g(x_j))\varphi_i(g(j))\right)=0,\\j=0,1,2,\ldots,n,g(x_j)=g(j)\end{cases}\tag{17}$$

Here also, Eq. (17) gives $(n+1)$ system of equations involving $(n+1)$ unknown variables that can be resolved by any known method. Substituting the estimated constants $c_0,c_1,\cdots,c_n$ in Eq. (16) one can get the solution approximation for the original Eq. (15).

6. NUMERICAL RESULTS

Here, the proposed methods are implemented for both linear and nonlinear examples of FODDEs. The first two examples are solved by Legendre Galerkin method, and examples 3 and 4 are solved by using Legendre Collocation Method. Solutions of these examples are thereafter compared with those from Legendre Pseudospectral method.

Example 1. Consider the nonlinear FODDE of the form [28]:

$$\begin{cases}D^{1.5}u(x)=g(x)+u\left(x-\frac{1}{2}\right)+u^3(x),\\g(x)=\frac{2}{\Gamma(1.5)}x^{0.5}-\left(x-\frac{1}{2}\right)^2-x^6,\\u(0)=0,\ u(1)=1.\end{cases}\tag{18}$$

Let us consider two terms guess solution as

$$\begin{aligned}u(x)&=h(x)\left[c_0\varphi_0+c_1\varphi_1\right]\\&=h(x)\left(\sum_{i=0}^{1}c_i\varphi_i\right)\end{aligned}\tag{19}$$

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where $h(x) = x$ which will control the boundary conditions.

The residual

$$R = \begin{cases} D^{1.5}\left(h(x)\left(\sum_{i=0}^{1} c_i \varphi_i \right) \right) \\ -\left(h(x-0.5)\left(\sum_{i=0}^{1} c_i \varphi_i (x-0.5) \right) \right) \\ -\left(h(x)\left(\sum_{i=0}^{1} c_i \varphi_i \right) \right)^3 - f(x) \end{cases} \quad (20)$$

where $\phi_0 = 1$, and $\phi_1 = 2x - 1$ are the shifted Legendre polynomials in the domain [0 1].

Using the function $h(x)$, ϕ_0 and ϕ_1 in Eq. (20), we have

$$R = \begin{pmatrix} \frac{8}{\sqrt{\pi}} c_1 x^{0.5} - c_0 (x-0.5) \\ -c_1 (2(x-0.5)^2 - (x-0.5)) \\ -\{c_0 x + c_1 (2x^2 - x)\}^3 \\ -\frac{2}{\Gamma(1.5)} x^{0.5} + (x-0.5)^2 + x^6 \end{pmatrix}. \quad (21)$$

Now by using Legendre Galerkin Method, we have

$$\begin{cases} \int_0^1 (R\varphi_0) dx = 0, \\ \int_0^1 (R\varphi_1) dx = 0. \end{cases}$$

$$\Rightarrow \begin{cases} 2.842344445 c_1 - 1.730696032 \\ -0.09285714286 c_1^3 - 0.35 c_0 c_1^2 \\ -0.45 c_0^2 c_1 - 0.25 c_0^3 = 0 \end{cases} \quad (22) \text{ and}$$

$$\begin{pmatrix} .7684688891 c_1 - 0.4080439683 \\ -0.07857142857 c_1^3 - 0.2785714286 c_0 c_1^2 \\ -0.35 c_0^2 c_1 - 0.15 c_0^3 - 0.1666666667 c_0 = 0 \end{pmatrix}. \quad (23)$$

Solving the above two nonlinear systems of equations, we obtain

$$\begin{cases} c_0 = 0.5000000001 \\ c_1 = 0.5000000002. \end{cases}$$

So, the following solves the FODDE:

$$\left. \begin{aligned} u(x) &= h(x)\left[c_0 \varphi_0 + c_1 \varphi_1 \right] \\ &= x \begin{bmatrix} 0.5000000001 \\ +0.5000000002(2x-1) \end{bmatrix} \\ &= 1.0000000004 x^2 - 0.0000000001 x \end{aligned} \right\}. \quad (24)$$

