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A COMPARATIVE ANALYSIS OF THE SEVERAL MATRIX FACTORIZATION PROCESS IN IMAGE RECONSTRUCTION AND A HOMOGENEOUS APPROACH TO THE FOURIER SERIES

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ABSTRACT

Over the last few decades, image reconstruction has become an interesting field for the development of computer-based applications. The Decomposition of a matrix or matrix factorization is one of the most important components in many engineering and scientific applications. This technique is used to decompose one matrix into more than one matrix. One can efficiently solve a system of equations based on matrix factorization, and this, in turn, is the foundation of the inverse matrix, which is a major component of several important algorithms. Matrix factorizations are widely applied in situations that involve solving linear systems, numerical linear algebra, rank estimation, image processing, image reconstruction etc. This paper attempts to analyze the techniques of matrix factorization or decomposition techniques used in image reconstruction based on their advantages, disadvantages, limitations and computation complexity. Some techniques are examined and a comparative evaluation of these strategies is presented. This paper also shows the homogeneity between the Fourier series and matrix factorization process in image reconstruction.

Keywords: *Matrix Factorization, Image Reconstruction, Fourier Series, Eigen Decomposition, Cholesky Decomposition, LU Decomposition, SVD.*

1. INTRODUCTION

In today's world, information is the key, and with the rapid change in information, and computer technology, problems need to be solved faster than ever. As every day more and more data are offered to the users, demand grows for computer systems capable of processing the data faster than before. Speed is crucial; data must be processed fast so that services can be provided in much less time by using powerful computers. Simulations of engineering and scientific problems need the faster speed of calculation; this is the reason behind the fact that the software developers have been dealing with parallel computing technologies and platforms while providing rich as well as novel experiences. Matrix decompositions are in use for finding least-square solutions and linear equations. In engineering, subjects like machine learning, big data analysis, collaborative filtering, Fuzzy modeling [1], and computer vision use matrix decompositions [2, 3, 4]. The size of the data has been growing exponentially, and as a result, demand for feasible methods to analyze large datasets has also been growing significantly. Factorization may be an important step in the process of such an analysis of large datasets [5, 6, 7]. Two primary reasons behind the complexities in analyzing large data structures are computational complicacy and high consumption of memory.

Matrix decomposition is one of the basic operations in the computational application. Matrix factorization refers to the process that produces the decomposition that transforms the given matrix into the canonical form [8]. Both applied statistics and linear algebra, which have extreme significance in engineering and science, are fundamentally based on matrix decomposition. Two of the main aspects of the process of matrix decomposition are analytical simplicity and computational convenience. In reality, the feasibility of matrix computations such as matrix determinant, matrix inversion, least-square fitting, and solving linear systems is low. Hence, it is always easier when a complex matrix computation problem is broken into multiple simple parts. Such a mechanism is found to be significantly useful in solving diagonal and triangular systems.

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An image contains redundant information, and it needs more space for storage. Thus the image compression plays a vital role in medical imaging [9]. Compression of data is essential in data storage, and it is of two types, namely lossless and lossy decomposition. The original data can be retrieved without losing of the compressed data in lossless data [10]. Lossy compression techniques entail a certain information loss, and more noise is introduced to the compressed image [11]. The neighboring pixel is correlated in images, and spatial values are obtained by the redundancy between the neighboring pixelvalues. The size of the image before and after decomposition has been calculated and also the compression ratio is calculated for the images[12].

In [13], the SVD has been used for MRI [14] image reconstruction. The performance of the reconstructed image is still useable in medical diagnosis. The eigen decomposition method is used in [15] for color-image reconstruction, where the principal component analysis with eign vectors is used for feature extraction. In a reconstruction of the attenuation map for the single-photon emission computed tomography (SPECT) [16], the Cholesky decomposition is used to improve performance. The LU decomposition is used in face recognition where the sample and the transpose of the sample have been decomposed by LU decomposition method [17]. In [18], the QR-decomposition algorithm is used for the image reconstruction of computed tomography (CT), where the factorized matrices enabled noise analysis.

