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MODULAR IRREDUCIBLE REPRESENTATIONS OF THE F_pW_4-SUBMODULES $N_{F_p}(\lambda,\mu)$ OF THE SPECHT MODULES $S_{F_p}(\lambda,\mu)$ AS LINEAR CODES WHERE W_4 IS THE WEYL GROUP OF TYPE B_4

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ABSTRACT

The modular representations of the F_pW_n-Specht modules $S_K(\lambda,\mu)$ as linear codes is given in our paper [5]. In this paper we are concerning of finding the linear codes of the representations of the irreducible F_pW_4-submodules $N_{F_p}(\lambda,\mu)$ of the F_pW_4-Specht modules $S_{F_p}(\lambda,\mu)$ for each pair of partitions (λ,μ) of a positive integer $n = 4$, where $F_p = GF(p)$ is the Galois field (finite field) of order p, and p is a prime number greater than or equal to 3. We will find in this paper a generator matrix of a subspace $Y_{(p)}^{((2,1),(1))}$ representing the irreducible F_pW_4-submodules $N_{F_p}((2,1),(1))$ of the F_pW_4-Specht modules $S_{F_p}((2,1),(1))$ and give the linear code of $Y_{(p)}^{((2,1),(1))}$ for each prime number p greater than or equal to 3. Then we will give the linear codes of all the subspaces $Y_{(p)}^{(\lambda,\mu)}$ for all pair of partitions (λ,μ) of a positive integer $n = 4$, and for each prime number p greater than or equal to 3.

We mention that some of the ideas of this work in this paper have been influenced by that of Adalbert Kerber and Axel Kohnert [12], even though that their paper is about the symmetric group and this paper is about the Weyl groups of type B_n.

Keywords: Field of characteristic 0 (infinite field), Finite field $F_p = GF(p)$, Weyl group W_n of type B_n, group ring F_pW_n, F_pW_n-module, F_pW_n-submodule, pair of partitions (λ,μ) of a positive integer n, Specht polynomial, Specht module, (λ,μ)-tableau, standard (λ,μ)-tableau, vector space, subspace, generating matrix, linear code.

Remarks: Throughout this paper, let:

i- F_p be the Galois field (finite field) of order p ([7], p.429), that is $F_p = GF(p)$.

ii- K be a field which is infinite (of characteristic 0) or finite of order a prime number $p \geq 3$, and $x_1, x_2, \ldots, x_n$ be independent indeterminates over K.

iii- W_n be the Weyl group of type B_n, which is the group of all permutations w of $\{x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n\}$, such that $w(-x_i) = -w(x_i)$, for each $i = 1, 2, \ldots, n$.

iv- KW_n be the group ring of W_n with coefficients in K. KW_n is also a group algebra of W_n over K.

1. INTRODUCTION

There are many types of Weyl groups which are:

(1) The infinite family of Weyl groups of type A_n, namely symmetric groups, (2) The infinite family of Weyl groups of type B_n, namely hyperoctahedral groups, (3) The infinite family of Weyl groups of type C_n, (4) The infinite family of Weyl groups of type D_n, (5) The Weyl groups of type G_2, (6) The Weyl groups of type F_4, (7) The Weyl groups of types E_6, E_7, and E_8 (see [8], p.40; [11], p.134; and [14], p.36). In this paper we are concern with the Weyl groups of type B_n, and the connection of the representations of the Weyl groups W_n of type B_n with the linear codes and more precisely we are concern with the modular irreducible representations of the F_pW_n-submodules $N_{F_p}(\lambda,\mu)$ of the F_pW_n- Specht modules $S_{F_p}(\lambda,\mu)$ as linear codes when n = 4, and for each prime number p greater than or equal to 3.

2. PRELIMINARIES

Definition 2.1. Let $\{y_1,\dots,y_r\} \subseteq \{\pm x_1, \dots, \pm x_n\}$, such that $y_i \neq \pm y_j$ for each $i, j = 1,\dots,r$ and $i \neq j$, then we define

$$\Delta_1(y_1,\dots,y_r)=\begin{cases}\prod_{1\le i<j\le r}(y_j^2-y_i^2)\prod_{\ell=1}^{r}y_\ell & if\ r>1\\ y_1, & if\ r=1\end{cases}$$

$$\Delta_2(y_1,\dots,y_r)=\begin{cases}\prod_{1\le i<j\le r}(y_j^2-y_i^2) & if\ r>1\\ 1 & if\ r=1\end{cases}$$

([2], p.8 and [4], p.15).

Example 2.2. $\Delta_2(x_2,x_7,x_1)=(x_1^2-x_2^2)(x_1^2-x_7^2)(x_7^2-x_2^2)=x_1^4x_7^2-x_1^4x_2^2+x_1^2x_2^4-x_1^2x_7^4+x_2^2x_7^4-x_2^4x_7^2$, and $\Delta_1(x_2,x_7,x_1)=x_1x_2x_7\cdot(\Delta_2(x_2,x_7,x_1))= x_1^5x_2x_7^3-x_1^5x_2^3x_7+x_1^3x_2^5x_7-x_1^3x_2x_7^5+x_1x_2^3x_7^5-x_1x_2^5x_7^3$.

Definition 2.3. Let $(\lambda,\mu)=((\lambda_1,\lambda_2,\dots,\lambda_s),(\mu_1,\mu_2,\dots,\mu_t))$ be a pair of partitions of a positive integer n, and let $Z^{(\lambda,\mu)}$ be any (λ,μ)-tableau, then:

$$f(Z^{(\lambda,\mu)})=\begin{cases}f_1(Z^\lambda) & if\quad |\mu|=0\\ f_2(Z^\mu) & if\quad |\lambda|=0\\ f_1(Z^\lambda).\,f_2(Z^\mu) & \text{otherwise}\end{cases}$$

such that:

$$f_1(Z^\lambda)=\prod_{j=1}^{\lambda_1}\Delta_1(Z^{(\lambda,\mu)}(1,j,1),\dots,Z^{(\lambda,\mu)}(\lambda'_j,j,1))$$

where λ'_j is the number of the indeterminates in the j^{th} column of the first tableau Z^λ, and

$$f_2(Z^\mu)=\prod_{j=1}^{\mu_1}\Delta_2(Z^{(\lambda,\mu)}(1,j,2),\dots,Z^{(\lambda,\mu)}(\mu'_j,j,2))$$

where μ'_j is the number of the indeterminates in the j^{th} column of the second tableau Z^μ, $f(Z^{(\lambda,\mu)})$ is called the Specht polynomial of (λ,μ)-tableau $Z^{(\lambda,\mu)}$ ([2], p.9 and [4], p.15).

Example 2.4. Let $Z^{((2,1),(1))}$ be the following $((2,1),(1))$-tableau:

$$\begin{matrix}x_3 & x_1\\ x_4 & \end{matrix}\quad ;\quad -x_2\,.$$

$f(Z^{((2,1),(1))}) = \prod_{j=1}^{\lambda_1}\Delta_1(Z^{((2,1),(1))}(1,j,1), \dots, Z^{((2,1),(1))}(\lambda'_j,j,1))\prod_{j=1}^{\mu_1}\Delta_2(Z^{((2,1),(1))}(1,j,2), \dots, Z^{((2,1),(1))}(\mu'_j,j,2))$

$= \Delta_1(Z^{((2,1),(1))}(1,1,1),\ Z^{((2,1),(1))}(2,1,1))\cdot \Delta_1(Z^{((2,1),(1))}(1,2,1))\cdot\Delta_2(Z^{((2,1),(1))}(1,1,2))= \Delta_1(x_3,\ x_4)\cdot\Delta_1(x_1)\cdot\Delta_2(-x_2)= (x_4^2-x_3^2)x_3x_4\cdot x_1\cdot 1 = x_1x_3x_4^3-x_1x_3^3x_4$.

Definition 2.5. Let $(\lambda,\mu) = ((\lambda_1,\lambda_2,\ldots,\lambda_s),(\mu_1,\mu_2,\ldots,\mu_t))$ be a pair of partitions of a positive integer n, and let $Z^{(\lambda,\mu)}$ be any (λ,μ)-tableau. Then the cyclic KW_n-module $S_K(\lambda,\mu)$ generated over KW_n by $f(Z^{(\lambda,\mu)})$ (i.e., $S_K(\lambda,\mu) = KW_n f(Z^{(\lambda,\mu)})$) is called the Specht module over K corresponding to the pair of partitions (λ,μ) of n ([2], p.10 and [4], p.16).

Theorem 2.6. Let $(\lambda,\mu) = ((\lambda_1,\lambda_2,\ldots,\lambda_s),(\mu_1,\mu_2,\ldots,\mu_t))$ be a pair of partitions of a positive integer n. Then there are exactly $\dfrac{n!}{H_{\lambda,\mu}}$ distinct (λ,μ)-standard tableaux where $H_{\lambda,\mu} = H_\lambda . H_\mu$, such that $H_\lambda = \prod_{i=1}^{s}\prod_{j=1}^{\lambda_i} h_{ij}$, where $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, and $H_\mu = \prod_{i=1}^{t}\prod_{j=1}^{\mu_i} e_{ij}$, where $e_{ij} = \mu_i + \mu'_j - i - j + 1$ ([2], p.20 & p.21 and [4], p.13).

Theorem 2.7. Let $(\lambda,\mu) = ((\lambda_1,\lambda_2,\ldots,\lambda_s),(\mu_1,\mu_2,\ldots,\mu_t))$ be a pair of partitions of a positive integer n. Then the Specht module $S_K(\lambda,\mu)$ has a K-basis $B(\lambda,\mu) = \{ f(Z^{(\lambda,\mu)}) \mid Z^{(\lambda,\mu)}$ is a standard (λ,μ)-tableau$\}$, and $\dim_K S_K(\lambda,\mu) = \dfrac{n!}{H_{\lambda,\mu}}$ ([2], p.21, [3], p.305 and [4], p.17).

Theorem 2.8. ([1], p.68 & p.87)

Let $((m-2,2),(n-m))$ be a pair of partitions of a positive integer n, where $4 \le m \le n$.

1. If p divides neither $(m-1)$ nor $(m-2)$. Then $S_K((m-2,2),(n-m))$ is irreducible KW_n-module.
2. If p divides either $(m-1)$ or $(m-2)$. Then the KW_n-module $S_K((m-2,2),(n-m))$ has the following composition series: $0 \subset N_K((m-2,2),(n-m)) \subset S_K((m-2,2),(n-m))$.

Theorem 2.9. ([2], p.64)

Let K be a field of characteristic p not equal to 2.

1. If p does not divide m, then
$S_K((m-r+1,1^{r-1}),(n-m))$,
$S_K((m-r+1,1^{r-1}),(1^{n-m}))$,
$S_K((n-m),(m-r+1,1^{r-1}))$, and
$S_K((1^{n-m}),(m-r+1,1^{r-1}))$ are irreducible KW_n-modules where $0 < r \le m \le n$.

2. If p divides m, then we have the following
1. composition series:

$$0 \subset N_K((m-r+1,1^{r-1}),(n-m)) \subset S_K((m-r+1,1^{r-1}),(n-m)),$$

$$0 \subset N_K((m-r+1,1^{r-1}),(1^{n-m})) \subset S_K((m-r+1,1^{r-1}),(1^{n-m})),$$

$$0 \subset N_K((n-m),(m-r+1,1^{r-1})) \subset S_K((n-m),(m-r+1,1^{r-1})),$$

$$0 \subset N_K((1^{n-m}),(m-r+1,1^{r-1})) \subset S_K((1^{n-m}),(m-r+1,1^{r-1})).$$

2. (i.e., $S_K((m-r+1,1^{r-1}),(n-m))$,
$S_K((m-r+1,1^{r-1}),(1^{n-m}))$,
$S_K((n-m),(m-r+1,1^{r-1}))$, and
$S_K((1^{n-m}),(m-r+1,1^{r-1}))$
are reducible KW_n-modules and each one of them has only one proper irreducible KW_n-module), where $1 < r \le m-1 \le n$, and $S_K((m),(n-m))$, $S_K((m),(1^{n-m}))$, $S_K((1^m),(n-m))$, and $S_K((1^m),(1^{n-m}))$ are irreducible KW_n- modules.

