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CONFORMABLE DECOMPOSITION METHOD FOR TIME-SPACE FRACTIONAL INTERMEDIATE SCALAR TRANSPORTATION MODEL

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ABSTRACT

This paper considers the analytical solutions of a time-space fractional intermediate scalar transportation model via the application of Conformable Decomposition Algorithm. The method is a blend of Adomian Decomposition coupled with fractional derivative defined in conformable sense; herein referred to as CADM. Illustrative examples (cases) are considered in order to clarify the effectiveness of the proposed method, and the solutions are presented in infinite series form with high level of convergence to the exact form of solution.

Keywords: *Advection-Dispersion Model, Adomian Decomposition Method, Fractional Calculus, Conformable Derivative*

1. INTRODUCTION

The Advection Dispersion equation (ADE) also known as Advection diffusion equation is a well-known method in applied engineering and physics. It is mostly applied in the area of transportation modelling. This equation can be used to solve and analyse time and space differences in a particle activity [1-2]. Mostly, the results of this particular equation equipped with boundary conditions require the application of numerical methods. Meanwhile, the corresponding model has been investigated by many scholars using the stochastic, analytic and numerical approaches [2-4]. It is a generally accepted fact that in life fluids moves through the combined effect of advection and diffusion motion. The coupling of Fractional Complex Transform (FCT) with modified version of differential transform method has been applied for exact solutions of time fractional ADE [5].

Several authors have also examined the governing equation that is classically and traditionally used to model the dissolve solutes, mainly for this equation to be well positioned some important assumptions must be valid [6-7].

ADE has been used to illustrate a dynamical system, for example, a groundwater pollution model, coupled with a generalised analytical solution for one-dimensional solute transport in countable spatial domain. This method can also take two-dimensional form, which can be approximated but it poses a lot of challenges for the researcher and it is equally very important to consider. This is in recent time has actually motivated a lot of strong research work [8-9].

Over the years, many scholars have developed Fractional Advection-Dispersion Equation (FADE) from the advection-dispersion equation, some solved a notable number of problems using the finite difference approach, while some considered an open channel shallow water, and some set of scholars have also combined ADE with KDV which was solved numerically by the use of (FCCS) scheme and their results proffered significant solutions [9-16].

The classical advection-dispersion model takes the form:

$$\begin{cases} \dfrac{\partial w}{\partial \tau} = -u_1 \dfrac{\partial w}{\partial \xi} + u_2 \dfrac{\partial^2 w}{\partial \xi^2}, \xi > 0 \\ w(\xi,0) = w_0 \end{cases} \quad (1)$$

where $w(\xi,\tau) = w$ represents the dissolved concentration, u_1 and u_2 are Darcy velocity and dispersion coefficient respectively.

Sayed and Behiry and Raslan [17] extended (1) to time-fractional form:

$$\begin{cases} \dfrac{\partial}{\partial t} h(x,\tau) = \phi(\xi,\beta) D_x^\beta h(x,\tau), \tau > 0, \ x \in (0,1) \\ \phi(\xi,\beta) = \phi = \xi(\beta-1) - (2-\beta) \\ h(x,0) = e^{-x}. \end{cases} \quad (2)$$

In their work [17], they used (2) to describe the intermediate process that occurs between advection and dispersion using fractional derivative in the Caputo sense with the aid of Adomian's decomposition method. In this paper, (1) and (2) will be considered for extension in the direction of time-space fractional derivative as regards conformable view of fractional derivative. Hence, time-space fractional intermediate scalar transportation model of the form:

$$\begin{cases} D_t^\alpha h(x,t) = \phi(\xi,\beta) D_x^\beta h(x,t), t > 0, \ x \in (0,1) \\ \phi(\xi,\beta) = \phi = \xi(\beta-1) - (2-\beta), \\ \beta \in [1,2], \alpha \in (0,1] \\ h(x,0) = h_0. \end{cases} \quad (3)$$

In considering the solutions of classical, and fractional differential models, various solution methods include the views of [18-31]. Here, Adomian Decomposition Method coupled with fractional derivative defined in conformable sense (CADM) is mainly applied for the first time, regarding analytical solution of a time-space fractional intermediate scalar transportation model.