The factorization techniques discussed for image reconstruction using all decomposed or factorized matrices or singular values. Therefore, image reconstruction requires much time, processing power and hence not computationally efficient. However, the singular top K-vectors captured most of the variations. Therefore, instead of using all the singular vectors, the image can be retrieved with top K-singular vectors, which leads to efficient image reconstruction. The approximation of a complex wave can be made by using a summation of sines and cosines waves, i.e., Fourier series; which is analogous to the Matrix reconstruction process. Therefore, the homogeneity of the Fourier series and matrix factorization can be easily illustrated.

2. MATRIX FACTORIZATION MODELS

In linear algebra, as discussed earlier, that factorizing matrix refers to finding two or more matrices. The original matrix can be arrived by multiplying these matrices. There are many types of matrix decompositions. Five decomposition models have been discussed in this paper.

2.1 QR decomposition
2.2 SVD decomposition
2.3 LU decomposition
2.4 Cholesky Decomposition
2.5 Eigen Decomposition

2.1 QR Decomposition

The QR factorization or decomposition is the process of decomposing a matrix into a triangular matrix, and an orthogonal matrix. A=QR, with Q being an orthogonal matrix (i.e.), and R is an upper triangular matrix. The factorization is unique when A is non-singular. There are various methods available for estimating the factorization of the QR. The Gram-Schmidt method is one of them.

In this method, columns of matrix A will be considered as the vectors.

Say,

$$\mathbf{A} = [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3 \ldots\ldots\ldots\ldots\mathbf{v}_n]$$

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

$$\mathbf{u}_2 = \frac{\mathbf{e}_2}{\|\mathbf{e}_2\|} \text{ Where, } \mathbf{e}_2 = \mathbf{v}_2 - (\mathbf{u}_1.\mathbf{v}_2)\mathbf{u}_1$$

$$\mathbf{u}_3 = \frac{\mathbf{e}_3}{\|\mathbf{e}_3\|}$$

Where, $\mathbf{e}_3 = \mathbf{v}_3 - ((\mathbf{u}_1.\mathbf{v}_3)\mathbf{u}_1 + (\mathbf{u}_2.\mathbf{v}_3)\mathbf{u}_2)$

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$$\mathbf{u}_i = \frac{\mathbf{e}_i}{\|\mathbf{e}_i\|}$$

Where,

$$e_i = v_i - ((u_1.v_i)u_1 + (u_2.v_i)u_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + (u_{i-1}.v_i)u_{i-1})$$

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$$\mathbf{u}_k = \frac{\mathbf{e}_k}{\|\mathbf{e}_k\|}$$

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Where,

$$e_k = v_k - ((u_1.v_k)u_1 + (u_2.v_k)u_2 + \dots\dots\dots\dots + (u_{k-1}.v_k)u_{k-1})$$

The QR factorization that results is

$$\mathbf{A} = [\mathbf{v_1\ v_2\ v_3 \dots\dots\dots v_n}]$$

$$= [e_1 e_2\ e_3 ..e_n] \begin{bmatrix} a_1.e_1 & a_2.e_1 & - & - & a_n.e_1 \\ 0 & a_2.e_2 & - & - & a_n.e_2 \\ 0 & 0 & & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & a_n.e_n \end{bmatrix}$$

$$= QR \text{ -----------(i)}$$

The matrix A is defined by AS=QR in QR decomposition method. $Q \in R^{N \times M}$ has orthonormal columns, $S \in R^{M \times M}$ is a permutation matrix, and $R \in R^{M \times M}$ is upper triangular. The QR decomposition is exceptionally calculated by the permutation matrix S. The values $|R(kk)|$ on the diagonal of R, called the R-values, are in descending order and they lean to follow the singular values of A. [19,20]

Another method is the Householder method, which is more robust and faster[21].

Householder Reflector:

Let $\mathbf{P} \in \mathbf{C}^{n \times n}$ is a matrix

$\mathbf{P = 1 - 2uu^*}$; $\mathbf{u} \in \mathbf{C^n}$ and Reflector Householder is: $\|\mathbf{u}\| = \mathbf{1}$

Householder Reflector properties:

i. $\mathbf{P = P^*}$; that is a hermitian matrix

ii. Since $\mathbf{P^2 = I}$, that is orthogonal and $\mathbf{P^{-1} = P^* = P}$

iii. A vector $\mathbf{O(n)}$ operation; If u is given: $\mathbf{Px = x - 2u(u^*x)}$

A QR factorization is considered to be a crucial tool to resolve linear systems of equations since the inevitability of unitary matrices and property like good error generation. There are many advantages of QR factorization. One can find the least square solution when there is no exact solution.