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3. THE SPECHT POLYNOMIALS OF THE STANDARD ((2,1),(1))- TABLEAUX

$\dim_K S_K((2,1),(1)) = \dfrac{4!}{3\cdot 1\cdot 1\cdot 1} = 8$, and thus we have eight standard ((2,1),(1))-tableaux, which are:

$$Z_1^{((2,1,(1))} = \begin{array}{cc} x_1 & x_3 \\ x_2 & \end{array} ; \; x_4 ,$$

$$Z_2^{((2,1),(1))} = \begin{array}{cc} x_1 & x_2 \\ x_3 & \end{array} ; \; x_4 ,$$

$$Z_3^{((2,1),(1))} = \begin{array}{cc} x_1 & x_4 \\ x_2 & \end{array} ; \; x_3 ,$$

$$Z_4^{((2,1),(1))} = \begin{array}{cc} x_1 & x_2 \\ x_4 & \end{array} ; \; x_3 ,$$

$$Z_5^{((2,1),(1))} = \begin{array}{cc} x_1 & x_4 \\ x_3 & \end{array} ; \; x_2 ,$$

$$Z_6^{((2,1),(1))} = \begin{array}{cc} x_1 & x_3 \\ x_4 & \end{array} ; \; x_2 ,$$

$$Z_7^{((2,1),(1))} = \begin{array}{cc} x_2 & x_4 \\ x_3 & \end{array} ; \; x_1 ,$$

$$Z_8^{((2,1),(1))} = \begin{array}{cc} x_2 & x_3 \\ x_4 & \end{array} ; \; x_1 .$$

The corresponding Specht polynomials are:

$$f(Z_1^{((2,1),(1))}) = (x_2^2 - x_1^2)\, x_1x_2x_3 = x_1x_2^3x_3 - x_1^3x_2x_3 ,$$

$$f(Z_2^{((2,1),(1))}) = (x_3^2 - x_1^2)\, x_1x_2x_3 = x_1x_2x_3^3 - x_1^3x_2x_3 ,$$

$$f(Z_3^{((2,1),(1))}) = (x_2^2 - x_1^2)\, x_1x_2x_4 = x_1x_2^3x_4 - x_1^3x_2x_4 ,$$

$$f(Z_4^{((2,1),(1))}) = (x_4^2 - x_1^2)\, x_1x_2x_4 = x_1x_2x_4^3 - x_1^3x_2x_4 ,$$

$$f(Z_5^{((2,1),(1))}) = (x_3^2 - x_1^2)\, x_1x_3x_4 = x_1x_3^3x_4 - x_1^3x_3x_4 ,$$

$$f(Z_6^{((2,1),(1))}) = (x_4^2 - x_1^2)\, x_1x_3x_4 = x_1x_3x_4^3 - x_1^3x_3x_4 ,$$

$$f(Z_7^{((2,1),(1))}) = (x_3^2 - x_2^2)\, x_2x_3x_4 = x_2x_3^3x_4 - x_2^3x_3x_4 ,$$

$$f(Z_8^{((2,1),(1))}) = (x_4^2 - x_2^2)\, x_2x_3x_4 = x_2x_3x_4^3 - x_2^3x_3x_4 .$$

The above polynomials $f(Z_1^{((2,1),(1))})$, $f(Z_2^{((2,1),(1))}), \ldots, f(Z_8^{((2,1),(1))})$ mod 3 will be:

$$f_{(3)}(Z_1^{((2,1),(1))}) = x_1x_2^3x_3 + 2x_1^3x_2x_3 ,$$

$$f_{(3)}(Z_2^{((2,1),(1))}) = x_1x_2x_3^3 + 2x_1^3x_2x_3 ,$$

$$f_{(3)}(Z_3^{((2,1),(1))}) = x_1x_2^3x_4 + 2x_1^3x_2x_4 ,$$

$$f_{(3)}(Z_4^{((2,1),(1))}) = x_1x_2x_4^3 + 2x_1^3x_2x_4 ,$$

$$f_{(3)}(Z_5^{((2,1),(1))}) = x_1x_3^3x_4 + 2x_1^3x_3x_4 ,$$

$$f_{(3)}(Z_6^{((2,1),(1))}) = x_1x_3x_4^3 + 2x_1^3x_3x_4 ,$$

$$f_{(3)}(Z_7^{((2,1),(1))}) = x_2x_3^3x_4 + 2x_2^3x_3x_4 ,$$

$$f_{(3)}(Z_8^{((2,1),(1))}) = x_2x_3x_4^3 + 2x_2^3x_3x_4 .$$

The above polynomials $f(Z_1^{((2,1),(1))})$, $f(Z_2^{((2,1),(1))}), \ldots, f(Z_8^{((2,1),(1))})$ mod 5 will be:

$$f_{(5)}(Z_1^{((2,1),(1))}) = x_1x_2^3x_3 + 4x_1^3x_2x_3 ,$$

$$f_{(5)}(Z_2^{((2,1),(1))}) = x_1x_2x_3^3 + 4x_1^3x_2x_3 ,$$

$$f_{(5)}(Z_3^{((2,1),(1))}) = x_1x_2^3x_4 + 4x_1^3x_2x_4 ,$$

$$f_{(5)}(Z_4^{((2,1),(1))}) = x_1x_2x_4^3 + 4x_1^3x_2x_4 ,$$

$$f_{(5)}(Z_5^{((2,1),(1))}) = x_1x_3^3x_4 + 4x_1^3x_3x_4 ,$$

$$f_{(5)}(Z_6^{((2,1),(1))}) = x_1x_3x_4^3 + 4x_1^3x_3x_4 ,$$

$f_{(5)}(Z_7^{((2,1),(1))}) = x_2 x_3^3 x_4 + 4x_2^3 x_3 x_4$,

$f_{(5)}(Z_8^{((2,1),(1))}) = x_2 x_3 x_4^3 + 4x_2^3 x_3 x_4$.

4. THE SYMMETRIZED SPECHT POLYNOMIALS OF THE STANDARD ((2,1),(1))- TABLEAUX

Definition 4.1. Let $Z_\ell^{(\lambda,\mu)}$ be any (λ,μ)-tableau. Then $\boldsymbol{\underline{R}(Z_\ell^{(\lambda,\mu)})}$ will be defined as the set of all permutations w belong to the Weyl group W_n of type B_n, which permute the variables in each row of Z_ℓ^{λ} and in each row of Z_ℓ^{μ} without changing the sign of any variable in $Z_\ell^{(\lambda,\mu)}$, i.e., $\underline{R}(Z_\ell^{(\lambda,\mu)}) = \{ w \in W_n \mid w Z_\ell^{(\lambda,\mu)}(i, j_1, 1) = Z_\ell^{(\lambda,\mu)}(i, j_2, 1),\ i = 1, \ldots, s$ and $1 \le j_1, j_2 \le \lambda_i$; and $w Z_\ell^{(\lambda,\mu)}(i, j_1, 2) = Z_\ell^{(\lambda,\mu)}(i, j_2, 2),\ i = 1, \ldots, t$ and $1 \le j_1, j_2 \le \mu_i \}$.

Definition 4.2. Let $Z_\ell^{(\lambda,\mu)}$ be any (λ,μ)-tableau, then the **symmetrized Specht polynomial** $\boldsymbol{f\left[Z_\ell^{(\lambda,\mu)}\right]}$ will be defined by $f\left[Z_\ell^{(\lambda,\mu)}\right] = \sum_{w \in \underline{R}(Z_\ell^{(\lambda,\mu)})} f\left(w Z_\ell^{(\lambda,\mu)}\right)$.

If we take the coefficients of the polynomial $f\left[Z_\ell^{(\lambda,\mu)}\right]$ modulo a prime number p, then $f\left[Z_\ell^{(\lambda,\mu)}\right]$ will be denoted by $\boldsymbol{f_{(p)}\left[Z_\ell^{(\lambda,\mu)}\right]}$, which will be called the **p - reduced symmetrized Specht polynomial** of the (λ,μ)-tableau $Z_\ell^{(\lambda,\mu)}$.

Remark 4.3. The F_pW_n-module generated by any p-reduced symmetrized Specht polynomial $f_{(p)}\left[Z_\ell^{(\lambda,\mu)}\right]$ will be denoted by $N_{F_p}(\lambda,\mu)$, i.e., $N_{F_p}(\lambda,\mu) = F_p W_n f_{(p)}\left[Z_\ell^{(\lambda,\mu)}\right]$. $N_{F_p}(\lambda,\mu)$ will be irreducible $F_p W_n$ - submodule of the Specht module $S_{F_p}(\lambda,\mu)$, where F_p is a field of order p.

The pair of partitions ((2,1),(1)) of 4 have the following symmetrized Specht polynomials of the standard ((2,1),(1))-tableaux:

$$f\left[Z_1^{((2,1),(1))}\right] = f\left(Z_1^{((2,1),(1))}\right) + f\left((x_1\ x_3) Z_1^{((2,1),(1))}\right)$$

$$= \left(x_1 x_2^3 x_3 - x_1^3 x_2 x_3\right) + (x_1\ x_3)\left(x_1 x_2^3 x_3 - x_1^3 x_2 x_3\right)$$

(since $f\left(w\, Z_\ell^{(\lambda,\mu)}\right) = w\, f\left(Z_\ell^{(\lambda,\mu)}\right)\ \forall Z_\ell^{(\lambda,\mu)} \in T^{(\lambda,\mu)}$ and $\forall$ w $\in W_n$ by [4], Remark 1.6.7, p.16)

$$= x_1 x_2^3 x_3 - x_1^3 x_2 x_3 + x_1 x_2^3 x_3 - x_1 x_2 x_3^3$$
$$= 2x_1 x_2^3 x_3 - x_1^3 x_2 x_3 - x_1 x_2 x_3^3,$$

$$f\left[Z_2^{((2,1),(1))}\right] = f\left(Z_2^{((2,1),(1))}\right) + f\left((x_1\ x_2) Z_2^{((2,1),(1))}\right)$$

$$= \left(x_1 x_2 x_3^3 - x_1^3 x_2 x_3\right) + (x_1\ x_2)\left(x_1 x_2 x_3^3 - x_1^3 x_2 x_3\right)$$
$$= x_1 x_2 x_3^3 - x_1^3 x_2 x_3 + x_1 x_2 x_3^3 - x_1 x_2^3 x_3$$
$$= 2x_1 x_2 x_3^3 - x_1^3 x_2 x_3 - x_1 x_2^3 x_3,$$

$$f\left[Z_3^{((2,1),(1))}\right] = f\left(Z_3^{((2,1),(1))}\right) + f\left((x_1\ x_4) Z_3^{((2,2),(1))}\right)$$

$$= \left(x_1 x_2^3 x_4 - x_1^3 x_2 x_4\right) + (x_1\ x_4)\left(x_1 x_2^3 x_4 - x_1^3 x_2 x_4\right)$$
$$= x_1 x_2^3 x_4 - x_1^3 x_2 x_4 + x_1 x_2^3 x_4 - x_1 x_2 x_4^3$$
$$= 2x_1 x_2^3 x_4 - x_1^3 x_2 x_4 - x_1 x_2 x_4^3,$$

$$f\left[Z_4^{((2,1),(1))}\right] = f\left(Z_4^{((2,1),(1))}\right) + f\left((x_1\ x_2) Z_4^{((2,1),(1))}\right)$$

$$= \left(x_1 x_2 x_4^3 - x_1^3 x_2 x_4\right) + (x_1\ x_2)\left(x_1 x_2 x_4^3 - x_1^3 x_2 x_4\right)$$

$= x_1x_2x_4^3 - x_1^3x_2x_4 + x_1x_2x_4^3 - x_1x_2^3x_4$

$= 2x_1x_2x_4^3 - x_1^3x_2x_4 - x_1x_2^3x_4,$

$f\left[Z_5^{((2,1),(1))}\right] = f\left(Z_5^{((2,1),(1))}\right) + f\left((x_1\ x_4)Z_5^{((2,1),(1))}\right)$

$= \left(x_1x_3^3x_4 - x_1^3x_3x_4\right) + (x_1\ x_4)\left(x_1x_3^3x_4 - x_1^3x_3x_4\right)$

$= x_1x_3^3x_4 - x_1^3x_3x_4 + x_1x_3^3x_4 - x_1x_3x_4^3$

$= 2x_1x_3^3x_4 - x_1^3x_3x_4 - x_1x_3x_4^3,$

$f\left[Z_6^{((2,1),(1))}\right] = f\left(Z_6^{((2,1),(1))}\right) + f\left((x_1\ x_3)Z_6^{((2,1),(1))}\right)$

$= \left(x_1x_3x_4^3 - x_1^3x_3x_4\right) + (x_1\ x_3)\left(x_1x_3x_4^3 - x_1^3x_3x_4\right)$

$= x_1x_3x_4^3 - x_1^3x_3x_4 + x_1x_3x_4^3 - x_1x_3^3x_4$

$= 2x_1x_3x_4^3 - x_1^3x_3x_4 - x_1x_3^3x_4,$

$f\left[Z_7^{((2,1),(1))}\right] = f\left(Z_7^{((2,1),(1))}\right) + f\left((x_2\ x_4)Z_7^{((2,1),(1))}\right)$

$= \left(x_2x_3^3x_4 - x_2^3x_3x_4\right) + (x_2\ x_4)\left(x_2x_3^3x_4 - x_2^3x_3x_4\right)$

$= x_2x_3^3x_4 - x_2^3x_3x_4 + x_2x_3^3x_4 - x_2x_3x_4^3$

$= 2x_2x_3^3x_4 - x_2^3x_3x_4 - x_2x_3x_4^3,$

$f\left[Z_8^{((2,1),(1))}\right] = f\left(Z_8^{((2,1),(1))}\right) + f\left((x_2\ x_3)Z_8^{((2,1),(1))}\right)$

$= \left(x_2x_3x_4^3 - x_2^3x_3x_4\right) + (x_2\ x_3)\left(x_2x_3x_4^3 - x_2^3x_3x_4\right)$

$= x_2x_3x_4^3 - x_2^3x_3x_4 + x_2x_3x_4^3 - x_2x_3^3x_4$

$= 2x_2x_3x_4^3 - x_2^3x_3x_4 - x_2x_3^3x_4.$

Then $f\left[Z_1^{((2,1),(1))}\right]$, $f\left[Z_2^{((2,1),(1))}\right]$, … , $f\left[Z_8^{((2,1),(1))}\right]$ mod 3 will be:

$f_{(3)}\left[Z_1^{((2,1),(1))}\right] = 2x_1x_2^3x_3 + 2x_1^3x_2x_3 + 2x_1x_2x_3^3,$

$f_{(3)}\left[Z_2^{((2,1),(1))}\right] = 2x_1x_2x_3^3 + 2x_1^3x_2x_3 + 2x_1x_2^3x_3,$

$f_{(3)}\left[Z_3^{((2,1),(1))}\right] = 2x_1x_2^3x_4 + 2x_1^3x_2x_4 + 2x_1x_2x_4^3,$