The structure of the remaining parts of the paper will be as follows, we have in section 2: a brief notion of conformable differential operator and its properties, section 3 is on the proposed solution method (CADM), section 4 contains the application while section 5 is on concluding remarks.

2. BASIC NOTIONS OF CONFORMABLE DIFFERENTIAL OPERATORS [32-35]

Definition (1): For a function $h:[0,\infty) \to \mathbb{R}$, the conformable derivative of h of order $\alpha \in [0,\infty)$ is defined as:

$$C_\alpha(h)(t) = \lim_{\varepsilon \to 0} \left(\frac{h(t + \varepsilon t^{1-\alpha}) - h(t)}{\varepsilon} \right), \ \forall t \ge 0. \quad (4)$$

2.1 Properties of Conformable Differential Operators CDOs

Let $h = h(t)$, $h_1 = h_1(t)$ and $h_2 = h_2(t)$ be $\alpha-$differentiable functions at $t > 0$ for α. Then the following hold:

(P1): $C_\alpha(\lambda_1 h_1 \pm \lambda_2 h_2) = \begin{cases} \lambda_1 C_\alpha(h_1) \\ \pm \lambda_2 C_\alpha(h_2), \\ \lambda_1, \lambda_2 \in (-\infty,\infty). \end{cases}$

(P2): $C_\alpha(h = \lambda) = 0, \ \lambda \in (-\infty,\infty)$.

(P3): $C_\alpha(h_1 h_2) = h_1 C_\alpha(h_2) + h_2 C_\alpha(h_1)$.

(P4): $C_\alpha\left(\dfrac{h_1}{h_2}\right) = \dfrac{h_2 C_\alpha(h_1) - h_1 C_\alpha(h_2)}{h_2^2}$.

(P5): $C_\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}, \ \forall \gamma \in (-\infty,\infty)$.

(P6): $C_\alpha(h(t)) = t^{\gamma-1} h'(t), \ h'(t) = \dfrac{dh(\cdot)}{dt}$.

(P7): Suppose further that $h(t)$ is an n-times differentiable function at t, then:

$$C_\alpha(h(t)) = \begin{cases} t^{[\alpha]-\alpha} h^{([\alpha])}(t) \\ \alpha \in (n, n+1], \end{cases}$$

where $[\alpha]$ denotes the smallest integer such that $[\alpha] \ge \alpha$.

3. THE CONFORMABLE SENSE OF THE DECOMPOSITION METHOD [32-35]

Consider a general fractional (nonlinear) partial differential equation (NLFDE) of the form:

$$L_\alpha\left(h(x,t)\right)+R\left(h(x,t)\right)+N\left(h(x,t)\right)=q(x,t) \tag{5}$$

where $L_\alpha(\cdot)$ denotes a linear operator based on conformable derivative of order α, with respect to t, such that $\alpha\in(n,n+1]$, R is the remaining part of the linear conformable differential operator, N denotes the nonlinear operator, while $q(x,t)$ is the associated non-homogeneous part (source term).

Suppose $L_\alpha(\cdot)=C_\alpha(\cdot)$ is invertible such that $L_\alpha^{-1}(\cdot)$ exists, then (5) becomes:

$$\left\{\begin{array}{l} C_\alpha\left(h(x,t)\right)+R\left(h(x,t)\right) \\ +N\left(h(x,t)\right)\end{array}\right\}=q(x,t) \ . \tag{6}$$

Hence, by the differential property of the conformable derivative (P7), we have:

$$\left\{\begin{array}{l} t^{[\alpha]-\alpha}h^{([\alpha])}(t)+R\left(h(x,t)\right) \\ +N\left(h(x,t)\right)\end{array}\right\}=q(x,t). \tag{7}$$

$$\therefore \quad t^{[\alpha]-\alpha}\frac{\partial^{[\alpha]}h(x,t)}{\partial t^\alpha}=\left\{\begin{array}{l} q(x,t) \\ -\left(\begin{array}{l} R\left(h(x,t)\right) \\ +N\left(h(x,t)\right)\end{array}\right)\end{array}\right\}. \tag{8}$$