One may see that the exact solution of Eq. (18) agrees precisely by taking two terms. The numerical results of the present solution and solution solved by Legendre pseudospectral method are given in table 1. The behavior of the exact and present solutions of this example has been presented in figure 1

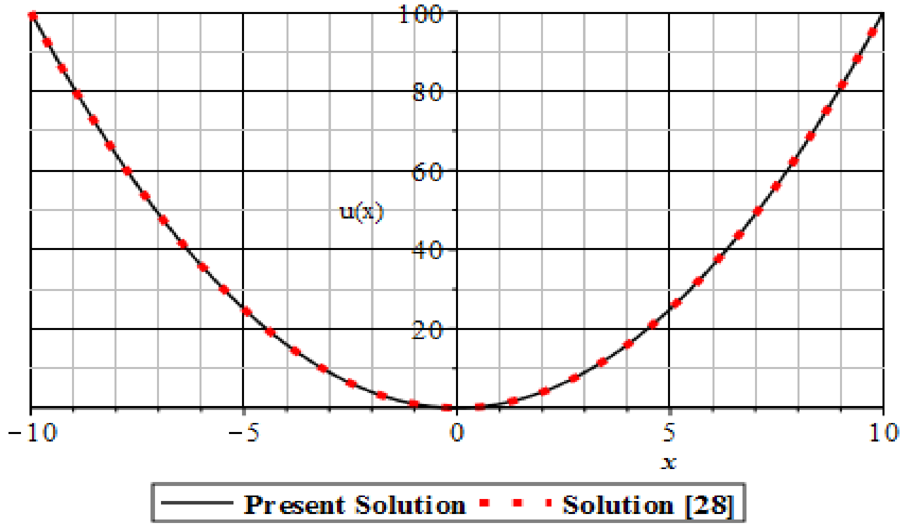

Figure 1: Graphic Of Exact And Present Solutions Of Example 1.

Table 1: Comparison Of The Present Solution With Ref. [28]

x	Present solution $n=1$	solution [28]
0.0	0.000	0.00
0.1	0.0099	0.01
0.2	0.040	0.04
0.3	0.090	0.09
0.4	0.160	0.16
0.5	0.250	0.25
0.6	0.360	0.36
0.7	0.490	0.49
0.8	0.640	0.64

Example 2. Here, the following form of linear FODDE is considered [28]:

$$\left.\begin{cases} D^{1.5}u(x) = -\left(\begin{array}{l} u(x) + u\left(\dfrac{x}{2}\right) \\ +\dfrac{7}{8}x^3 + \dfrac{6}{\Gamma(2.5)}x^{1.5} \end{array} \right), \\ u(0)=0,\ u(1)=1. \end{cases}\right\}, \quad (25)$$

Here we take 4 term solution as till 3 term solution it was not converging.

As such let us take

$$\left.\begin{aligned} u(x) &= h(x)\left[a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3\right] \\ &= h(x)\left(\sum_{i=0}^{3} a_i\varphi_i(x)\right) \end{aligned}\right\}, \quad (26)$$

where $h(x) = x$, which will control the boundary conditions.

The residual

$$\left.\begin{aligned} R = {} & D^{1.5}\left(h(x)\left(\sum_{i=0}^{3} a_i\varphi_i(x)\right)\right) \\ & + h(x)\left(\sum_{i=0}^{3} a_i\varphi_i(x)\right) \\ & - h\left(\frac{x}{2}\right)\left(\sum_{i=0}^{3} a_i\varphi_i\left(\frac{x}{2}\right)\right) - \frac{7}{8}x^3 - \frac{6}{\Gamma(2.5)}x^{1.5} \end{aligned}\right\}. \quad (27)$$

where $\phi_0, \phi_1, \phi_2 \,\&\, \phi_3$ are the shifted Legendre polynomials of order zero, one, two, and three.