$f_{(3)}\left[Z_4^{((2,1),(1))}\right] = 2x_1x_2x_4^3 + 2x_1^3x_2x_4 + 2x_1x_2^3x_4,$

$f_{(3)}\left[Z_5^{((2,1),(1))}\right] = 2x_1x_3^3x_4 + 2x_1^3x_3x_4 + 2x_1x_3x_4^3,$

$f_{(3)}\left[Z_6^{((2,1),(1))}\right] = 2x_1x_3x_4^3 + 2x_1^3x_3x_4 + 2x_1x_3^3x_4,$

$f_{(3)}\left[Z_7^{((2,1),(1))}\right] = 2x_2x_3^3x_4 + 2x_2^3x_3x_4 + 2x_2x_3x_4^3,$

$f_{(3)}\left[Z_8^{((2,1),(1))}\right] = 2x_2x_3x_4^3 + 2x_2^3x_3x_4 + 2x_2x_3^3x_4.$

Let $b_1 = f_{(3)}\left[Z_1^{((2,1),(1))}\right] = f_{(3)}\left[Z_2^{((2,1),(1))}\right],$

$b_2 = f_{(3)}\left[Z_3^{((2,1),(1))}\right] = f_{(3)}\left[Z_4^{((2,1),(1))}\right],$

$b_3 = f_{(3)}\left[Z_5^{((2,1),(1))}\right] = f_{(3)}\left[Z_6^{((2,1),(1))}\right],$

$b_4 = f_{(3)}\left[Z_7^{((2,1),(1))}\right] = f_{(3)}\left[Z_8^{((2,1),(1))}\right],$

Then $B_{(3)}^{((2,1),(1))} = \{b_1, b_2, b_3, b_4\}$ is a basis of the submodule $N_{F_3}((2,1),(1)) = F_3W_4\ f_{(3)}\left[Z_1^{((2,1),(1))}\right]$ of the Specht module $S_{F_3}((2,1),(1)) = F_3W_4\ f_{(3)}\left(Z_1^{((2,1),(1))}\right)$.

$f\left[Z_1^{((2,1),(1))}\right]$, … , $f\left[Z_8^{((2,1),(1))}\right]$ mod 5 will be:

$f_{(5)}\left[Z_1^{((2,1),(1))}\right] = 2x_1x_2^3x_3 + 4x_1^3x_2x_3 + 4x_1x_2x_3^3,$

$f_{(5)}\left[Z_2^{((2,1),(1))}\right] = 2x_1x_2x_3^3 + 4x_1^3x_2x_3 + 4x_1x_2^3x_3,$

$f_{(5)}\left[Z_3^{((2,1),(1))}\right] = 2x_1x_2^3x_4 + 4x_1^3x_2x_4 + 4x_1x_2x_4^3,$

$f_{(5)}\left[Z_4^{((2,1),(1))}\right] = 2x_1x_2x_4^3 + 4x_1^3x_2x_4 + 4x_1x_2^3x_4,$

$f_{(5)}\left[Z_5^{((2,1),(1))}\right] = 2x_1x_3^3x_4 + 4x_1^3x_3x_4 + 4x_1x_3x_4^3,$

$f_{(5)}\left[Z_6^{((2,1),(1))}\right] = 2x_1x_3x_4^3 + 4x_1^3x_3x_4 + 4x_1x_3^3x_4,$

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$$f_{(5)}\left[Z_7^{((2,1),(1))}\right]=2x_2x_3^3x_4+4x_2^3x_3x_4+4x_2x_3x_4^3,$$

$$f_{(5)}\left[Z_8^{((2,1),(1))}\right]=2x_2x_3x_4^3+4x_2^3x_3x_4+4x_2x_3^3x_4.$$

Let $b_i = f_{(5)}\left[Z_i^{((2,1),(1))}\right],\ i = 1, \dots, 8,$ then $B_{(5)}^{((2,1),(1))} = \{b_1, b_2, \dots, b_8\}$ is a basis of the submodule $N_{F_5}((2,1),(1)) = F_5W_4\, f_{(5)}\left[Z_1^{((2,1),(1))}\right]$ of the Specht module $S_{F_5}((2,1),(1)) = F_5W_4\, f_{(5)}\left(Z_1^{((2,1),(1))}\right)$.

5. THE SUBSPACE $Y_{(3)}^{((2,1),(1))}$ AS A LINEAR CODE

The symmetrized Specht polynomials $f\left[Z_1^{((2,1),(1))}\right], \dots, f\left[Z_8^{((2,1),(1))}\right]$ will be written in terms of $f\left(Z_1^{((2,1),(1))}\right), \dots, f\left(Z_8^{((2,1),(1))}\right)$ as follows:

$$\begin{aligned}
f\left[Z_1^{((2,1),(1))}\right] &= 2x_1x_2^3x_3-x_1^3x_2x_3-x_1x_2x_3^3\\
&= 2x_1x_2^3x_3-x_1^3x_2x_3-x_1^3x_2x_3\\
&\quad -x_1x_2x_3^3+x_1^3x_2x_3\\
&= 2f\left(Z_1^{((2,1),(1))}\right)-f\left(Z_2^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_2^{((2,1),(1))}\right] &= 2x_1x_2x_3^3-x_1^3x_2x_3-x_1x_2^3x_3\\
&= 2x_1x_2x_3^3-x_1^3x_2x_3-x_1^3x_2x_3\\
&\quad -x_1x_2^3x_3+x_1^3x_2x_3\\
&= -f\left(Z_1^{((2,1),(1))}\right)+2f\left(Z_2^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_3^{((2,1),(1))}\right] &= 2x_1x_2^3x_4-x_1^3x_2x_4-x_1x_2x_4^3\\
&= 2x_1x_2^3x_4-x_1^3x_2x_4-x_1^3x_2x_4\\
&\quad -x_1x_2x_4^3+x_1^3x_2x_4\\
&= 2f\left(Z_3^{((2,1),(1))}\right)-f\left(Z_4^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_4^{((2,1),(1))}\right] &= 2x_1x_2x_4^3-x_1^3x_2x_4-x_1x_2^3x_4\\
&= 2x_1x_2x_4^3-x_1^3x_2x_4-x_1^3x_2x_4\\
&\quad -x_1x_2^3x_4+x_1^3x_2x_4\\
&= -f\left(Z_3^{((2,1),(1))}\right)+2f\left(Z_4^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_5^{((2,1),(1))}\right] &= 2x_1x_3^3x_4-x_1^3x_3x_4-x_1x_3x_4^3\\
&= 2x_1x_3^3x_4-x_1^3x_3x_4-x_1^3x_3x_4\\
&\quad -x_1x_3x_4^3+x_1^3x_3x_4\\
&= 2f\left(Z_5^{((2,1),(1))}\right)-f\left(Z_6^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_6^{((2,1),(1))}\right] &= 2x_1x_3x_4^3-x_1^3x_3x_4-x_1x_3^3x_4\\
&= 2x_1x_3x_4^3-x_1^3x_3x_4-x_1^3x_3x_4\\
&\quad -x_1x_3^3x_4+x_1^3x_3x_4\\
&= -f\left(Z_5^{((2,1),(1))}\right)+2f\left(Z_6^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_7^{((2,1),(1))}\right] &= 2x_2x_3^3x_4-x_2^3x_3x_4-x_2x_3x_4^3\\
&= 2x_2x_3^3x_4-x_2^3x_3x_4-x_2^3x_3x_4\\
&\quad -x_2x_3x_4^3+x_2^3x_3x_4\\
&= 2f\left(Z_7^{((2,1),(1))}\right)-f\left(Z_8^{((2,1),(1))}\right),
\end{aligned}$$

$$\begin{aligned}
f\left[Z_8^{((2,1),(1))}\right] &= 2x_2x_3x_4^3-x_2^3x_3x_4-x_2x_3^3x_4\\
&= 2x_2x_3x_4^3-x_2^3x_3x_4-x_2^3x_3x_4\\
&\quad -x_2x_3^3x_4+x_2^3x_3x_4\\
&= -f\left(Z_7^{((2,1),(1))}\right)+2f\left(Z_8^{((2,1),(1))}\right),
\end{aligned}$$

$f\left[Z_1^{((2,1),(1))}\right], \dots, f\left[Z_8^{((2,1),(1))}\right]$ mod 3 will be:

$$\begin{aligned}
f_{(3)}\left[Z_1^{((2,1),(1))}\right] &= 2\,f_{(3)}\left(Z_1^{((2,1),(1))}\right)\\
&\quad + 2\,f_{(3)}\left(Z_2^{((2,1),(1))}\right),
\end{aligned}$$

$$f_{(3)}\left[Z_2^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_1^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_2^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_3^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_3^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_4^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_4^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_3^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_4^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_5^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_5^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_6^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_6^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_5^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_6^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_7^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_7^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_8^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_8^{((2,1),(1))}\right] = 2 f_{(3)}\left(Z_7^{((2,1),(1))}\right) + 2 f_{(3)}\left(Z_8^{((2,1),(1))}\right).$$

The above polynomials modulo 3 give the following matrix:

$$\begin{bmatrix} 2&2&0&0&0&0&0&0\\ 2&2&0&0&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&0&0&2&2\\ 0&0&0&0&0&0&2&2 \end{bmatrix} \xrightarrow[\substack{R_2+2R_1\to R_5\\ R_4+2R_3\to R_6\\ R_6+2R_5\to R_7\\ R_8+2R_7\to R_8}]{\substack{2R_1\to R_1\\ 2R_3\to R_2\\ 2R_5\to R_3\\ 2R_7\to R_4}} \begin{bmatrix} 1&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The above matrix will give the following generator matrix ([9], p.2 & [10], p.49) of the subspace $Y_{(3)}^{((2,1),(1))}$ (which represents the submodule $N_{F_3}((2,1),(1))$) of the vector space F_3^8 (which represents the Specht module $S_{F_3}((2,1),(1))$):

$$\chi_{(3)}^{((2,1),(1))} = \begin{bmatrix} 1&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1 \end{bmatrix}.$$

The four rows of the above generator matrix $\chi_{(3)}^{((2,1),(1))}$ are representing the elements of the basis $B_{(3)}^{((2,1),(1))} = \{b_1, b_2, b_3, b_4\}$ of the submodule $N_{F_3}((2,1),(1)) = F_3 W_4 f_{(3)}\left[Z_1^{((2,1),(1))}\right]$ of the Specht module $S_{F_3}((2,1),(1)) = F_3 W_4 f_{(3)}\left(Z_1^{((2,1),(1))}\right)$, where :

$$b_1 = f_{(3)}\left(Z_1^{((2,2),(1))}\right) + f_{(3)}\left(Z_2^{((2,2),(1))}\right),$$

$$b_2 = f_{(3)}\left(Z_3^{((2,2),(1))}\right) + f_{(3)}\left(Z_4^{((2,2),(1))}\right),$$

$$b_3 = f_{(3)}\left(Z_5^{((2,2),(1))}\right) + f_{(3)}\left(Z_6^{((2,2),(1))}\right),$$

$$b_4 = f_{(3)}\left(Z_7^{((2,2),(1))}\right) + f_{(3)}\left(Z_8^{((2,2),(1))}\right).$$

The four-dimensional subspace $Y_{(3)}^{((2,1),(1))}$ (which represents the submodule $N_{F_3}((2,1),(1))$) of the vector space F_3^8 (which represents the Specht module $S_{F_3}((2,1),(1))$) can be considered as a linear (8, 4, 2, 3)-code ([6], p.16), where 8 means that each vector of this subspace has 8 coordinates, and 4 means that the dimension of this subspace $Y_{(3)}^{((2,1),(1))}$ is 4, and 2 means that the minimum number of nonzero coordinates of any nonzero element of the subspace $Y_{(3)}^{((2,1),(1))}$ is 2 (the minimum distance of this code $Y_{(3)}^{((2,1),(1))}$ is 2 ([13], p.195)), and 3 means that this subspace $Y_{(3)}^{((2,1),(1))}$ is over a field of order 3.