2.2 Singular Value Decomposition

Singular Value Decomposition (SVD) is a heavily used method to find multiple matrices from a single matrix.

Taking a matrix *A* of order *m* x *n in a* field *K*. It breaks down into three matrices: U, S, V [22,23].

$$\mathbf{A_{mn} = U_{mm} S_{mn} V_{nn}^T} \text{ ---------------------(ii)}$$

Here,

$\mathbf{A} \rightarrow$ Given Matrix

$\mathbf{U} \rightarrow$ Matrix of Eigen Vectors of the Matrix $\mathbf{AA^T}$

$\mathbf{V} \rightarrow$ Matrix of Eigen Vectors of the Matrix $\mathbf{A^T A}$

$\mathbf{S} \rightarrow$ Roots of non zeroEigen Values of the real Matrix $\mathbf{AA^T}$ or $\mathbf{A^T A}$

$\mathbf{V^T} \rightarrow$ Transpose of the Matrix $\mathbf{V}$

Unit vectors: $\|\mathbf{u_i}\| = \mathbf{1}$ and $\|\mathbf{v_i}\| = \mathbf{1}$

Where, $\mathbf{U = \{u_1, u_2, u_3, \dots\dots\dots\dots, u_n\}}$ and $\mathbf{V = \{v_1, v_2, v_3, \dots\dots\dots\dots, v_n\}}$ are orthogonal vectors ($\mathbf{U^T U = VV^T = I}$) and $\mathbf{S = \{\sigma_1, \sigma_2, \sigma_3, \dots\dots\dots, \sigma_n)}$ is a matrix whose diagonal entries are the singular values $(\lambda_i = \sigma_i^2)$ into decreasing order $\sigma_{1,} \geq \sigma_2 \geq \sigma_3 \geq \dots\dots\dots \geq \sigma_n$,which is real and positive. In SVD matrix *U* and *V* are observed to be real when the matrix A is real as well. In SVD, a matrix can be very closely approximated when the singular values' number is decreased. This feature can be used in case of data compression where truncated forms of *U, S, and V* can be stored in place of *A*. Singular values give information on the the image's intensity. More information can be obtained about an image if the singular value is increased. To reconstruct a high-resolution image, the maximum of an eigenvalue is calculated, and A matrix of the respective singular values is then used. Singular value needs to be chosen precisely as it determines the intensity of the information related to the reconstructed image. Consequently, scholars have given more importance to singular values and improved the method of selecting them [24, 25].

SVD is capable of offering low-rank approximation while considering that singular value, which is the largest one, and saves the greatest amount of energy in the image. Sub rank approximations can be obtained from this low-rank approximation [26, 27]. The matrix A can be

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approximated and represented as matrix $A_k$where k is the specific rank. There are several advantages of using SVD in matrix approximation. One of such benefits is stored only the approximation A_k of a matrix in place of the entire matrix A. This is similar to applications like water making and image compression.

2.3 LU decomposition

In LU decomposition, a matrix is factorized into two triangular matrices: lower and upper triangular matrix. LU method, as compared to the Gauss method, is found to be more efficient, but more complicated as well in the process of solving the equation system [28, 29].