The inverse operator is defined as follows:

$$L_\alpha^{-1}(\cdot)=\int_o^t\int_o^{\tau_1}\cdots\int_o^{\tau_{n-1}}\frac{1}{\tau^{[\alpha]-\alpha}}(\cdot)\,d\tau_n d\tau_{n-1}\cdots d\tau_1. \tag{9}$$

So, applying (9) to both sides of (8) gives:

$$L_\alpha^{-1}\left(t^{[\alpha]-\alpha}\frac{\partial^{[\alpha]}h(x,t)}{\partial t^\alpha}\right)=L_\alpha^{-1}\left(\begin{array}{l} q(x,t) \\ -\left(\begin{array}{l} R\left(h(x,t)\right) \\ +N\left(h(x,t)\right)\end{array}\right)\end{array}\right). \tag{10}$$

$$\Rightarrow h(x,t)=\left\{\begin{array}{l} h(x,0)+L_\alpha^{-1}\left(q(x,t)\right) \\ -L_\alpha^{-1}\left(R\left(h(x,t)\right)+N\left(h(x,t)\right)\right)\end{array}\right\}. \tag{11}$$

For the decomposition of the solution, we write:

$$h(x,t)=\sum_{n=0}^{\infty}h_n(x,t) \tag{12}$$

while the nonlinear term with the Adomian polynomials A_n is defined as:

$$N\left(h(x,t)\right)=\sum_{n=0}^{\infty}A_n. \tag{13}$$

and A_n (the Adomian polynomials) is given as:

$$A_n=\frac{1}{n!}\frac{\partial^n}{\partial\varsigma^n}\left[N\left(\sum_{i=0}^{n}\varsigma^i h_i\right)\right]_{\varsigma=0} . \tag{14}$$

Thus, using (12-14) in (10) gives:

$$\sum_{n=0}^{\infty}h_n(x,t)=\left\{\begin{array}{l} h(x,0)+L_t^{-1}\left\{q(x,t)\right\} \\ -L_t^{-1}\left\{\left(\begin{array}{l} R\left(\sum_{n=0}^{\infty}h_n(x,t)\right) \\ +N\left(\sum_{n=0}^{\infty}A_n\right)\end{array}\right)\right\}\end{array}\right\}. \tag{15}$$

Hence, in recursive relation, we have:

$$\left\{\begin{array}{l} h_0=h(x,0)+L_t^{-1}\left\{q(x,t)\right\} \\ h_{n+1}=-L_t^{-1}\left\{\left(R(h_n)+A_n\right)\right\}, n\geq 0.\end{array}\right. \tag{16}$$

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and $h(x,t)$ is therefore confirmed as:

$$h(x,t)=\lim_{n\to\infty}\sum_{n=0}^{\infty}h_n \tag{17}$$

4. APPLICATIONS AND ILLUSTRATIVE EXAMPLES

Here, the CADM as proposed above will be applied to some time-space fractional advection-dispersion model (TSFADM) as follows:

Example 1: Consider the following form of TSFADM:

$$\begin{cases} D_t^{\alpha}h(x,t)=\phi(\xi,\beta)D_x^{\beta}h(x,t), t>0,\ x\in(0,1) \\ \phi(\xi,\beta)=\phi=\xi(\beta-1)-(2-\beta) \\ h(x,0)=e^{-x}. \end{cases} \tag{18}$$

Procedure: Let $D_t^{\alpha}h(\cdot)=C_{\alpha}h(\cdot)$ be applied to (18). Thus,

$$\begin{cases} C_{\alpha}h(x,t)=\phi D_x^{\beta}h(x,t) \\ h(x,0)=e^{-x}. \end{cases}, \tag{19}$$

By (P6), we have:

$$t^{1-\alpha}\frac{\partial h(x,t)}{\partial t}=\phi D_x^{\beta}h(x,t). \tag{20}$$