6. THE SUBSPACE $Y_{(5)}^{((2,1),(1))}$ AS A LINEAR CODE

$f\left[Z_1^{((2,1),(1))}\right], \dots, f\left[Z_8^{((2,1),(1))}\right]$ mod 5 will be:

$$f_{(5)}\left[Z_1^{((2,1),(1))}\right] = 2\, f_{(5)}\left(Z_1^{((2,1),(1))}\right) + 4\, f_{(5)}\left(Z_2^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_2^{((2,1),(1))}\right] = 4\, f_{(5)}\left(Z_1^{((2,1),(1))}\right) + 2\, f_{(5)}\left(Z_2^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_3^{((2,1),(1))}\right] = 2\, f_{(5)}\left(Z_3^{((2,1),(1))}\right) + 4\, f_{(5)}\left(Z_4^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_4^{((2,1),(1))}\right] = 4\, f_{(5)}\left(Z_3^{((2,1),(1))}\right) + 2\, f_{(5)}\left(Z_4^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_5^{((2,1),(1))}\right] = 2\, f_{(5)}\left(Z_5^{((2,1),(1))}\right) + 4\, f_{(5)}\left(Z_6^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_6^{((2,1),(1))}\right] = 4\, f_{(5)}\left(Z_5^{((2,1),(1))}\right) + 2\, f_{(5)}\left(Z_6^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_7^{((2,1),(1))}\right] = 2\, f_{(5)}\left(Z_7^{((2,1),(1))}\right) + 4\, f_{(5)}\left(Z_8^{((2,1),(1))}\right),$$

$$f_{(5)}\left[Z_8^{((2,1),(1))}\right] = 4\, f_{(5)}\left(Z_7^{((2,1),(1))}\right) + 2\, f_{(5)}\left(Z_8^{((2,1),(1))}\right).$$

The above polynomials modulo 5 give the following matrix:

$$\begin{bmatrix} 2&4&0&0&0&0&0&0\\ 4&2&0&0&0&0&0&0\\ 0&0&2&4&0&0&0&0\\ 0&0&4&2&0&0&0&0\\ 0&0&0&0&2&4&0&0\\ 0&0&0&0&4&2&0&0\\ 0&0&0&0&0&0&2&4\\ 0&0&0&0&0&0&4&2 \end{bmatrix}$$

$$\xrightarrow[\substack{2R_6+R_5\to R_6\\ 2R_8+R_7\to R_8}]{\substack{2R_2+R_1\to R_2\\ 2R_4+R_3\to R_4}}$$

$$\chi_{(5)}^{((2,1),(1))} = \begin{bmatrix} 2&4&0&0&0&0&0&0\\ 0&3&0&0&0&0&0&0\\ 0&0&2&4&0&0&0&0\\ 0&0&0&3&0&0&0&0\\ 0&0&0&0&2&4&0&0\\ 0&0&0&0&0&3&0&0\\ 0&0&0&0&0&0&2&4\\ 0&0&0&0&0&0&0&3 \end{bmatrix},$$

which is the generator matrix of the subspace $Y_{(5)}^{((2,1),(1))}$ of the vector space F_5^8. The eight rows of the above generator matrix $\chi_{(5)}^{((2,1),(1))}$ are representing the elements of the basis $B_{(5)}^{((2,1),(1))} = \{b_1, b_2, \dots, b_8\}$ of the submodule $N_{F_5}((2,1),(1)) = F_5\, W_4\, f_{(5)}\left[Z_1^{((2,1),(1))}\right]$ of the Specht module $S_{F_5}((2,1),(1)) = F_5\, W_4\, f_{(5)}\left(Z_1^{((2,1),(1))}\right)$, where:

$$b_1 = 2 f_{(5)}\left(Z_1^{((2,1),(1))}\right) + 4 f_{(5)}\left(Z_2^{((2,1),(1))}\right),$$

$$b_2 = 3\, f_{(5)}\left(Z_2^{((2,1),(1))}\right),$$

$b_3 = 2 f_{(5)}\left(Z_3^{((2,1),(1))}\right) + 4 f_{(5)}\left(Z_4^{((2,1),(1))}\right),$

$b_4 = 3 f_{(5)}\left(Z_4^{((2,1),(1))}\right),$

$b_5 = 2 f_{(5)}\left(Z_5^{((2,1),(1))}\right) + 4 f_{(5)}\left(Z_6^{((2,1),(1))}\right),$

$b_6 = 3 f_{(5)}\left(Z_6^{((2,1),(1))}\right),$

$b_7 = 2 f_{(5)}\left(Z_7^{((2,1),(1))}\right) + 4 f_{(5)}\left(Z_8^{((2,1),(1))}\right),$

$b_8 = 3 f_{(5)}\left(Z_8^{((2,1),(1))}\right).$

The eight-dimensional subspace $Y_{(5)}^{((2,1),(1))}$ (which represents the submodule $N_{F_5}((2,1),(1))$) of the vector space F_5^8 (which represents the Specht module $S_{F_5}((2,1),(1))$) can be considered as a linear (8, 8, 1, 5)-code.

7. THE *p*-MODULAR IRREDUCIBLE REPRESENTATIONS FOR THE SUBMODULES $N_{F_p}(\lambda,\mu)$ OF THE SPECHT MODULES $S_{F_p}(\lambda,\mu)$ CORRESPONDING TO ALL PAIRS OF PARTITIONS (λ,μ) OF 4 AS LINEAR CODES WHEN *p* IS A PRIME NUMBER AND $p \geq 3$

The linear codes of the representations of the submodules $N_{F_p}(\lambda,\mu)$ of the Specht modules $S_{F_p}(\lambda,\mu)$ corresponding to all pairs of partitions (λ,μ) of 4, when $p \geq 3$, are as follows:

1) For the pair of partitions $((4),(\ ))$, we have that $m(S) = \dim_{F_p} S_{F_p}((4),(\ )) = \dfrac{4!}{4\cdot 3\cdot 2\cdot 1} = 1,$ and thus we have only one standard $((4),(\ ))$- tableau $Z_1^{((4),(\ ))}$ whose Specht polynomial is $f_{(p)}\left(Z_1^{((4),(\ ))}\right) = x_1x_2x_3x_4$.

(i) If $p = 3$, then $f_{(3)}\left[Z_1^{((4),(\ ))}\right] = 4!(\text{mod } 3)\, f_{(3)}\left(Z_1^{((4),(\ ))}\right) = 24 \ (\text{mod } 3) \cdot x_1x_2x_3x_4 = 0$ (since 24 (mod 3) = 0).

Thus $k_3 = \dim_{F_3} N_{F_3}((4),(\ )) = \dim_{F_3} F_3 W_4\ f_{(3)}\left[Z_1^{((4),(\ ))}\right] = 0,$ hence the minimum distance d_3 does not exist. Therefore, the subspace $Y_{(3)}^{((4),(\ ))}$ (which represents the submodule $N_{F_3}((4),(\ ))$) of the vector space F_3 (which represents the Specht module $S_{F_3}((4),(\ ))$) is a linear (1, 0, – , 3)-code.

(ii) If $p \geq 5$, then $f_{(p)}\left[Z_1^{((4),(\ ))}\right] = 4!(\text{mod } p)\, f_{(p)}\left(Z_1^{((4),(\ ))}\right) = 24 (\text{mod } p) \cdot x_1x_2x_3x_4 \neq 0$ (since 24 (mod p) $\neq$ 0).

Thus $k_p = \dim_{F_p} N_{F_p}((4),(\ )) = \dim_{F_p} F_p W_4\, f_{(p)}\left[Z_1^{((4),(\ ))}\right] = 1,$ since $N_{F_p}((4),(\ ))$ is a nontrivial submodule of the Specht module $S_{F_p}((4),(\ ))$, hence the minimum distance $d_p = 1$. Therefore, the subspace $Y_{(p)}^{((4),(\ ))}$ (which represents the submodule $N_{F_p}((4),(\ ))$) of the vector space F_p (which represents the Specht

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module $S_{F_p}((4),(\ ))$ is a linear $(1, 1, 1, p)$-code.

2) For the pair of partitions $((3,1),(\ ))$, we have that $m(S) = \dim_{F_p} S_{F_p}((3,1),(\ )) = \dfrac{4!}{4\cdot 2\cdot 1\cdot 1} = 3$, and thus we have 3 standard $((3,1),(\ ))$-tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((3,1),(\ ))}\right) = x_1x_2x_3x_4^3 + (p-1)\,x_1^3x_2x_3x_4,$$

$$f_{(p)}\left(Z_2^{((3,1),(\ ))}\right) = x_1x_2x_3^3x_4 + (p-1)\,x_1^3x_2x_3x_4,$$

$$f_{(p)}\left(Z_3^{((3,1),(\ ))}\right) = x_1x_2^3x_3x_4 + (p-1)\,x_1^3x_2x_3x_4.$$

If $p \geq 3$, then by theorem 2.9 (1), we have that $S_{F_p}((3,1),(\ ))$ is irreducible F_pW_4-module. Hence $N_{F_p}((3,1),(\ )) = F_pW_4\, f_{(p)}\left[Z_1^{((3,1),(\ ))}\right] = S_{F_p}((3,1),(\ ))$, since $f_{(p)}\left[Z_1^{((3,1),(\ ))}\right] = 6\ (\mathrm{mod}\ p)\, f_{(p)}\left(Z_1^{((3,1),(\ ))}\right) + (p-2)\, f_{(p)}\left(Z_2^{((3,1),(\ ))}\right) + (p-2)\, f_{(p)}\left(Z_3^{((3,1),(\ ))}\right) = 6\ (\mathrm{mod}\ p)\cdot x_1x_2x_3x_4^3 + (p-2)x_1x_2x_3^3x_4 + (p-2)\cdot x_1x_2^3x_3x_4 + (p-2)\,x_1^3x_2x_3x_4 \neq 0.$

Thus $k_p = \dim_{F_p} N_{F_p}((3,1),(\ )) = \dim_{F_p} S_{F_p}((3,1),(\ )) = 3$, and the minimum distance $d_p = 1$, since each Specht polynomial belongs to $S_{F_p}((3,1),(\ )) = N_{F_p}((3,1),(\ ))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((3,1),(\ ))}$ (which represents the submodule $N_{F_p}((3,1),(\ ))$ of the vector space F_p^3 (which represents the Specht module $S_{F_p}((3,1),(\ ))$) is a linear $(3, 3, 1, p)$-code.

3) For the pair of partitions $((2,2),(\ ))$, we have that $m(S) = \dim_{F_p} S_{F_p}((2,2),(\ )) = \dfrac{4!}{3\cdot 2\cdot 2\cdot 1} = 2$, and thus we have 2 standard $((2,2),(\ ))$-tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((2,2),(\ ))}\right) = x_1x_2x_3^3x_4^3 + (p-1)\cdot x_1x_2^3x_3^3x_4 + (p-1)\,x_1^3x_2x_3x_4^3 + x_1^3x_2^3x_3x_4,$$
$$f_{(p)}\left(Z_2^{((2,2),(\ ))}\right) = x_1x_2^3x_3x_4^3 + (p-1)\cdot x_1x_2^3x_3^3x_4 + (p-1)\,x_1^3x_2x_3x_4^3 + x_1^3x_2x_3^3x_4.$$

(i) If $p = 3$, then:

$$f_{(3)}\left[Z_1^{((2,2),(\ ))}\right] = x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3^3x_4 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3 = f_{(3)}\left(Z_1^{((2,2),(\ ))}\right) + f_{(3)}\left(Z_2^{((2,2),(\ ))}\right),$$

$$f_{(3)}\left[Z_2^{((2,2),(\ ))}\right] = x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3^3x_4 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3 = f_{(3)}\left(Z_1^{((2,2),(\ ))}\right) + f_{(3)}\left(Z_2^{((2,2),(\ ))}\right).$$

Thus $N_{F_3}((2,2),(\ ))$ generated by the polynomial $f_{(3)}\left[Z_1^{((2,2),(\ ))}\right]$ only which means that $k_3 = \dim_{F_3} N_{F_3}((2,2),(\ )) = 1$, and the minimum distance $d_3 = 2$, since

$$f_{(3)}\left[Z_1^{((2,2),(\ ))}\right] = f_{(3)}\left(Z_1^{((2,2),(\ ))}\right) + f_{(3)}\left(Z_2^{((2,2),(\ ))}\right).$$

Therefore, the subspace $Y_{(3)}^{((2,2),(\ ))}$ (which represents the submodule $N_{F_3}((2,2),(\ ))$) of the vector space F_3^2 (which represents the Specht module $S_{F_3}((2,2),(\ ))$) is a linear (2, 1, 2, 3)- code.

(ii) If $p \geq 5$, then by theorem 2.8 (1), we have that $S_{F_p}((2,2),(\ ))$ is irreducible F_pW_4 - module.

Hence $N_{F_p}((2,2),(\ ))$ = $F_p W_4\ f_{(p)}\left[Z_1^{((2,2),(\ ))}\right]$ = $S_{F_p}((2,2),(\ ))$, since $f_{(p)}\left[Z_1^{((2,2),(\ ))}\right] = 4x_1^3x_2^3x_3x_4 + (p-2)x_1^3x_2x_3^3x_4 + (p-2)x_1^3x_2x_3x_4^3 + (p-2)x_1x_2^3x_3^3x_4 + (p-2)\cdot x_1x_2^3x_3x_4^3 + 4\,x_1x_2x_3^3x_4^3 = 4\ f_{(p)}\left(Z_1^{((2,2),(\ ))}\right) + (p-2)\cdot f_{(p)}\left(Z_2^{((2,2),(\ ))}\right) \neq 0.$

Thus $k_p = \dim_{F_p} N_{F_p}((2,2),(\ ))$ $= \dim_{F_p} S_{F_p}((2,2),(\ )) = 2$, and $d_p = 1$, since $f_{(p)}\left(Z_1^{((2,2),(\ ))}\right) \in S_{F_p}((2,2),(\ )) = N_{F_p}((2,2),(\ ))$.

Therefore for each $p \geq 5$, the subspace $Y_{(p)}^{((2,2),(\ ))}$ (which represents the submodule $N_{F_p}((2,2),(\ ))$) of the vector space F_p^2 (which represents the Specht module $S_{F_p}((2,2),(\ ))$) is a linear $(2, 2, 1, p)$- code.