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$$\begin{cases}\varphi_0 = 1, \varphi_1 = 2x-1,\\ \varphi_2 = 6x^2-6x+1,\\ \varphi_3 = 20x^3-30x^2+12x-1.\end{cases} \tag{28}$$

Substituting the shifted Legendre polynomials given in [28] and $h(x)=x$, in Eq. (27), we have

$$R=\begin{cases}\frac{8}{\sqrt{\pi}}a_1x^{0.5}+\frac{48}{\sqrt{\pi}}a_2x^{1.5}-\frac{24}{\sqrt{\pi}}a_2x^{0.5}\\ +\frac{900}{\sqrt{\pi}}a_3x^{2.5}-\frac{240}{\sqrt{\pi}}a_3x^{1.5}+\frac{48}{\sqrt{\pi}}a_3x^{0.5}\\ +a_0x+a_1(2x^2-x)+a_2(6x^3-6x^2+x)\\ +a_3(20x^4-30x^3+12x^2-x)\\ -\left(\frac{x}{2}\right)\left(\begin{array}{l}a_0+a_a(x-1)+a_2\left(6\left(\frac{x}{2}\right)^2-3x+1\right)\\ +a_3\left(20\left(\frac{x}{2}\right)^3-30\left(\frac{x}{2}\right)^2+6x-1\right)\end{array}\right)\\ -\frac{7}{8}x^3-\frac{6}{\Gamma(2.5)}x^{1.5}.\end{cases}$$

Using Legendre Galerkin Method, we have

$$\left\{\int_0^1(R\varphi_0)dx=0.\right.$$

$$\Rightarrow\begin{pmatrix}-2.024156667+3.259011112a_1\\ +1.867906671a_2\\ +108.9066882a_3+0.250a_0=0\end{pmatrix}, \tag{29}$$

$$\begin{cases}\int_0^1(R\varphi_1)dx=0,\\ \Rightarrow 0.7684688891\ a_1+2.957900954\ a_2\\ +60.9761205\ a_3\\ -0.9049957145+0.8333333333\ a_0=0\end{cases} \tag{30}$$

$$\int_0^1(R\varphi_2)dx=0,$$

$$\Rightarrow\begin{pmatrix}-0.3597174606a_1+18.94535806a_3\\ +0.8862457145a_2-0.1297217461\end{pmatrix}=0, \tag{31}$$

$$\int_0^1(R\varphi_3)dx=0,$$

$$\Rightarrow$$

$$\begin{pmatrix}0.28657248\ 69a_1+2.177646\ 906a_3\\ -0.9536542\ 570a_2+0.1565613\ 278=0\end{pmatrix}, \tag{32}$$

Solving Eqs. (29) to (32) we have

$$a_0 = 0.3333333297$$
$$a_1 = 0.4999999998$$
$$a_2 = 0.1666666667$$
$$a_3 = 2\times10^{(-12)} \cong 0.$$

The following solution solves the original Eq. (25):

$$\left.\begin{array}{l}u(x)=x\left[a_0\varphi_0+a_1\varphi_1+a_2\varphi_2+a_3\varphi_3\right]\\ \quad =x\begin{bmatrix}0.3333333297\\ +0.4999999998(2x-1)\\ +0.1666666667(6x^2-6x+1)\end{bmatrix}\\ \quad =(-3.4\times10^{-9})x+(-6\times10^{-10})x^2+x^3\\ \quad \cong x^3\end{array}\right\}.$$

One may see that the exact solution of Eq. (25) agrees by taking four terms. The numerical results of the present solution and the solution solved by Legendre Pseudospectral method are given in table 2. The behavior of the exact and present solutions of this example has been represented in figure 2.