So, operating $L_{\alpha}^{-1}(\cdot)=\int_0^t \frac{1}{\tau^{1-\alpha}}(\cdot)\,d\tau$ on both sides of (20) gives:

$$h(x,t)=h(x,0)+L_{\alpha}^{-1}\left(\phi D_x^{\beta}h(x,t)\right). \tag{21}$$

By decomposing $h(x,t)$, we have:

$$\sum_{n=0}^{\infty}h_n(x,t)=h(x,0)+L_{\alpha}^{-1}\left(\phi\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\sum_{n=0}^{\infty}h_n(x,t)\right)\right). \tag{22}$$

Thus,

$$\begin{cases} h_0=h(x,0) \\ h_{n+1}=L_{\alpha}^{-1}\left(\phi\frac{\partial^{\beta}}{\partial x^{\beta}}\left(h_n(x,t)\right)\right), \\ n\geq 0. \end{cases} \tag{23}$$

Therefore, the recursive relation in (23) yields:

$$\begin{cases} h_0=h(x,0) \\ h_1=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_0)\right) \\ h_2=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_1)\right) \\ h_3=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_2)\right) \\ h_4=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_3)\right) \\ \vdots \\ h_k=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_{k-1})\right) \end{cases} \tag{24}$$

whence, for: $h_0=e^{-x}$, the following are obtained:

$$\begin{cases} h_0=h(x,0)=e^{-rx} \\ h_1=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_0)\right)=\phi(-1)^{\beta}e^{-x}\frac{t^{\alpha}}{\alpha}, \\ h_2=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_1)\right)=\phi^2(-1)^{2\beta}e^{-x}\frac{t^{2\alpha}}{2!\alpha}, \\ h_3=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_2)\right)=\phi^3(-1)^{3\beta}e^{-x}\frac{t^{3\alpha}}{3!\alpha}, \\ h_4=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_3)\right)=\phi^4(-1)^{4\beta}e^{-x}\frac{t^{4\alpha}}{4!\alpha}, \\ \vdots \\ h_k=\phi L_{\alpha}^{-1}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}(h_{k-1})\right)=\phi^k(-1)^{k\beta}e^{-x}\frac{t^{k\alpha}}{k!\alpha}, \\ k\in N^{+}\cup\{0\}. \end{cases} \tag{25}$$

Hence,

$$h_k(x,t)=\sum_{n=0}^{k}h_n=\sum_{n=0}^{k}\phi^n(-1)^{n\beta}e^{-x}\frac{t^{n\alpha}}{n!\alpha}. \tag{26}$$

$$\therefore\ h(x,t)=e^{-x}\sum_{n=0}^{\infty}\phi^n(-1)^{n\beta}\frac{t^{n\alpha}}{n!\alpha}$$
$$=e^{-x}\sum_{n=0}^{\infty}\begin{pmatrix}\xi(\beta-1)\\-(2-\beta)\end{pmatrix}^n(-1)^{n\beta}\frac{t^{n\alpha}}{n!\alpha}. \tag{27}$$

Equation (27) yields the solution of (18) in analytical form corresponding to the time-space fractional advection-dispersion.

In what follows, we present the graphical views of the solution for different values associated with the model parameters. This is considered for integer and fractional orders as contained in Fig. 1 through Fig. 8.

If $\beta\to1$, and $\alpha\to1$ (for integral order, not fractional order) then the Eq. (1) reduces to pure advection equation (one may see Eq. (16) of refer [17]) and for this Eq. (1) reduces to

$$h(x,t)=e^{-x+t}. \tag{28}$$

If $\beta\to2$, and $\alpha\to1$ (for integral order, not fractional order) then the Eq. (1) reduces to pure diffusion equation (one may see Eq. (17) of refer [17]) and for this Eq. (1) reduces to