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$LU = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

---------------------------------------(iii)

A system of linear equations can be solved with LU decomposition. The following steps are to solve a problem using the LU decomposition.

i. Create an equation $\mathbf{Ax} = \mathbf{b}$.
ii. Look for the decomposition of LU for A. This will bring into the equation $\mathbf{(LU)x} = \mathbf{b}$.
iii. Let there be $\mathbf{y} = \mathbf{Ux}$. Then figure out $\mathbf{Ly} = \mathbf{b}$ for y.
iv. A solution is eventually established for $\mathbf{Ax} = \mathbf{b}$

The LU decomposition is used to decompose of the data, and it removes the noise present in an image. The size of the image before and after decomposition has been calculated, and also the compression ratio is calculated for the images [12]. LU decomposition used the Dolittle algorithm to delete column from left to right. The solution is a lower triangular matrix unit which can be used to store the original matrix. This algorithm needs a floating-point computation of $2n^3/3$. The Block LU decomposition algorithm has divided the whole matrix into smaller blocks so that analysis can be performed simultaneously to boost efficiency. There are certain drawbacks to the fact that an ideal solution is not feasible (least squares). It could possibly have been chaotic. Cholesky decomposition is then analyzed in order to find an approximate solution.

2.4. **Cholesky Decomposition**

A matrix (symmetric, positive definite) is factorized into two matrices, one is a lower triangular matrix, and the other is its transpose.

Let A is a matrix of order 3x3

$$\mathbf{A} = (\mathbf{a_{ij}}) = \begin{pmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{pmatrix}$$

$$A = LL^T$$

$$= \begin{pmatrix} \mathbf{l_{11}} & \mathbf{0} & \mathbf{0} \\ \mathbf{l_{21}} & \mathbf{l_{22}} & \mathbf{0} \\ \mathbf{l_{31}} & \mathbf{l_{32}} & \mathbf{l_{33}} \end{pmatrix} \begin{pmatrix} \mathbf{l_{11}} & \mathbf{l_{21}} & \mathbf{l_{31}} \\ \mathbf{0} & \mathbf{l_{22}} & \mathbf{l_{32}} \\ \mathbf{0} & \mathbf{0} & \mathbf{l_{33}} \end{pmatrix}$$

$$= \begin{pmatrix} l^2_{11} & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l^2_{21}+l^2_{22} & l_{31}l_{21}+l_{32}l_{22} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l^2_{31}+l^2_{32}+l^2_{33} \end{pmatrix}$$

------------(iv)

The diagonal elements $\mathbf{l_{kk}}$ are measured in a form,

$$\mathbf{l_{11}} = \sqrt{\mathbf{a_{11}}}$$

$$\mathbf{l_{22}} = \sqrt{\mathbf{a_{22}} - \mathbf{l^2_{21}}}$$

$$\mathbf{l_{33}} = \sqrt{\mathbf{a_{33}} - (\mathbf{l^2_{31}} + \mathbf{l^2_{32}})}$$

Or in general:

$$\mathbf{l_{kk}} = \sqrt{\mathbf{a_{kk}} - \sum_{j=1}^{k-1} \mathbf{l^2_{kj}}}$$

There is also a type of calculation for the elements below the diagonal (where i> k)

$$\mathbf{l_{21}} = \frac{\mathbf{1}}{\mathbf{l_{11}}}\mathbf{a_{21}}$$

$$\mathbf{l_{31}} = \frac{\mathbf{1}}{\mathbf{l_{11}}}\mathbf{a_{31}}$$

$$\mathbf{l_{32}} = \frac{\mathbf{1}}{\mathbf{l_{22}}}(\mathbf{a_{32}} - \mathbf{l_{31}l_{21}})$$

That can be expressed in general formulas as well,

$$\mathbf{l_{ik}} = \frac{\mathbf{1}}{\mathbf{l_{kk}}}(\mathbf{a_{ik}} - \sum_{j=1}^{k-1}\mathbf{l_{ij}l_{kj}})$$

And another approach to factorize a Matrix by Cholesky Method [30,31] is

$$A = LDL^T$$

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L stands for unit lower triangular matrix and D indicates diagonal matrix

2.4 Eigen Decomposition

Eigen decomposition [32] refers to the factorization of a square matrix, from which a collection of eigen values and vectors are obtained.

The eigen decomposition only can be done if the matrix is diagonalizable and square. i.e.

$$D=P^{-1}AP=\begin{bmatrix} \lambda_1 & 0 & 0 & - & 0 \\ 0 & \lambda_2 & 0 & - & 0 \\ 0 & 0 & \lambda_3 & - & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \text{--------(v)}$$

Let A be the given Matrix; then A can be decomposed as

$$A = PDP^{-1} \quad \text{------------------------(vi)}$$

D is the diagonal matrix in which the diagonal elements are the eigen values of A, and P consists of its eigen vector of the given matrix

.