4) For the pair of partitions $((2,1,1),(\ ))$, we have that $m(S) = \dim_{F_p} S_{F_p}((2,1,1),(\ ))$ $= \dfrac{4!}{4\cdot 1\cdot 2\cdot 1} = 3$, and thus we have 3 standard $((2,1,1),(\ ))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((2,1,1),(\ ))}\right) = x_1x_2x_3^3x_4^5 + (p-1)\cdot x_1^3x_2x_3x_4^5 + (p-1)x_1^5x_2x_3^3x_4 + (p-1)\cdot x_1x_2^5x_3^3x_4 + x_1^5x_2x_3x_4^3 + x_1x_2^3x_3^5x_4^3 ... $$

$$f_{(p)}\left(Z_1^{((2,1,1),(\ ))}\right) = x_1x_2x_3^3x_4^5 + (p-1)\cdot x_1^3x_2x_3x_4^5 + (p-1)x_1^5x_2x_3^3x_4 + (p-1)\cdot x_1x_2^5x_3^3x_4 + x_1^5x_2x_3x_4^3 + x_1x_2x_3^5x_4^3 ,$$

$$f_{(p)}\left(Z_2^{((2,1,1),(\ ))}\right) = x_1x_2^3x_3x_4^5 + (p-1)\cdot x_1^3x_2x_3x_4^5 + (p-1)x_1^5x_2^3x_3x_4 + (p-1)\cdot x_1x_2^5x_3^3x_4 + x_1^5x_2x_3x_4^3 + x_1x_2^3x_3^5x_4 ,$$

$$f_{(p)}\left(Z_3^{((2,1,1),(\ ))}\right) = x_1x_2^3x_3^5x_4 + (p-1)\cdot x_1^3x_2x_3^5x_4 + (p-1)x_1^5x_2^3x_3x_4 + (p-1)\cdot x_1x_2^5x_3^3x_4 + x_1^5x_2x_3^3x_4 + x_1x_2^3x_3^5x_4 .$$

If $p \geq 3$, then by theorem 2.9 (1), we have that $S_{F_p}((2,1,1),(\ ))$ is irreducible F_pW_4 - module. Hence $N_{F_p}((2,1,1),(\ )) =$ $= F_pW_4 f_{(p)}\left[Z_1^{((2,1,1),(\ ))}\right] = S_{F_p}((2,1,1),(\ ))$, since $f_{(p)}\left[Z_1^{((2,1,1),(\ ))}\right] = (p-1)x_1^5x_2x_3^3x_4 + (p-1)x_1x_2^5x_3^3x_4 + x_1^3x_2x_3^5x_4 + x_1x_2^3x_3^5x_4 + x_1^5x_2x_3x_4^3 + x_1x_2^5x_3x_4^3 + (p-2)x_1x_2x_3^5x_4^3 + (p-1)x_1^3x_2x_3x_4^5 + (p-1)x_1x_2^3x_3x_4^5 + 2x_1x_2x_3^3x_4^5 = 2\ f_{(p)}\left(Z_1^{((2,1,1),(\ ))}\right) + (p-1)f_{(p)}\left(Z_2^{((2,1,1),(\ ))}\right) + f_{(p)}\left(Z_3^{((2,1,1),(\ ))}\right) \neq 0$. Thus $k_p = \dim_{F_p} N_{F_p}((2,1,1),(\ )) = \dim_{F_p} S_{F_p}((2,1,1),(\ )) = 3$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((2,1,1),(\ ))}\right) \in S_{F_p}((2,1,1),(\ )) = N_{F_p}((2,1,1),(\ ))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((2,1,1),(\ ))}$ (which represents the submodule $N_{F_p}((2,1,1),(\ ))$ of the vector space F_p^3 (which represents the Specht module $S_{F_p}((2,1,1),(\ ))$ is a linear $(3, 3, 1, p)$-code.

5) For the pair of partitions $((1,1,1,1),(\ ))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1,1,1,1),(\ ))$ $= \dfrac{4!}{4\cdot 3\cdot 2\cdot 1} = 1$. Thus we have only one standard $((1,1,1,1),(\ ))$-tableau whose Specht polynomial is:

$$f_{(p)}\left(Z_1^{((1,1,1,1),(\ ))}\right) = x_1x_2^3x_3^5x_4^7 + (p-1)\cdot x_1^3x_2x_3^5x_4^7 + (p-1)x_1^5x_2^3x_3x_4^7 + (p-1)\cdot x_1^7x_2^3x_3^5x_4 + (p-1)x_1x_2^5x_3^3x_4^7 + (p-1)\cdot x_1x_2^7x_3^5x_4^3 + (p-1)x_1x_2^3x_3^7x_4^5 + x_1^5x_2x_3^3x_4^7 + x_1^3x_2^5x_3x_4^7 + x_1^7x_2x_3^5x_4^3 + x_1^3x_2^7x_3^5x_4 + x_1^7x_2^3x_3x_4^5 + x_1^5x_2^3x_3^7x_4 + x_1x_2^7x_3^3x_4^5 + x_1x_2^5x_3^7x_4^3 + x_1^3x_2x_3^7x_4^5 + x_1^5x_2^7x_3x_4^3 + x_1^7x_2^5x_3^3x_4 + (p-1)x_1^7x_2x_3^3x_4^5 + (p-1)\cdot x_1^5x_2^7x_3^3x_4 + (p-1)x_1^7x_2^5x_3x_4^3 + (p-1)\cdot x_1^3x_2^7x_3x_4^5 + (p-1)x_1^5x_2^7x_3^3x_4 + (p-1)x_1^3x_2^5x_3^7x_4,$$

and $f_{(p)}\left[Z_1^{((1,1,1,1),(\ ))}\right] = f_{(p)}\left(i\ Z_1^{((1,1,1,1),(\ ))}\right) = f_{(p)}\left(Z_1^{((1,1,1,1),(\ ))}\right)$, for each $p \geq 3$. Hence $N_{F_p}((1,1,1,1),(\ )) = S_{F_p}((1,1,1,1),(\ ))$.

Thus $k_p = \dim_{F_p} N_{F_p}((1,1,1,1),(\ )) = \dim_{F_p} S_{F_p}((1,1,1,1),(\ )) = 1$, and $d_p = 1$, since $f_{(p)}\left(Z_1^{((1,1,1,1),(\ ))}\right) \in S_{F_p}((1,1,1,1),(\ ))$ $= N_{F_p}((1,1,1,1),(\ ))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((1,1,1,1),(\ ))}$ (which represents the submodule $N_{F_p}((1,1,1,1),(\ ))$ of the vector space F_p (which represents the Specht module $S_{F_p}((1,1,1,1),(\ ))$ is a linear $(1, 1, 1, p)$-code.

6) For the pair of partitions $((3),(1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((3),(1)) = \dfrac{4!}{3\cdot 2\cdot 1\cdot 1}$ $= 4$.

(i) If $p = 3$, then $f_{(3)}\left[Z_1^{((3),(1))}\right] = 3!\ (\text{mod } 3)\ f_{(3)}\left(Z_1^{((3),(1))}\right) = 6\ (\text{mod } 3)\cdot x_1x_2x_3 = 0$ (since $6\ (\text{mod } 3) = 0$), where $Z_1^{((3),(1))}$ is a standard $((3),(1))$-tableau.

Thus $k_3 = \dim_{F_3} N_{F_3}((3),(1)) = \dim_{F_3} F_3W_4\, f_{(3)}\left[Z_1^{((3),(1))}\right] = 0$, hence the minimum distance d_3 does not exist.

Therefore, the subspace $Y_{(3)}^{((3),(1))}$ (which represents the submodule $N_{F_3}((3),(1))$ of the vector space F_3^4 (which represents the Specht module $S_{F_3}((3),(1))$ is a linear $(4, 0, -, 3)$-code.

(ii) If $p \geq 5$, then by theorem 2.9 (2), we have that $S_{F_p}((3),(1))$ is irreducible F_pW_4-module. Hence $N_{F_p}((3),(1)) =$

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$F_p W_4 f_{(p)}\left[Z_1^{((3),(1))}\right] = S_{F_p}((3),(1))$,

since $f_{(p)}\left[Z_1^{((3),(1))}\right] = 3!\ (\text{mod}\, p)\cdot$

$f_{(p)}\left(Z_1^{((3),(1))}\right) = 6\ (\text{mod}\, p)\, x_1 x_2 x_3 \neq 0$ (since $6\ (\text{mod}\, p) \neq 0$).

Thus $k_p = \dim_{F_p} N_{F_p}((3),(1)) = \dim_{F_p} S_{F_p}((3),(1)) = 4$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((3),(1))}\right) \in S_{F_p}((3),(1)) = N_{F_p}((3),(1))$.

Therefore for $p \geq 5$, the subspace $Y_{(p)}^{((3),(1))}$ (which represents the submodule $N_{F_p}((3),(1))$) of the vector space F_p^4 (which represents the Specht module $S_{F_p}((3),(1))$) is a linear $(4, 4, 1, p)$-code.

7) For the pair of partitions $((2,1),(1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((2,1),(1)) =$

$$= \frac{4!}{3\cdot 1\cdot 1\cdot 1} = 8.$$

(i) If $p = 3$, then:

$$f_{(3)}\left(Z_1^{((2,1),(1))}\right) = x_1 x_2 x_3^3 + 2x_1^3 x_2 x_3,$$
$$f_{(3)}\left(Z_2^{((2,1),(1))}\right) = x_1 x_2^3 x_3 + 2x_1^3 x_2 x_3,$$
$$f_{(3)}\left(Z_3^{((2,1),(1))}\right) = x_1 x_2 x_4^3 + 2x_1^3 x_2 x_4,$$
$$f_{(3)}\left(Z_4^{((2,1),(1))}\right) = x_1 x_2^3 x_4 + 2x_1^3 x_2 x_4,$$
$$f_{(3)}\left(Z_5^{((2,1),(1))}\right) = x_1 x_3 x_4^3 + 2x_1^3 x_3 x_4,$$
$$f_{(3)}\left(Z_6^{((2,1),(1))}\right) = x_1 x_3^3 x_4 + 2x_1^3 x_3 x_4,$$
$$f_{(3)}\left(Z_7^{((2,1),(1))}\right) = x_2 x_3 x_4^3 + 2x_2^3 x_3 x_4,$$
$$f_{(3)}\left(Z_8^{((2,1),(1))}\right) = x_2 x_3^3 x_4 + 2x_2^3 x_3 x_4,$$

and the 3-reduced symmetrized Specht polynomials are:

$$f_{(3)}\left[Z_1^{((2,1),(1))}\right] = 2x_1^3 x_2 x_3 + 2x_1 x_2^3 x_3 + 2x_1 x_2 x_3^3 = 2f_{(3)}\left(Z_1^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_2^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_2^{((2,1),(1))}\right] = 2x_1^3 x_2 x_3 + 2x_1 x_2^3 x_3 + 2x_1 x_2 x_3^3 = 2f_{(3)}\left(Z_1^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_2^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_3^{((2,1),(1))}\right] = 2x_1^3 x_2 x_4 + 2x_1 x_2^3 x_4 + 2x_1 x_2 x_4^3 = 2f_{(3)}\left(Z_3^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_4^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_4^{((2,1),(1))}\right] = 2x_1^3 x_2 x_4 + 2x_1 x_2^3 x_4 + 2x_1 x_2 x_4^3 = 2f_{(3)}\left(Z_3^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_4^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_5^{((2,1),(1))}\right] = 2x_1^3 x_3 x_4 + 2x_1 x_3^3 x_4 + 2x_1 x_3 x_4^3 = 2f_{(3)}\left(Z_5^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_6^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_6^{((2,1),(1))}\right] = 2x_1^3 x_3 x_4 + 2x_1 x_3^3 x_4 + 2x_1 x_3 x_4^3 = 2f_{(3)}\left(Z_5^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_6^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_7^{((2,1),(1))}\right] = 2x_2^3 x_3 x_4 + 2x_2 x_3^3 x_4 + 2x_2 x_3 x_4^3 = 2f_{(3)}\left(Z_7^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_8^{((2,1),(1))}\right),$$

$$f_{(3)}\left[Z_8^{((2,1),(1))}\right] = 2x_2^3 x_3 x_4 + 2x_2 x_3^3 x_4 + 2x_2 x_3 x_4^3 = 2f_{(3)}\left(Z_7^{((2,1),(1))}\right) + 2f_{(3)}\left(Z_8^{((2,1),(1))}\right).$$

The above polynomials $f_{(3)}\left[Z_1^{((2,1),(1))}\right], \ldots, f_{(3)}\left[Z_8^{((2,1),(1))}\right]$ give the following matrix:

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$$\begin{bmatrix} 2&2&0&0&0&0&0&0\\ 2&2&0&0&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&0&0&2&2\\ 0&0&0&0&0&0&2&2 \end{bmatrix} \xrightarrow[\substack{R_2+2R_1\to R_5\\ R_4+2R_3\to R_6\\ R_6+2R_5\to R_7\\ R_8+2R_7\to R_8}]{\substack{R_1\to R_1\\ R_3\to R_2\\ R_5\to R_3\\ R_7\to R_4}} \begin{bmatrix} 2&2&0&0&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&0&0&2&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The first 4 rows of the matrix above form a basis of the subspace $Y_{(3)}^{((2,1),(1))}$ (which represents the submodule $N_{F_3}((2,1),(1))$).

Hence $k_3 = \dim_{F_3} N_{F_3}((2,1),(1)) = 4$, and the minimum distance $d_3 = 2$.

Therefore, the subspace $Y_{(3)}^{((2,1),(1))}$ which represents the submodule $N_{F_3}((2,1),(1))$ of the vector space F_3^8 (which represents the Specht module $S_{F_3}((2,1),(1))$) is a linear $(8, 4, 2, 3)$-code.