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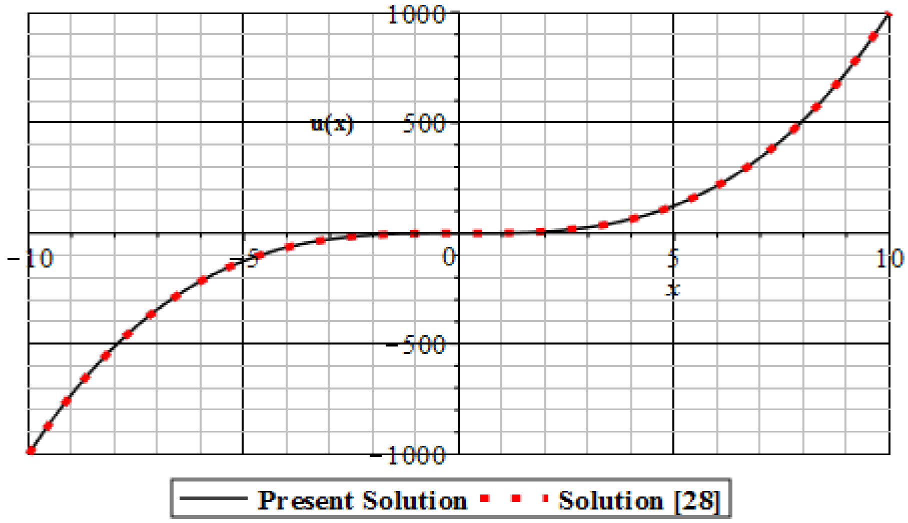

Figure 2: Graphic Of Exact And Present Solutions Of Example 2.

Table 2: Comparison Of The Present Solution With Ref. [28].

x	Present solution			Ref.[28]
	$n=1$	$n=2$	$n=3$	
0.0	0.0	0.0	0.0	0.0
0.1	1812.00292×10^{-3}	9.99999×10^{-4}	1.00000×10^{-3}	1.00000×10^{-3}
0.2	3595.02821×10^{-3}	7.99999×10^{-3}	8.00000×10^{-3}	8.00000×10^{-3}
0.3	5349.07587×10^{-3}	$26.99999*10^{-3}$	$27.00000*10^{-3}$	$27.00000*10^{-3}$
0.4	7074.14590×10^{-3}	$63.99999*10^{-3}$	$64.00000*10^{-3}$	$64.00000*10^{-3}$
0.5	$8770.23829*10^{-3}$	$124.99999*10^{-3}$	$125.00001*10^{-3}$	$125.00000*10^{-3}$
0.6	$1043.73530*10^{-2}$	$215.99999*10^{-3}$	$216.00001*10^{-3}$	$216.00000*10^{-3}$
0.7	$1207.54901*10^{-2}$	$342.99999*10^{-3}$	$343.00001*10^{-3}$	$343.00000*10^{-3}$
0.8	$1368.46496*10^{-2}$	$511.99999*10^{-3}$	$512.00001*10^{-3}$	$512.00000*10^{-3}$

Example 3. In this case, we consider the nonlinear fractional order delay differential equation [28]

$$\begin{cases} D^{1.5}u(x) = g(x) + u(x-0.5) + u^3(x), \\ g(x) = \dfrac{2}{\Gamma(1.5)}x^{0.5} - (x-0.5)^2 - x^6, \\ u(0)=0,\ u(1)=1. \end{cases} \quad (33)$$

Now taking two terms guess solution, we have

$$u(x) = h(x)\left[a_0\varphi_0 + a_1\varphi_1\right] = h(x)\left(\sum_{i=0}^{1} a_i\varphi_i(x)\right)$$

where $h(x) = x$ which will control the boundary conditions.