It is important to measure eigen values and eigen vectors in an image processing systems such as pattern recognition [33], image compression [34], face recognition [35] etc. The concept of eigen values is used to measure image sharpness [36].

3. EXPERIMENT RESULTS

3.1 Reconstructing a complex wave by adding more sine waves in different phases

A waveform can be established from a precisely selected set of sine wave components, on the contrary, a given complex periodic signal can be fragmented into a sequence of sine wave components for analysis. Eigenfaces are a superset of Fourier series, one of the fundamental ideas regarding mathematical computation and signal processing. This ideology must be explored to understand the basic functionality of Eigen faces. A signal when represented in a linear combination of complex sinusoids, it is known as Fourier series. Therefore, representing an image as a linear combination of a sequence of basis images (sines and cosines were based in the Fourier series) has to be the intent. The idea of Eigen faces is quite similar, however, not precisely the same.

Mathematically, a periodic signal $f(t)$ with period $T = 2L$ may be represented by a Fourier series, as follows:

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{a_0}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi}{L}t)+b_n\sin(\frac{n\pi}{L}t))}_{\text{AC value}}$$

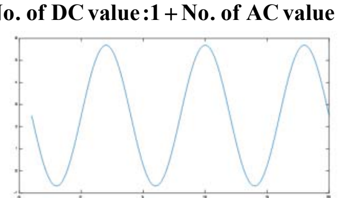
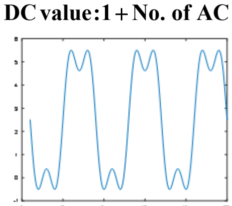
-------------(vii)

An example of an original complex wave [figure 10]

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

[Priod T =2L= 8]

The example as mentioned above were implemented in MATLAB (R2018a)

$f(t) = 2.5 + \frac{10}{\pi}\sin\frac{\pi t}{4}$ [No. of DC value :1 + No. of AC value :1]  *Figure 1*	$f(t) = 2.5 + \frac{10}{\pi}(\sin\frac{\pi t}{4} + \frac{1}{3}\sin\frac{3\pi t}{4})$ [No. of DC value :1 + No. of AC value :2]  *Figure 2*
$f(t) = 2.5 + \frac{10}{\pi}(\sin\frac{\pi t}{4} + \frac{1}{3}\sin\frac{3\pi t}{4} + \frac{1}{5}\sin\frac{5\pi t}{4})$ [No. of DC value :1 + No. of AC value :3]	$f(t) = 2.5 + \frac{10}{\pi}\sin\frac{\pi t}{4} + \frac{10}{3\pi}\sin\frac{3\pi t}{4} + \frac{10}{5\pi}\sin\frac{5\pi t}{4} + \frac{10}{7\pi}\sin\frac{7\pi t}{4}$ [No. of DC value :1 + No. of AC value :4]

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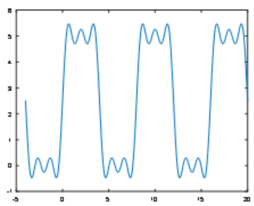

Figure 3

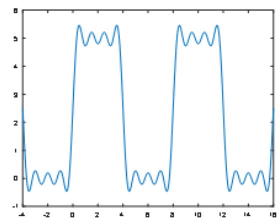

Figure 4

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{25}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

[No. of DC value : 1 + No. of AC value : 25]

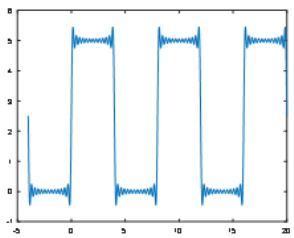

Figure 5

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{50}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

[No. of DC value : 1 + No. of AC value : 50]

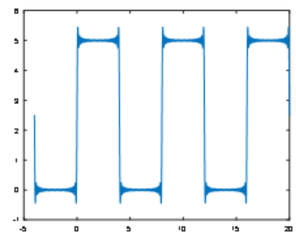

Figure 6

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{100}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