(ii) If $p \geq 5$, then by theorem 2.9 (1), we have that $S_{F_p}((2,1),(1))$ is irreducible F_pW_4- module. Hence $N_{F_p}((2,1),(1))$ $= F_pW_4 f_{(p)}\left[Z_1^{((2,1),(1))}\right] = S_{F_p}((2,1),(1))$, since $f_{(p)}\left[Z_1^{((2,1),(1))}\right]$ $= (p-1)x_1^3x_2x_3 + (p-1)x_1x_2^3x_3 + 2x_1x_2x_3^3 = 2f_{(p)}\left(Z_1^{((2,1),(1))}\right) + (p-1)\cdot f_{(p)}\left(Z_2^{((2,1),(1))}\right) \neq 0.$

Thus $k_p = \dim_{F_p} N_{F_p}((2,1),(1)) = \dim_{F_p} S_{F_p}((2,1),(1)) = 8$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((2,1),(1))}\right) \in S_{F_p}((2,1),(1)) = N_{F_p}((2,1),(1))$. Therefore for each $p \geq 5$, the subspace $Y_{(p)}^{((2,1),(1))}$ (which represents the submodule $N_{F_p}((2,1),(1))$) of the vector space F_p^8 (which represents the Specht module $S_{F_p}((2,1),(1))$) is a linear $(8, 8, 1, p)$-code.

8) For the pair of partitions $((1,1,1),(1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1,1,1),(1))$ $= \dfrac{4!}{3\cdot 2\cdot 1\cdot 1} = 4$. Thus we have 4 standard $((1,1,1),(1))$-tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((1,1,1),(1))}\right) = x_1x_2^3x_3^5 + (p-1)x_1^3x_2x_3^5 + (p-1)x_1^5x_2^3x_3 + (p-1)x_1x_2^5x_3^3 + x_1^5x_2x_3^3 + x_1^3x_2^5x_3,$$

$$f_{(p)}\left(Z_2^{((1,1,1),(1))}\right) = x_1x_2^3x_4^5 + (p-1)x_1^3x_2x_4^5 + (p-1)x_1^5x_2^3x_4 + (p-1)x_1x_2^5x_4^3 + x_1^5x_2x_4^3 + x_1^3x_2^5x_4,$$

$$f_{(p)}\left(Z_3^{((1,1,1),(1))}\right) = x_1x_3^3x_4^5 + (p-1)x_1^3x_3x_4^5 + (p-1)x_1^5x_3^3x_4 + (p-1)x_1x_3^5x_4^3 + x_1^5x_3x_4^3 + x_1^3x_3^5x_4,$$

$$f_{(p)}\left(Z_4^{((1,1,1),(1))}\right) = x_2x_3^3x_4^5 + (p-1)\,x_2^3x_3x_4^5 + (p-1)\,x_2^5x_3^3x_4 + (p-1)\,x_2x_3^5x_4^3 + x_2^5x_3x_4^3 + x_2^3x_3^5x_4,$$

and $f_{(p)}\left[Z_1^{((1,1,1),(1))}\right] = f_{(p)}\left(i\,Z_1^{((1,1,1),(1))}\right) = f_{(p)}\left(Z_1^{((1,1,1),(1))}\right)$, for each $p \geq 3$. Hence $N_{F_p}((1,1,1),(1)) = F_p W_4 f_{(p)}\left[Z_1^{((1,1,1),(1))}\right] = F_p W_4 f_{(p)}\left(Z_1^{((1,1,1),(1))}\right) = S_{F_p}((1,1,1),(1))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((1,1,1),(1)) = \dim_{F_p} S_{F_p}((1,1,1),(1)) = 4$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((1,1,1),(1))}\right) \in S_{F_p}((1,1,1),(1)) = N_{F_p}((1,1,1),(1))$. Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((1,1,1),(1))}$ (which represents the submodule $N_{F_p}((1,1,1),(1))$) of the vector space F_p^4 (which represents the Specht module $S_{F_p}((1,1,1),(1))$) is a linear $(4, 4, 1, p)$- code.

9) For the pair of partitions $((2),(2))$, we have that $m(S) = \dim_{F_p} S_{F_p}((2),(2)) = \dfrac{4!}{2\cdot1\cdot2\cdot1} = 6$. Thus we have 6 standard $((2),(2))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((2),(2))}\right) = x_1x_2,$$
$$f_{(p)}\left(Z_2^{((2),(2))}\right) = x_1x_3,$$
$$f_{(p)}\left(Z_3^{((2),(2))}\right) = x_1x_4,$$
$$f_{(p)}\left(Z_4^{((2),(2))}\right) = x_2x_3,$$
$$f_{(p)}\left(Z_5^{((2),(2))}\right) = x_2x_4,$$
$$f_{(p)}\left(Z_6^{((2),(2))}\right) = x_3x_4,$$

and $f_{(p)}\left[Z_1^{((2),(2))}\right] = 4\ (\mathrm{mod}\ p)\ x_1x_2 = 4\ (\mathrm{mod}\ p)\ f_{(p)}\left(Z_1^{((2),(2))}\right)$, for each $p \geq 3$. Hence $N_{F_p}((2),(2)) = F_p W_4 f_{(p)}\left[Z_1^{((2),(2))}\right] = F_p W_4 f_{(p)}\left(Z_1^{((2),(2))}\right) = S_{F_p}((2),(2))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((2),(2)) = \dim_{F_p} S_{F_p}((2),(2)) = 6$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((2),(2))}\right) \in S_{F_p}((2),(2)) = N_{F_p}((2),(2))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((2),(2))}$ (which represents the submodule $N_{F_p}((2),(2))$) of the vector space F_p^6 (which represents the Specht module $S_{F_p}((2),(2))$) is a linear $(6, 6, 1, p)$- code.

10) For the pair of partitions $((1,1),(2))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1,1),(2)) = \dfrac{4!}{2\cdot1\cdot2\cdot1} = 6$. Thus we have 6 standard $((1,1),(2))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((1,1),(2))}\right) = x_1x_2^3 + (p-1)\,x_1^3x_2,$$
$$f_{(p)}\left(Z_2^{((1,1),(2))}\right) = x_1x_3^3 + (p-1)\,x_1^3x_3,$$
$$f_{(p)}\left(Z_3^{((1,1),(2))}\right) = x_1x_4^3 + (p-1)\,x_1^3x_4,$$
$$f_{(p)}\left(Z_4^{((1,1),(2))}\right) = x_2x_3^3 + (p-1)\,x_2^3x_3,$$

$$f_{(p)}\left(Z_5^{((1,1),(2))}\right) = x_2 x_4^3 + (p-1)\, x_2^3 x_4,$$

$$f_{(p)}\left(Z_6^{((1,1),(2))}\right) = x_3 x_4^3 + (p-1)\, x_3^3 x_4,$$

and $f_{(p)}\left[Z_1^{((1,1),(2))}\right] = 2x_1 x_2^3 + (p-2)\cdot x_1^3 x_2 = 2 f_{(p)}\left(Z_1^{((1,1),(2))}\right)$, for each $p \geq 3$.

Hence $N_{F_p}((1,1),(2)) = F_p W_4 f_{(p)}\left[Z_1^{((1,1),(2))}\right] = F_p W_4 f_{(p)}\left(Z_1^{((1,1),(2))}\right) = S_{F_p}((1,1),(2))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((1,1),(2)) = \dim_{F_p} S_{F_p}((1,1),(2)) = 6$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((1,1),(2))}\right) \in S_{F_p}((1,1),(2)) = N_{F_p}((1,1),(2))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((1,1),(2))}$ (which represents the submodule $N_{F_p}((1,1),(2))$) of the vector space F_p^6 (which represents the Specht module $S_{F_p}((1,1),(2))$) is a linear $(6, 6, 1, p)$- code.

11) For the pair of partitions $((2),(1,1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((2),(1,1)) = \dfrac{4!}{2\cdot 1\cdot 2\cdot 1} = 6$. Thus we have 6 standard $((2),(1,1))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((2),(1,1))}\right) = x_1 x_2 x_4^2 + (p-1)\, x_1 x_2 x_3^2,$$

$$f_{(p)}\left(Z_2^{((2),(1,1))}\right) = x_1 x_3 x_4^2 + (p-1)\, x_1 x_2^2 x_3,$$

$$f_{(p)}\left(Z_3^{((2),(1,1))}\right) = x_1 x_3^2 x_4 + (p-1)\, x_1 x_2^2 x_4,$$

$$f_{(p)}\left(Z_4^{((2),(1,1))}\right) = x_2 x_3 x_4^2 + (p-1)\, x_1^2 x_2 x_3,$$

$$f_{(p)}\left(Z_5^{((2),(1,1))}\right) = x_2 x_3^2 x_4 + (p-1)\, x_1^2 x_2 x_4,$$

$$f_{(p)}\left(Z_6^{((2),(1,1))}\right) = x_2^2 x_3 x_4 + (p-1)\, x_1^2 x_3 x_4,$$

and $f_{(p)}\left[Z_1^{((2),(1,1))}\right] = 2x_1 x_2 x_4^2 + (p-2)\cdot x_1 x_2 x_3^2 = 2 f_{(p)}\left(Z_1^{((2),(1,1))}\right)$, for each $p \geq 3$. Hence $N_{F_p}((2),(1,1)) = F_p W_4 f_{(p)}\left[Z_1^{((2),(1,1))}\right] = F_p W_4 f_{(p)}\left(Z_1^{((2),(1,1))}\right) = S_{F_p}((2),(1,1))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((2),(1,1)) = \dim_{F_p} S_{F_p}((2),(1,1)) = 6$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((2),(1,1))}\right) \in S_{F_p}((2),(1,1)) = N_{F_p}((2),(1,1))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((2),(1,1))}$ (which represents the submodule $N_{F_p}((2),(1,1))$) of the vector space F_p^6 (which represents the Specht module $S_{F_p}((2),(1,1))$) is a linear $(6, 6, 1, p)$- code.

12) For the pair of partitions $((1,1),(1,1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1,1),(1,1)) = \dfrac{4!}{2\cdot 1\cdot 2\cdot 1} = 6$. Thus we have 6 standard $((1,1),(1,1))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((1,1),(1,1))}\right) = x_1 x_2^3 x_4^2 + (p-1) x_1 x_2^3 x_3^2 + (p-1)\, x_1^3 x_2 x_4^2 + x_1^3 x_2 x_3^2,$$

$$f_{(p)}\left(Z_2^{((1,1),(1,1))}\right) = x_1 x_3^3 x_4^2 + (p-1) x_1 x_2^2 x_3^3 + (p-1)\, x_1^3 x_3 x_4^2 + x_1^3 x_2^2 x_3,$$

$$f_{(p)}\left(Z_3^{((1,1),(1,1))}\right) = x_1 x_3^2 x_4^3 + (p-1) x_1 x_2^2 x_4^3 + (p-1)\, x_1^3 x_3^2 x_4 + x_1^3 x_2^2 x_4,$$

$f_{(p)}\left(Z_4^{((1,1),(1,1))}\right) = x_2x_3^3x_4^2 + (p-1)x_1^2x_2x_3^3 + (p-1)x_2^3x_3x_4^2 + x_1^2x_2^3x_3,$

$f_{(p)}\left(Z_5^{((1,1),(1,1))}\right) = x_2x_3^2x_4^3 + (p-1)x_1^2x_2x_4^3 + (p-1)x_2^3x_3^2x_4 + x_1^2x_2^3x_4,$

$f_{(p)}\left(Z_6^{((1,1),(1,1))}\right) = x_2^2x_3x_4^3 + (p-1)x_1^2x_3x_4^3 + (p-1)x_2^2x_3^3x_4 + x_1^2x_3^3x_4,$

and $f_{(p)}\left[Z_1^{((1,1),(1,1))}\right] = f_{(p)}\left(i\,Z_1^{((1,1),(1,1))}\right) = f_{(p)}\left(Z_1^{((1,1),(1,1))}\right)$, for each $p \geq 3$. Hence $N_{F_p}((1,1),(1,1)) = F_pW_4 f_{(p)}\left[Z_1^{((1,1),(1,1))}\right] = F_pW_4 f_{(p)}\left(Z_1^{((1,1),(1,1))}\right) = S_{F_p}((1,1),(1,1))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((1,1),(1,1)) = \dim_{F_p} S_{F_p}((1,1),(1,1)) = 6$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((1,1),(1,1))}\right) \in S_{F_p}((1,1),(1,1)) = N_{F_p}((1,1),(1,1))$. Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((1,1),(1,1))}$ (which represents the submodule $N_{F_p}((1,1),(1,1))$) of the vector space F_p^6 (which represents the Specht module $S_{F_p}((1,1),(1,1))$) is a linear $(6, 6, 1, p)$- code.

13) For the pair of partitions $((1),(3))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1),(3)) = \dfrac{4!}{1\cdot 3\cdot 2\cdot 1} = 4$.

(i) If $p = 3$, then $f_{(3)}\left[Z_1^{((1),(3))}\right] = 3!\,(\text{mod }3)\, f_{(3)}\left(Z_1^{((1),(3))}\right) = 6\ (\text{mod }3)\cdot x_1 = 0$ (since $6\ (\text{mod }3) = 0$), where $Z_1^{((1),(3))}$ is a standard $((1),(3))$- tableau.

Thus $k_3 = \dim_{F_3} N_{F_3}((1),(3)) = \dim_{F_3} F_3 W_4\, f_{(3)}\left[Z_1^{((1),(3))}\right] = 0$, hence the minimum distance d_3 does not exist. Therefore, the subspace $Y_{(3)}^{((1),(3))}$ (which represents the submodule $N_{F_3}((1),(3))$) of the vector space F_3^4 (which represents the Specht module $S_{F_3}((3),(1))$) is a linear (4,0, – ,3)-code.

(ii) If $p \geq 5$, then by theorem 2.9 (2), we have that $S_{F_p}((1),(3))$ is irreducible F_pW_4-module. Hence $N_{F_p}((1),(3)) = F_p W_4\, f_{(p)}\left[Z_1^{((1),(3))}\right] = S_{F_p}((1),(3))$, since $f_{(p)}\left[Z_1^{((1),(3))}\right] = 3!\ (\text{mod }p)\cdot f_{(p)}\left(Z_1^{((1),(3))}\right) = 6\,(\text{mod }p)\ x_1 \neq 0$ (since $6\ (\text{mod }p) \neq 0$).