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$$\left.\begin{aligned} R = D^{1.5}&\left(h(x)\left(\sum_{i=0}^{1} a_i\varphi_i(x)\right)\right) \\ &-\left(h(x-0.5)\left(\sum_{i=0}^{1} a_i\varphi_i(x-0.5)\right)\right) \\ &-\left(h(x)\left(\sum_{i=0}^{1} a_i\varphi_i(x)\right)\right)^3 - f(x) = 0, \end{aligned}\right\} \quad (34)$$

The suitable collocation points which are the roots of the shifted Legendre polynomials $\phi_2(x)$ and their values may be taken as [28]

$$\left\{x_0 \frac{3-\sqrt{3}}{6} \text{ and } x_1 = \frac{3+\sqrt{3}}{6}\right\}. \quad (35)$$

As such x_0 in Eq. (34) gives

$$\begin{cases} 1.6195256 18a_1 + 0.2886751347a_0 \\ -(0.2113248653a_0 - 0.1220084679a_1)^3 \\ -0.9540113113 = 0. \end{cases} \quad (36)$$

Also at the point x_1, Eq. (34) becomes,

$$\begin{cases} 4.130344 52a_1 - 0.2886751347a_0 \\ -(0.7886751347a_0 + 0.4553418012a_1)^3 \\ -1.680183016 = 0 \end{cases}. \quad (37)$$

Solving the system of nonlinear Eqs. (36) and (37) we have

$$\left.\begin{aligned} a_0 &= 0.4999999992, \\ a_1 &= 0.4999999997 \end{aligned}\right\}$$

So

$$\left.\begin{aligned} u(x) &= x\begin{bmatrix} 0.4999999992 \\ +0.4999999997(2x-1) \end{bmatrix} \\ &= \left(-5\times10^{-10}\right)x + 0.9999999994x^2 \\ &\cong x^2 \end{aligned}\right\}$$

One may see that the exact solution of Eq. (33) agrees well by taking two terms only. The numerical results of the present solution and the solution solved by Legendre pseudospectral method [28] are given in table 3. The behavior of the exact and present solutions of this example has been presented in figure 3.

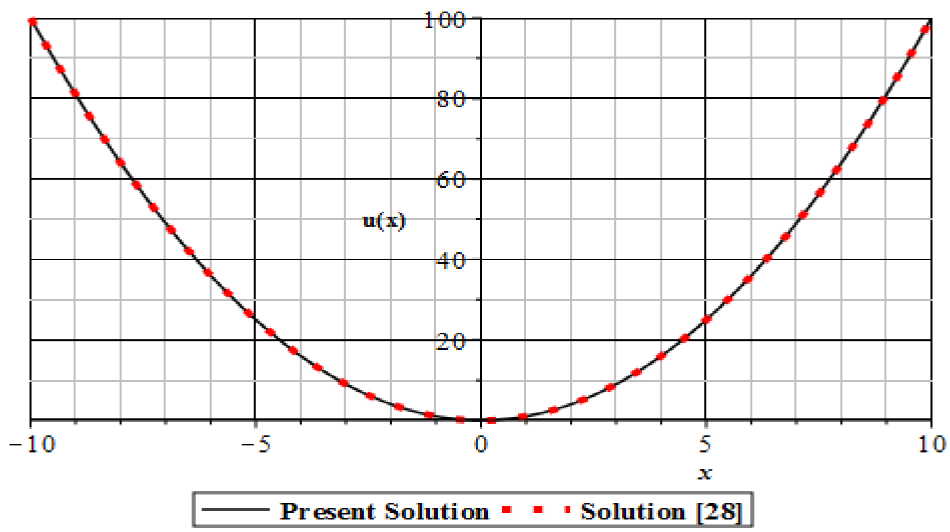

Figure 3: Graphic Of Exact And Present Solutions Of Example 3.