[No. of DC value : 1 + No. of AC value : 100]

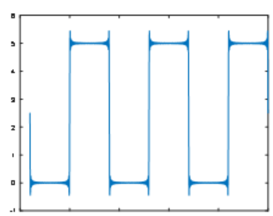

Figure 7

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{500}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

[No. of DC value : 1 + No. of AC value : 500]

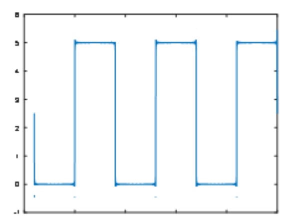

Figure 8

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{1000}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

[No. of DC value : 1 + No. of AC value : 1000]

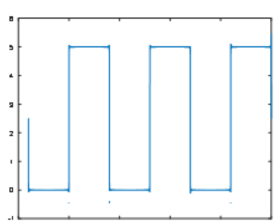

Figure 9

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{n}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{\text{AC value}}$$

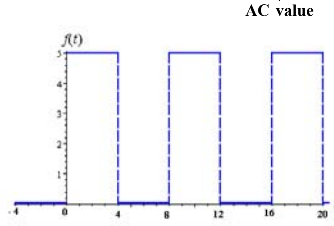

Figure 10 (Original Wave)

The approximation of a square wave can be made using a sequence of sines and cosines (summation result), as shown in the figures above. Anyone can see the gradual improvement of the square wave (image) as more AC signals are added. It is quite evident that we can rebuild the square wave explicitly using sines and cosines. The same method we can apply to image reconstruction.

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3.2 Reconstructing image with top-K singular vectors

An image is regarded as a function in which x and y are variables and ranges of magnitude between 0 and 255. The two-dimensional image $\mathbf{f(x,y)}$ is split in rows; M and columns N. The eigen values of such matrix demonstrate the key feature of the images.

A sizable matrix of grayscale values is called an image. Each image is assigned to a particular pixel and color. A matrix of m by n nomenclature has n entries repeating m times (a large number if an image is represented). An image is basically a large rectangular matrix. The grayscales of every pixel in the image are portrayed by the entries. We can consider a pixel to be a tiny square, wherefrom the lower-left corner, there are i steps across, and j steps upwards. It's grayscale is always a number and very often a whole number in the range ($\mathbf{0 \le a_{ij} < 256 = 2^8}$). Thus, when the computer writes 255 ($\mathbf{255 = 11111111}$) in binary notation, the number will have eight 1's. An image will become a m by n matrix with 256 values for each entry when it has m times n pixels where every pixel uses 8 bits for it's grayscale. Thus, it can be said that an image is a large matrix. 8 (m)(n) bits of information are required to copy an image accurately. An image in the RGB color model stores an image in three matrices, one each for Red, Green, and Blue color.

The singular top k-vectors capture most of the variation. Therefore, instead of using all the singular vectors and multiplying them, as shown in SVD decomposition, the image can be reconstructed with top K singular vectors.

A linear combination of an image A (say) can be expressed as

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots\ldots\ldots + \sigma_r u_r v_r^T \text{-----} \text{-----} + \sigma_k u_k v_k^T \quad \text{----------(viii)}$$

An approximate image can be gained by using fewer singular values. Say, the number of singular values is 10 i.e. $\sigma_1, \sigma_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\sigma_{10}.$ Then,

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots\ldots\ldots\ldots + \sigma_{10} u_{10} v_{10}^T$$

Using 5 singular values then

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots\ldots\ldots\ldots + \sigma_5 u_5 v_5^T$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots\ldots\ldots\ldots\ldots + \sigma_k u_k v_k^T$$

$$\Rightarrow A = A_1 + A_2 + A_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + A_k$$

That is, A matrix of rank k has exactly k singular values.

An experiment with an example

In our experiment, we have used a face 'lena.jpg' [figure 11]. First, the top 5 singular vectors (SV5)[figure 13] were used and the approximate image was retrieved using matrix multiplication, as shown below. Anyone looking at the reconstructed image will see that the top 5 components are not enough to reconstruct it. When someone uses the top 20 singular vectors [figure 16] and sees what the reconstructed image looks like. With the top 50 singular vectors [figure 19], one can see that the essence of the original image has been captured nicely. The quality of the reconstructed image would improve when using more top singular vectors. Here is a comparison of the reconstructed image using the different number of top components. Anyone can see the improvement in the quality of the image as one adds more singular vectors initially and then it kind of saturate suggesting that we don't gain much adding more components as the variance explained is smaller after the top components.