Thus $k_p = \dim_{F_p} N_{F_p}((1),(3)) = \dim_{F_p} S_{F_p}((1),(3)) = 4$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((1),(3))}\right) \in S_{F_p}((1),(3)) = N_{F_p}((1),(3))$. Therefore for $p \geq 5$, the subspace $Y_{(p)}^{((1),(3))}$ (which represents the submodule $N_{F_p}((1),(3))$) of the vector space F_p^4 (which represents the Specht module $S_{F_p}((1),(3))$) is a linear $(4, 4, 1, p)$- code.

14) For the pair of partitions $((1),(2,1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1),(2,1)) =$

$$= \frac{4!}{1\cdot 3\cdot 1\cdot 1} = 8.$$

(i) If $p = 3$, then:

$$f_{(3)}\left(Z_1^{((1),(2,1))}\right) = x_1x_4^2 + 2x_1x_2^2,$$
$$f_{(3)}\left(Z_2^{((1),(2,1))}\right) = x_1x_3^2 + 2x_1x_2^2,$$
$$f_{(3)}\left(Z_3^{((1),(2,1))}\right) = x_2x_4^2 + 2x_1^2x_2,$$
$$f_{(3)}\left(Z_4^{((1),(2,1))}\right) = x_2x_3^2 + 2x_1^2x_2,$$
$$f_{(3)}\left(Z_5^{((1),(2,1))}\right) = x_3x_4^2 + 2x_1^2x_3,$$
$$f_{(3)}\left(Z_6^{((1),(2,1))}\right) = x_2^2x_3 + 2x_1^2x_3,$$
$$f_{(3)}\left(Z_7^{((1),(2,1))}\right) = x_3^2x_4 + 2x_1^2x_4,$$
$$f_{(3)}\left(Z_8^{((1),(2,1))}\right) = x_2^2x_4 + 2x_1^2x_4,$$

and the 3-reduced symmetrized Specht polynomials are:

$$f_{(3)}\left[Z_1^{((1),(2,1))}\right] = 2x_1x_2^2 + 2x_1x_3^2 + 2x_1x_4^2 = 2f_{(3)}\left(Z_1^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_2^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_2^{((1),(2,1))}\right] = 2x_1x_2^2 + 2x_1x_3^2 + 2x_1x_4^2 = 2f_{(3)}\left(Z_1^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_2^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_3^{((1),(2,1))}\right] = 2x_1^2x_2 + 2x_2x_3^2 + 2x_2x_4^2 = 2f_{(3)}\left(Z_3^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_4^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_4^{((1),(2,1))}\right] = 2x_1^2x_2 + 2x_2x_3^2 + 2x_2x_4^2 = 2f_{(3)}\left(Z_3^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_4^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_5^{((1),(2,1))}\right] = 2x_1^2x_3 + 2x_2^2x_3 + 2x_3x_4^2 = 2f_{(3)}\left(Z_5^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_6^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_6^{((1),(2,1))}\right] = 2x_1^2x_3 + 2x_2^2x_3 + 2x_3x_4^2 = 2f_{(3)}\left(Z_5^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_6^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_7^{((1),(2,1))}\right] = 2x_1^2x_4 + 2x_2^2x_4 + 2x_3^2x_4 = 2f_{(3)}\left(Z_7^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_8^{((1),(2,1))}\right),$$
$$f_{(3)}\left[Z_8^{((1),(2,1))}\right] = 2x_1^2x_4 + 2x_2^2x_4 + 2x_3^2x_4 = 2f_{(3)}\left(Z_7^{((1),(2,1))}\right) + 2f_{(3)}\left(Z_8^{((1),(2,1))}\right),$$

The above polynomials $f_{(3)}\left[Z_1^{((1),(2,1))}\right], \ldots, f_{(3)}\left[Z_8^{((1),(2,1))}\right]$ give the following matrix:

$$\begin{bmatrix} 2&2&0&0&0&0&0&0\\ 2&2&0&0&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&0&0&2&2\\ 0&0&0&0&0&0&2&2 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2+2R_1\to R_5\\ R_4+2R_3\to R_6\\ R_6+2R_5\to R_7\\ R_8+2R_7\to R_8}]{\substack{R_1\to R_1\\ R_3\to R_2\\ R_5\to R_3\\ R_7\to R_4}}$$

$$\begin{bmatrix} 2&2&0&0&0&0&0&0\\ 0&0&2&2&0&0&0&0\\ 0&0&0&0&2&2&0&0\\ 0&0&0&0&0&0&2&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

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The first 4 rows of the matrix above form a basis of the subspace $Y_{(3)}^{((1),(2,1))}$ (which represents the submodule $N_{F_3}((1),(2,1))$).

Hence $k_3 = \dim_{F_3} N_{F_3}((1),(2,1)) = 4$, and the minimum distance $d_3 = 2$.

Therefore, the subspace $Y_{(3)}^{((1),(2,1))}$ which represents the submodule $N_{F_3}((1),(2,1))$ of the vector space F_3^8 (which represents the Specht module $S_{F_3}((1),(2,1))$ is a linear (8, 4, 2, 3)-code.

(ii) If $p \geq 5$, then by theorem 2.9 (1), we have that $S_{F_p}((1),(2,1))$ is irreducible $F_p W_4$-module. Hence $N_{F_p}((1),(2,1)) = F_p W_4 f_{(p)}\left[Z_1^{((1),(2,1))}\right] = S_{F_p}((1),(2,1))$, since $f_{(p)}\left[Z_1^{((1),(2,1))}\right] = (p-1)x_1x_2^2 + (p-1)x_1x_3^2 + 2x_1x_4^2 \neq 0$.

Thus $k_p = \dim_{F_p} N_{F_p}((1),(2,1)) = \dim_{F_p} S_{F_p}((1),(2,1)) = 8$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((1),(2,1))}\right) \in S_{F_p}((1),(2,1)) = N_{F_p}((1),(2,1))$. Therefore for each $p \geq 5$, the subspace $Y_{(p)}^{((1),(2,1))}$ (which represents the submodule $N_{F_p}((1),(2,1))$ of the vector space F_p^8 (which represents the Specht module $S_{F_p}((1),(2,1))$ is a linear (8, 8, 1, p)-code.

15) For the pair of partitions $((1),(1,1,1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((1),(1,1,1)) = \dfrac{4!}{1\cdot 3\cdot 2\cdot 1} = 4$. Thus we have 4 standard $((1),(1,1,1))$-tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((1),(1,1,1))}\right) = x_1x_3^2x_4^4 + (p-1)x_1x_2^2x_4^4 + (p-1)x_1x_2^4x_3^2 + (p-1)x_1x_3^4x_4^2 + x_1x_2^4x_4^2 + x_1x_2^2x_3^4,$$

$$f_{(p)}\left(Z_2^{((1),(1,1,1))}\right) = x_2x_3^2x_4^4 + (p-1)x_1^2x_2x_4^4 + (p-1)x_1^4x_2x_3^2 + (p-1)x_2x_3^4x_4^2 + x_1^4x_2x_4^2 + x_1^2x_2x_3^4,$$

$$f_{(p)}\left(Z_3^{((1),(1,1,1))}\right) = x_2^2x_3x_4^4 + (p-1)x_1^2x_3x_4^4 + (p-1)x_1^4x_2^2x_3 + (p-1)x_2^4x_3x_4^2 + x_1^4x_3x_4^2 + x_1^2x_2^4x_3,$$

$$f_{(p)}\left(Z_4^{((1),(1,1,1))}\right) = x_2^2x_3^4x_4 + (p-1)\cdot x_1^2x_3^4x_4 + (p-1)x_1^4x_2^2x_4 + (p-1)\cdot x_2^4x_3^2x_4 + x_1^4x_3^2x_4 + x_1^2x_2^4x_4,$$

and $f_{(p)}\left[Z_1^{((1),(1,1,1))}\right] = f_{(p)}\left(i\,Z_1^{((1),(1,1,1))}\right) = f_{(p)}\left(Z_1^{((1),(1,1,1))}\right)$, for each $p \geq 3$. Hence $N_{F_p}((1),(1,1,1)) = F_p W_4 f_{(p)}\left[Z_1^{((1),(1,1,1))}\right] = F_p W_4 f_{(p)}\left(Z_1^{((1),(1,1,1))}\right) = S_{F_p}((1),(1,1,1))$, for each $p \geq 3$.

Thus $k_p = \dim_{F_p} N_{F_p}((1),(1,1,1)) = \dim_{F_p} S_{F_p}((1),(1,1,1)) = 4$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((1),(1,1,1))}\right) \in S_{F_p}((1),(1,1,1)) = N_{F_p}((1),(1,1,1))$. Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((1),(1,1,1))}$ (which represents the submodule $N_{F_p}((1),(1,1,1))$ of the vector space F_p^4 (which represents the Specht module $S_{F_p}((1),(1,1,1))$ is a linear $(4, 4, 1, p)$- code.

16) For the pair of partitions $((\ ),(4))$, we have that $m(S)$ = $\dim_{F_p} S_{F_p}((\ ),(4))$ = $\frac{4!}{4\cdot3\cdot2\cdot1}=1$, and thus we have only one standard $((\ ),(4))$- tableau $Z_1^{((\ ),(4))}$ whose Specht polynomial is $f_{(p)}\left(Z_1^{((\ ),(4))}\right)=1$.

(i) If p = 3, then $f_{(3)}\left[Z_1^{((\ ),(4))}\right]$ $=4!(\text{mod}\,3)\, f_{(3)}\left(Z_1^{((\ ),(4))}\right)=24\ (\text{mod}\,3)$ $\cdot 1=0$ (since 24 (mod 3) = 0).

Thus k_3 = $\dim_{F_3} N_{F_3}((\ ),(4))$ = $\dim_{F_3} F_3 W_4\, f_{(3)}\left[Z_1^{((\ ),(4))}\right]=0$, hence the minimum distance d_3 does not exist.

Therefore, the subspace $Y_{(3)}^{((\ ),(4))}$ (which represents the submodule $N_{F_3}((\ ),(4))$) of the vector space F_3 (which represents the Specht module $S_{F_3}((\ ),(4))$) is a linear $(1, 0, -, 3)$-code.

(ii) If $p\geq 5$, then $f_{(p)}\left[Z_1^{((\ ),(4))}\right]$ = $4!\ (\text{mod}\,p)\, f_{(p)}\left(Z_1^{((\ ),(4))}\right)=24(\text{mod}\,p)$ $\cdot 1\neq 0$ (since $24\ (\text{mod}\,p)\neq 0$), where $Z_1^{((\ ),(4))}$ is the standard $((\ ),(4))$- tableau.

Thus k_p = $\dim_{F_p} N_{F_p}((\ ),(4))$ = $\dim_{F_p} F_p W_4\, f_{(p)}\left[Z_1^{((\ ),(4))}\right]=1$, since $N_{F_p}((\ ),(4))$ is a nontrivial submodule of the Specht module $S_{F_p}((\ ),(4))$, hence the minimum distance $d_p=1$.

Therefore, the subspace $Y_{(p)}^{((\ ),(4))}$ (which represents the submodule $N_{F_p}((\ ),(4))$) of the vector space F_p (which represents the Specht module $S_{F_p}((\ ),(4))$) is a linear $(1, 1, 1, p)$-code.

17) For the pair of partitions $((\ ),(3,1))$, we have that $m(S)$ = $\dim_{F_p} S_{F_p}((\ ),(3,1))$ = $\frac{4!}{4\cdot2\cdot1\cdot1}=3$, and thus we have 3 standard $((\ ),(3,1))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((\ ),(3,1))}\right)=x_4^2+(p-1)\,x_1^2,$$
$$f_{(p)}\left(Z_2^{((\ ),(3,1))}\right)=x_3^2+(p-1)\,x_1^2,$$
$$f_{(p)}\left(Z_3^{((\ ),(3,1))}\right)=x_2^2+(p-1)\,x_1^2.$$

If $p\geq 3$, then by theorem 2.9 (1), we have that $S_{F_p}((\ ),(3,1))$ is irreducible F_pW_4- module. Hence $N_{F_p}((\ ),(3,1))$ = $F_pW_4\ f_{(p)}\left[Z_1^{((\ ),(3,1))}\right]$ = $S_{F_p}((\ ),(3,1))$ since $f_{(p)}\left[Z_1^{((\ ),(3,1))}\right]=$ 6 (mod p) x_4^2 + $(p-2)\,x_3^2+(p-2)\,x_2^2+(p-2)\,x_1^2$ = 6 (mod p) $f_{(p)}\left(Z_1^{((\ ),(3,1))}\right)$ + $(p-2)\cdot$ $f_{(p)}\left(Z_2^{((\ ),(3,1))}\right)+(p-2)\ f_{(p)}\left(Z_3^{((\ ),(3,1))}\right)$ $\neq 0$. Thus k_p = $\dim_{F_p} N_{F_p}((\ ),(3,1))$ =

$\dim_{F_p} S_{F_p}((\ ),(3,1)) = 3$, and the minimum distance $d_p = 1$, since each Specht polynomial belongs to $S_{F_p}((\ ),(3,1)) = N_{F_p}((\ ),(3,1))$. Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((\ ),(3,1))}$ (which represents the submodule $N_{F_p}((\ ),(3,1))$) of the vector space F_p^3 (which represents the Specht module $S_{F_p}((\ ),(3,1))$) is a linear $(3, 3, 1, p)$- code.