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Table 3: Comparison Of The Present Solution With Ref. [28].

x	Present solution $n=1$	Solution [28]
0.0	0.000	0.00
0.1	0.0099	0.01
0.2	0.0399	0.04
0.3	0.0899	0.09
0.4	0.1599	0.16
0.5	0.2499	0.25
0.6	0.3599	0.36
0.7	0.4899	0.49
0.8	0.6399	0.64

Example 4. Here, the following form of linear FODDE is considered [28]:

$$\begin{cases} D^{1.5}u(x) = \begin{pmatrix} -u(x) + u\left(\dfrac{x}{2}\right) \\ +\dfrac{7}{8}x^3 + \dfrac{6}{\Gamma(2.5)}x^{1.5} \end{pmatrix}, \\ u(0)=0,\ u(1)=1. \end{cases} \tag{38}$$

Taking four terms guess solution, we have

$$\left. \begin{aligned} u(x) &= h(x)\left[a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3\right] \\ &= h(x)\left(\sum_{i=0}^{3} a_i\varphi_i(x)\right) \end{aligned} \right\}.$$

The Residual

$$\begin{cases} R = D^{1.5}\left(h(x)\left(\displaystyle\sum_{i=0}^{3} a_i\varphi_i(x)\right)\right) \\ \quad + h(x)\left(\displaystyle\sum_{i=0}^{3} a_i\varphi_i(x)\right) \\ \quad - h\left(\dfrac{x}{2}\right)\left(\displaystyle\sum_{i=0}^{3} a_i\varphi_i\left(\dfrac{x}{2}\right)\right) \\ \quad -\dfrac{7}{8}x^3 - \dfrac{6}{\Gamma(2.5)}x^{1.5} = 0 \end{cases} \tag{39}$$

where $h(x) = x$ will control the boundary conditions.

Here:

$$\begin{cases} \varphi_0 = 1, \\ \varphi_1 = 2x - 1, \\ \varphi_3 = 6x^2 - 6x + 1, \\ \varphi_3 = 20x^3 - 30x^2 + 12x - 1. \end{cases}$$

For collocation point, we use the roots of the shifted Legendre polynomials $\phi_5(x)$ and $\phi_2(x)$.

Accordingly putting:

$$x_0 = \frac{3-\sqrt{3}}{6}$$

in Eq. (39), we have

$$\begin{pmatrix} 0 = 2.036192284a_1 - 3.639528963a_2 \\ \quad + 9.80522842a_3 + 0.1056624327a_0 \\ \quad - 0.4467287920. \end{pmatrix} \tag{40}$$

Similarly substituting:

$$x_1 = \frac{3+\sqrt{3}}{6},\ x_2 = 0.5,\ \text{and}\ x_3 = 0.7692$$

in Eq. (39) we have the following respectively:

$$\begin{pmatrix} 0 = 4.547011186a_1 + 7.11339489a_2 \\ \quad + 209.2796436a_3 + 0.3943375673a_0 \\ \quad - 3.590517262, \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} 3.316538243a_1 - 0.218749996a_2 \\ +60.92879397a_3 + 0.25a_0 \\ -1.705144122 = 0, \end{pmatrix} \tag{42}$$

and

$$\begin{pmatrix} 4.461439776a_1 + 6.50525526a_2 \\ +195.4525972a_3 + 0.38460000a_0 \\ -3.443129097 = 0. \end{pmatrix} \tag{43}$$

Solving Eqs. (40) to (43) we get

$$\begin{cases} a_0 = 0.3333314797 \\ a_1 = 0.5000001672 \\ a_2 = 0.1666667028 \\ a_3 = -1.371310\times 10^{-9} \cong 0 \end{cases}$$

As such, the solution to Eq. (38) may be written as

$$\left.\begin{aligned} u(x) &= x\begin{bmatrix} -0.0000019847+\left(1.17\times10^{-7}\right)x \\ +\left(1.000000217\right)x^2 \end{bmatrix} \\ &\cong 1.000000217x^3. \end{aligned}\right\}$$

One may see again that the exact solution of Eq. (38) agrees fully by taking four terms. The numerical results of the present and exact solution solved by Legendre Pseudospectral method [28] are given in table 4. The behavior of the exact and present solutions of this example has also been represented in figure 4.