The line spectrum of the top 50 singular values is shown in the bar diagram along with their variance [Figure 12]

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Figure 11: Original Image

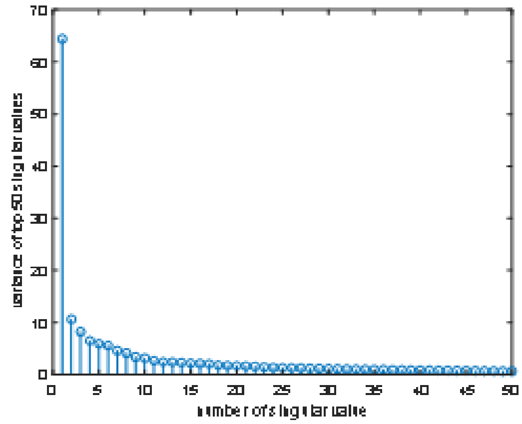
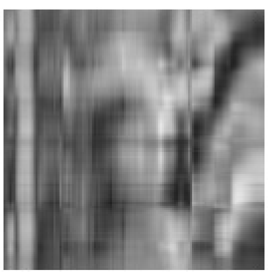


Figure 12: Number Of Singular Values With Variance

As the singular value (SV) increases, such as SV-5, SV-10, SV-15, SV-20, SV-30 etc., the images below gradually take the shape of the actual image. This can be shown appropriately in the figures below using Singular Value Decomposition (SVD).



SV-5
Figure 13

SV-10
Figure 14

SV-15
Figure 15

SV- 20
Figure 16

SV- 30
Figure 17

SV- 40
Figure 18

SV-50
Figure 19

SV-100
Figure 20

Figure: An example of how the image has been reconstructed using SVD

4. THE HOMOGENEITY OF THE FOURIER SERIES AND MATRIX FACTORIZATION IN IMAGE RECONSTRUCTION

Firstly, As it is seen in Section 3.1, a complex web is gradually presenting its own precisely structured web, while more sine waves are added with different frequencies and different amplitudes. In exactly the same way, It is clear that the more singular values are added, the clearer the image becomes and the more its original shape is revealed in section 3.2. The characteristics of the wave [figure 10] and picture [figure 11] are different in real life but both are images from the graphics point of view. From that point of view, it can be said that they are homogeneous.

5. DISCUSSION

The value of n equals to 50 in section 3.1 makes it clear that what the original shape of the web [figure 6] might look like. Similarly, when the value of K is taken as 50 in section 3.2, it [figure 19] gives a clear idea of whose image or which.

The reduction of the dimensions can be made to K rather than M (K<M) as an accurate reformation of the face is not necessary. This is achieved by selecting the K Eigen values, which comprise the largest associated eigen values, which now span a K dimensional, diminishing the space and the computational time.

There are M Eigen values. There are K most significant Eigen values by the use of which we can approximate a face with considerable accuracy (K<M). Thus we can consummate that if several eigen values are chosen between 20 to 30, a reassembled image is achieved which can effortlessly be identified

Singular Value Decomposition (SVD) technique is used in the image matrix to get the image. This example clearly shows that we can get the same looking image by omitting the smallest singular values. To summarize, we see how we can use SVD to decompose an image and reconstruct it.

To the naked eye, there seems to be no difference between reconstructed and original images. Mathematically, when the matches between the two images are calculated using PSNR and MSE, the difference in a numerical value is captured.