18) For the pair of partitions $((\ ),(2,2))$, we have that $m(S) = \dim_{F_p} S_{F_p}((\ ),(2,2)) = \frac{4!}{3\cdot 2\cdot 2\cdot 1} = 2$, and thus we have 2 standard $((\ ),(2,2))$- tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((\ ),(2,2))}\right) = x_3^2x_4^2 + (p-1)\,x_2^2x_3^2 + (p-1)\,x_1^2x_4^2 + x_1^2x_2^2,$$

$$f_{(p)}\left(Z_2^{((\ ),(2,2))}\right) = x_2^2x_4^2 + (p-1)\,x_2^2x_3^2 + (p-1)\,x_1^2x_4^2 + x_1^2x_3^2.$$

(i) If $p = 3$, then:

$$f_{(3)}\left[Z_1^{((\ ),(2,2))}\right] = x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2 = f_{(3)}\left(Z_1^{((\ ),(2,2))}\right) + f_{(3)}\left(Z_2^{((\ ),(2,2))}\right),$$

$$f_{(3)}\left[Z_2^{((\ ),(2,2))}\right] = x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2 = f_{(3)}\left(Z_1^{((\ ),(2,2))}\right) + f_{(3)}\left(Z_2^{((\ ),(2,2))}\right).$$

Thus $N_{F_3}((\ ),(2,2))$ generated by the polynomial $f_{(3)}\left[Z_1^{((\ ),(2,2))}\right]$ only which means that $k_3 = \dim_{F_3} N_{F_3}((\ ),(2,2)) = 1$, and the minimum distance $d_3 = 2$, since

$$f_{(3)}\left[Z_1^{((\ ),(2,2))}\right] = f_{(3)}\left(Z_1^{((\ ),(2,2))}\right) + f_{(3)}\left(Z_2^{((\ ),(2,2))}\right).$$

Therefore, the subspace $Y_{(3)}^{((\ ),(2,2))}$ (which represents the submodule $N_{F_3}((\ ),(2,2))$) of the vector space F_3^2 (which represents the Specht module $S_{F_3}((\ ),(2,2))$) is a linear $(2, 1, 2, 3)$- code.

(ii) If $p \geq 5$, then:

$$f_{(p)}\left[Z_1^{((\ ),(2,2))}\right] = 4x_1^2x_2^2 + (p-2)\cdot x_1^2x_3^2 + (p-2)\,x_1^2x_4^2 + (p-2)\,x_2^2x_3^2 + (p-2)\,x_2^2x_4^2 + 4x_3^2x_4^2,$$

$$f_{(p)}\left[Z_2^{((\ ),(2,2))}\right] = (p-2)\,x_1^2x_2^2 + 4x_1^2x_3^2 + (p-2)\,x_1^2x_4^2 + (p-2)\,x_2^2x_3^2 + 4x_2^2x_4^2 + (p-2)\,x_3^2x_4^2,$$

which implies that:

$$f_{(p)}\left[Z_1^{((\ ),(2,2))}\right] = 4\,f_{(p)}\left(Z_1^{((\ ),(2,2))}\right) + (p-2)\,f_{(p)}\left(Z_2^{((\ ),(2,2))}\right),$$

$$f_{(p)}\left[Z_2^{((\ ),(2,2))}\right] = (p-2)\cdot f_{(p)}\left(Z_1^{((\ ),(2,2))}\right) + 4\,f_{(p)}\left(Z_2^{((\ ),(2,2))}\right).$$

The above polynomials modulo p give the following matrix:

$$\begin{bmatrix} 4 & p-2 \\ p-2 & 4 \end{bmatrix} \xrightarrow[2R_2+R_1 \to R_2]{2R_1+R_2 \to R_1} \begin{bmatrix} 6\ (\bmod p) & 0 \\ 0 & 6\ (\bmod p) \end{bmatrix}.$$

The rows of the above matrix form a basis of the subspace $Y_{(p)}^{((\ ),(2,2))}$. Thus $k_p = \dim_{F_p} N_{F_p}((\ ),(2,2)) = 2$, and $d_p = 1$. Therefore for each $p \geq 5$, the subspace $Y_{(p)}^{((\ ),(2,2))}$ (which represents the submodule $N_{F_p}((\ ),(2,2))$) of the vector space F_p^2 (which represents the Specht module $S_{F_p}((\ ),(2,2))$) is a linear $(2, 2, 1, p)$-code.

19) For the pair of partitions $((\ ),(2,1,1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((\ ),(2,1,1)) = \frac{4!}{4\cdot 1\cdot 2\cdot 1} = 3$, and thus we have 3 standard $((\ ),(2,1,1))$-tableaux whose Specht polynomials are:

$$f_{(p)}\left(Z_1^{((\ ),(2,1,1))}\right) = x_3^2x_4^4 + (p-1)x_1^2x_4^4 + (p-1)x_1^4x_3^2 + (p-1)x_3^4x_4^2 + x_1^4x_4^2 + x_1^2x_3^4,$$

$$f_{(p)}\left(Z_2^{((\ ),(2,1,1))}\right) = x_2^2x_4^4 + (p-1)x_1^2x_4^4 + (p-1)x_1^4x_2^2 + (p-1)x_2^4x_4^2 + x_1^4x_4^2 + x_1^2x_2^4,$$

$$f_{(p)}\left(Z_3^{((\ ),(2,1,1))}\right) = x_2^2x_3^4 + (p-1)x_1^2x_3^4 + (p-1)x_1^4x_2^2 + (p-1)x_2^4x_3^2 + x_1^4x_3^2 + x_1^2x_2^4.$$

If $p \geq 3$, then by theorem 2.9 (1), we have that $S_{F_p}((\ ),(2,1,1))$ is irreducible F_pW_4- module. Hence $N_{F_p}((\ ),(2,1,1)) = F_pW_4 f_{(p)}\left[Z_1^{((\ ),(2,1,1))}\right] = S_{F_p}((\ ),(2,1,1))$, since $f_{(p)}\left[Z_1^{((\ ),(2,1,1))}\right] = (p-1)\, x_1^4x_3^2 + (p-1)\, x_2^4x_3^2 + x_1^2x_3^4 + x_2^2x_3^4 + x_1^4x_4^2 + x_2^4x_4^2 + (p-2)\, x_3^4x_4^2 + (p-1)\, x_1^2x_4^4 + (p-1)\, x_2^2x_4^4 + 2x_3^2x_4^4$

$$= 2\, f_{(p)}\left(Z_1^{((\ ),(2,1,1))}\right) + (p-1)\cdot f_{(p)}\left(Z_2^{((\ ),(2,1,1))}\right) + f_{(p)}\left(Z_3^{((\ ),(2,1,1))}\right) \neq 0.$$

Thus $k_p = \dim_{F_p} N_{F_p}((\ ),(2,1,1)) = \dim_{F_p} S_{F_p}((\ ),(2,1,1)) = 3$, and the minimum distance $d_p = 1$, since $f_{(p)}\left(Z_1^{((\ ),(2,1,1))}\right) \in S_{F_p}((\ ),(2,1,1)) = N_{F_p}((\ ),(2,1,1))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((\ ),(2,1,1))}$ (which represents the submodule $N_{F_p}((\ ),(2,1,1))$) of the vector space F_p^3 (which represents the Specht module $S_{F_p}((\ ),(2,1,1))$) is a linear $(3, 3, 1, p)$-code.

20) For the pair of partitions $((\ ),(1,1,1,1))$, we have that $m(S) = \dim_{F_p} S_{F_p}((\ ),(1,1,1,1)) = \frac{4!}{4\cdot 3\cdot 2\cdot 1} = 1$. Thus we have only one standard $((1,1,1,1),(\ ))$-tableau whose Specht polynomial is:

$$f_{(p)}\left(Z_1^{((\ ),(1,1,1,1))}\right) = x_2^2x_3^4x_4^6 + (p-1)\cdot x_1^2x_3^4x_4^6 + (p-1)\, x_1^4x_2^2x_4^6 + (p-1)\cdot x_1^6x_2^2x_3^4 + (p-1)\, x_2^4x_3^2x_4^6 + (p-1)\cdot$$

$x_2^6 x_3^4 x_4^2 + (p-1)\, x_2^2 x_3^6 x_4^4 + x_1^4 x_3^2 x_4^6 + x_1^2 x_2^4 x_4^6 + x_1^6 x_3^4 x_4^2 + x_1^2 x_2^6 x_3^4 + x_1^6 x_2^2 x_4^4 + x_1^4 x_2^2 x_3^6 + x_2^6 x_3^2 x_4^4 + x_2^4 x_3^6 x_4^2 + x_1^2 x_3^6 x_4^4 + x_1^4 x_2^6 x_4^2 +$

$x_1^6 x_2^4 x_3^2 + (p-1)\, x_1^6 x_3^2 x_4^4 + (p-1) \cdot x_1^4 x_3^6 x_4^2 + (p-1)\, x_1^6 x_2^4 x_4^2 + (p-1) \cdot x_1^2 x_2^6 x_4^4 + (p-1)\, x_1^4 x_2^6 x_3^2 + (p-1) \cdot x_1^2 x_2^4 x_3^6$, and

$f_{(p)}\left[Z_1^{((\ ),(1,1,1,1))}\right] = f_{(p)}\left(i\ Z_1^{((\ ),(1,1,1,1))}\right) = f_{(p)}\left(Z_1^{((\ ),(1,1,1,1))}\right)$ for each $p \geq 3$.

Hence $N_{F_p}((\ ),(1,1,1,1)) = S_{F_p}((\ ),(1,1,1,1))$.

Thus $k_p = \dim_{F_p} N_{F_p}((\ ),(1,1,1,1)) = \dim_{F_p} S_{F_p}((\ ),(1,1,1,1)) = 1$, and $d_p = 1$, since $f_{(p)}\left(Z_1^{((\ ),(1,1,1,1))}\right) \in S_{F_p}((\ ),(1,1,1,1)) = N_{F_p}((\ ),(1,1,1,1))$.

Therefore for each $p \geq 3$, the subspace $Y_{(p)}^{((\ ),(1,1,1,1))}$ (which represents the submodule $N_{F_p}((\ ),(1,1,1,1))$) of the vector space F_p (which represents the Specht module $S_{F_p}((\ ),(1,1,1,1))$) is a linear $(1, 1, 1, p)$-code.

Finally, we summarize the above linear codes in the following Table 1:

Table 1

No.	(λ, μ) ***of*** $n=4$	$m(S)$	k_3	d_3	$k_p, p \geq 5$	$d_p, p \geq 5$
1	((4),( ))	1	0	–	1	1
2	((3,1),( ))	3	3	1	3	1
3	((2,2),( ))	2	1	2	2	1
4	((2,1,1),( ))	3	3	1	3	1
5	((1,1,1,1),( ))	1	1	1	1	1
6	((3),(1))	4	0	–	4	1
7	((2,1),(1))	8	4	2	8	1
8	((1,1,1),(1))	4	4	1	4	1
9	((2),(2))	6	6	1	6	1
10	((1,1),(2))	6	6	1	6	1
11	((2),(1,1))	6	6	1	6	1
12	((1,1),(1,1))	6	6	1	6	1
13	((1),(3))	4	0	–	4	1
14	((1),(2,1))	8	4	2	8	1
15	((1),(1,1,1))	4	4	1	4	1
16	(( ),(4))	1	0	–	1	1
17	(( ),(3,1))	3	3	1	3	1
18	(( ),(2,2))	2	1	2	2	1
19	(( ),(2,1,1))	3	3	1	3	1

20	$((\ ),(1,1,1,1))$	1	1	1	1	1

where $m(S)$ is the dimension of the vector space $F_p^{m(S)}$ which represents the Specht module $S_{F_p}(\lambda,\mu) = F_p W_4 f_{(p)}\left(Z^{(\lambda,\mu)}\right)$, k_p is the dimension of the subspace $Y_{(p)}^{(\lambda,\mu)}$ of $F_p^{m(S)}$, where $Y_{(p)}^{(\lambda,\mu)}$ represents the irreducible $F_p W_4$-submodule $N_{F_p}(\lambda,\mu) = F_p W_4 f_{(p)}\left[Z^{(\lambda,\mu)}\right]$ of $S_{F_p}(\lambda,\mu)$, and d_p is the minimum distance, which is the least number of the nonzero coordinates in any nonzero vector of the subspace $Y_{(p)}^{(\lambda,\mu)}$.

8. CONCLUSIONS

When p is a prime number greater than or equal to 3 and $n = 4$, we conclude the following:

1) If (λ,μ) and $(\overline{\lambda},\overline{\mu})$ are two pairs of partitions of n, such that $\lambda = \overline{\mu}$ and $\mu = \overline{\lambda}$ and $Y_{(p)}^{(\lambda,\mu)}$ is a linear $(m(s), k_p, d_p, p)$-code, then $Y_{(p)}^{(\overline{\lambda},\overline{\mu})}$ is the same linear $(m(s), k_p, d_p, p)$-code.

2) If $(\lambda,\mu) = ((\lambda_1, \lambda_2, \ldots, \lambda_s), (\mu_1, \mu_2, \ldots, \mu_t))$ be a pair of partitions of n and p divides $(\lambda_1 - \lambda_2)!$ or $(\mu_1 - \mu_2)!$ then $Y_{(p)}^{(\lambda,\mu)}$ is a linear $(m(s), 0, -, p)$-code.

3) If $p \geq 5$, then $Y_{(p)}^{(\lambda,\mu)}$ is a linear $(m(s), k_p, d_p, p)$-code, where $k_p = m(s)$ and $d_p = 1$.

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