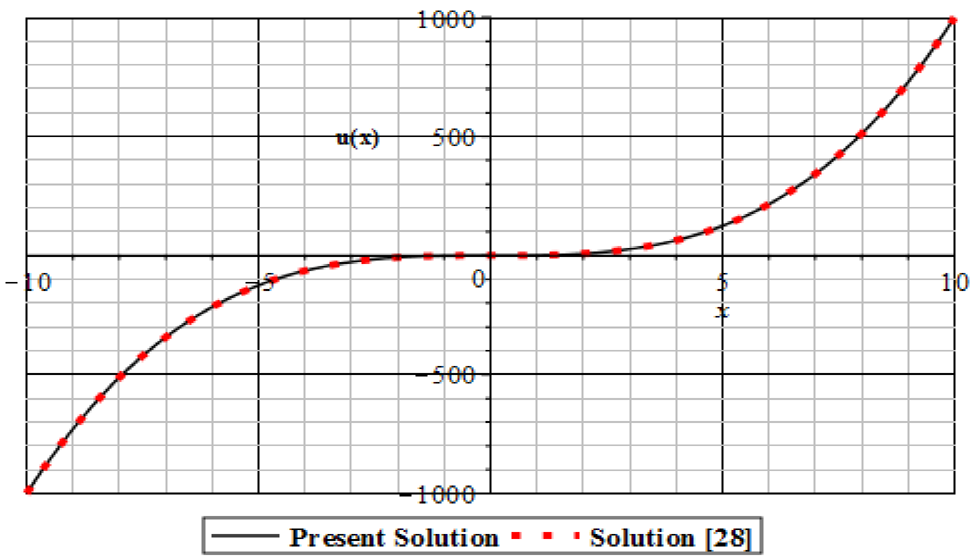

Figure 4: Graphic Of Exact And Present Solutions Of Example 4.

Table 4: Comparison Of The Present Solution With The Solution Of Ref. [28].

x	Present solution			Ref.[28]
	$n=1$	$n=2$	$n=3$	
0.0	0.0	0.0	0.0	0.0
0.1	1687.53452×10^{-3}	9.99999×10^{-4}	1.00000×10^{-3}	1.00000×10^{-3}
0.2	3349.86327×10^{-3}	7.99999×10^{-3}	8.00000×10^{-3}	8.00000×10^{-3}
0.3	4986.98624×10^{-3}	26.99999×10^{-3}	27.00000×10^{-3}	27.00000×10^{-3}
0.4	6598.90343×10^{-3}	63.99999×10^{-3}	64.00001×10^{-3}	64.00000×10^{-3}
0.5	8185.61485×10^{-3}	124.99999×10^{-3}	125.00000×10^{-3}	125.00000×10^{-3}
0.6	9747.12049×10^{-3}	215.99999×10^{-3}	216.00004×10^{-3}	216.00000×10^{-3}
0.7	1128.34203×10^{-2}	342.99999×10^{-3}	343.00007×10^{-3}	343.00000×10^{-3}
0.8	1279.45144×10^{-2}	511.99999×10^{-3}	512.00001×10^{-3}	512.00000×10^{-3}

7.0 CONCLUSIONS

In this present paper, two proposed methods have been successfully applied to the fractional delay differential equations (FDDEs). It is remarked to the best of our knowledge, that Legendre Galerkin and Legendre Collocation methods have not been used for solving the FDDEs. Furthermore, these well-known methods appear simpler in application for handling the titled problems by the use of the shifted Legendre polynomials. From the above results, one may draw the following conclusions:

(i) The present solutions are in excellent agreement with the exact solutions.
(ii) The accuracy of present methods may be improved by taking more terms of shifted Legendre polynomials in different other problems.
(iii) These methods are used in linear and nonlinear fractional differential equations, and the solutions are validated.

CONFLICT OF INTERESTS

No conflict of interest is declared by the authors.

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