5.1. Significance of Eigen Value Approach

To represent an image, one needs M Eigen vectors. Among these M Eigenvectors, only K Eigenvectors must be chosen so that K is always less than M to represent the face space spanned by images. As a consequence, faster face recognition and also a reduction in face space dimensionality are expected. Only K Eigenvectors can be selected with the highest Eigen values. Higher Eigen values tend to represent the highest face variation in the respective Eigenvector direction; thus, it is necessary to contemplate this Eigenvector to represent face space. As one can get negligible information from the lower eigen values, they can be neglected to reduce the dimension of free space. The consequence of selecting various values of K is provided in the results section. The selection of K values (maximum Eigen values) is of paramount importance to reduce high rates of error in the face reconstruction process. SVD is calculated more effectively for large matrices than Eigen-decomposition. SVD factors have both positive and negative inputs, but the components of non-negative matrix factorization (NMF) is purely positive.

5.2. Performance evaluation of Matrices factorization

Measure the closeness between the original image and the reconstructed image.One approach to this is Pixel by pixel comparison. That is not, however, a standard process. There are several methods for comparing the two images. PSNR, MSE, SSIM of QIUI, Histogram comparison, BP, LTP, LDP, LTrP and GLTrP that are used to detect similarities between images. Mean square error (MSE) and Peak signal-to-noise ratio (PSNR) [37, 38] is the best-known process. This ratio is used as an indicator of the consistency between the compressed image and the original image. The higher the value of PSNR, the better the quality of a compressed or reconstructed image. MSE is the cumulative square error between the original and the compressed image, while PSNR is a peak error measure. The lower the MSE value shows the lower the error. The following equations are used to find the values of MSE and PSNR for the original image and the new image.

$$\mathrm{MSE} = \frac{1}{M.N}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}[F(i,j)-I(i,j)]^2 \text{ ------(ix)}$$

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$$PSNR = 10\log_{10}\frac{(255)^2}{MSE} \quad \text{--------}(x)$$

Where:

I (i, j): Original image.
F (i, j): New image (de-noised image)
M and N: The number of rows and columns in the input images is M and N
255: Max pixel value of an image in grayscale that is used in this work which equals to 255

5.2.1. An image could be reconstructed from Eigen decomposition of a matrix if and only if the matrix is square. But not even every square matrix has an eigen decomposition. Some eigen values are found to be negative when reconstructing the image using the Eigen decomposition method. As the input matrix is not symmetrical, it gives the complex-valued eigen values; that make the use of eigen value decomposition more difficult. Only using the standard MATLAB library can overcome the perceived complexity.

5.2.2. It has been found that SVD-decomposition is a simpler task than Eigen value-decomposition method as SVD provides positive and real singular values that facilitate the application of simple filters and compression. The SVD can be applied to square, an arbitrary, reversible or non-reversible matrix. In SVD, the image is being restored or reconstructed even though the matrix rank is down, but it is not possible to reconstruct the image by reducing the rank of a matrix in QR, LU, Cholesky method.

5.2.3. The drawback of Cholesky decomposition is that matrix must be positive definite. There are some positive aspects of Cholesky decomposition, such as ignoring square roots and reducing data dependence.

5.2.4. The matrix symmetry lets a machine store more than half of its components in memory and reduces the number of activities by two factors compared to Gaussian elimination. Square-root activities are not required for LU-decomposition when using symmetric properties and It is therefore marginally faster than the Cholesky decomposition, but it allows the entire matrix to be processed.

6. CONCLUSIONS AND FUTURE WORKS

Matrix decomposition is a popular method for image reconstruction. However, all the decomposed matrices are not equally essential for image reconstruction. For computational efficiency, the reduced number of decomposed matrices or singular values can be used for image reconstruction, which is analogous to the AC values of Fourier series.

In the image reconstruction process, the Eigen value approach is quick as well as simple, and works effectively under a constrained setting. It is a realistic approach that addresses the problems related to image reconstruction. It can be reduced dimensionality by using Eigen value approach. The smaller the image space's dimensionality, the better is the image reconstruction.

It has also been observed that considering the Eigenvectors with higher K Eigen values in place of all M Eigenvectors does not significantly impact efficiency. Hence, considering the Eigen space for the images in lower dimensions is necessary. The other critical aspect is the acceptability of K, which is likely to be significant depending on application type as well as error rate. More studies need to be conducted on selecting the appropriate value of K. Depending on the image reconstruction application, this value of K can be varied. Hence, different approaches need to be tested to make the right choice of K.

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