$$h(x,t)=e^{-x+\xi t}. \tag{29}$$

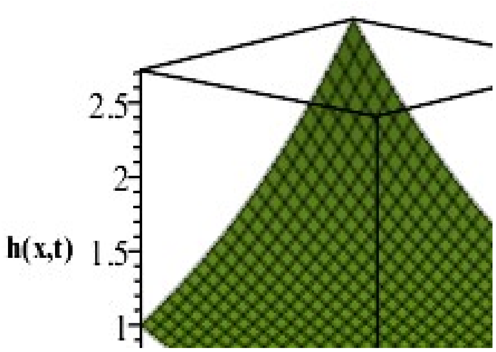

Figure 1. Plot of (27) at $\alpha=1,\ \&\ \beta=1$

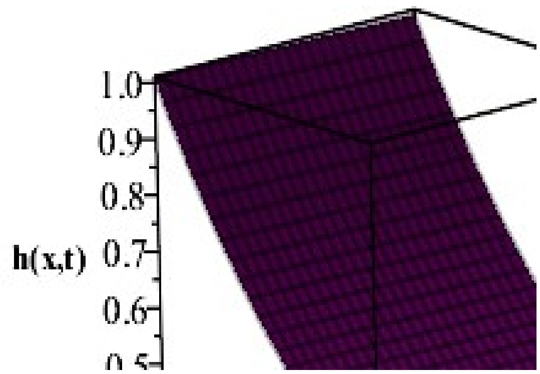

Figure 2. Plot of (27): $\alpha=1,\ \beta=2,\ \xi=0.01$.

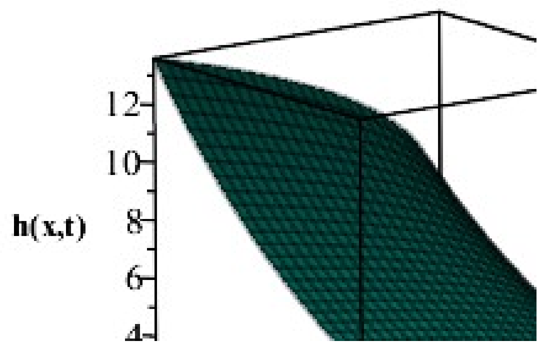

Figure 3. Plot of (27) at $\alpha=0.2, \&\ \beta=1$.

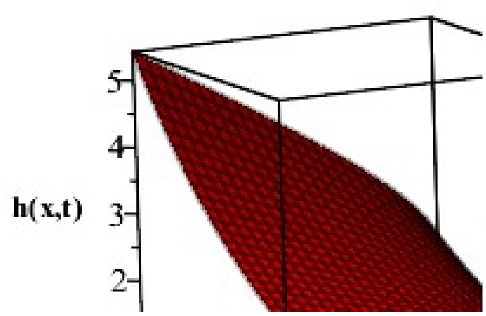

Figure 4. Plot of (27) at $\alpha=0.5, \&\ \beta=1$.

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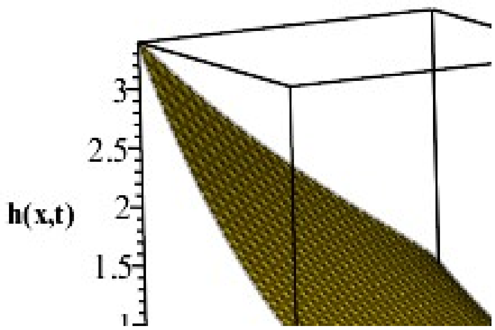

Figure 5. Plot of (27) at $\alpha = 0.8, \& \ \beta = 1$.

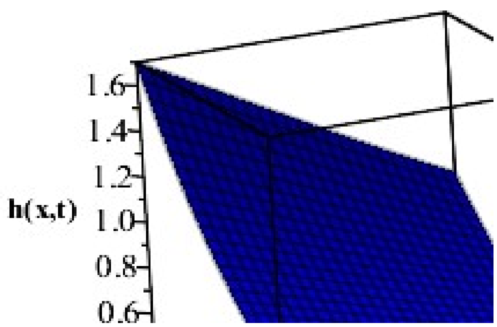

Figure 6. Plot of (27) at $\alpha = 1, \beta = 1.2$ and $\xi = 0.01$.

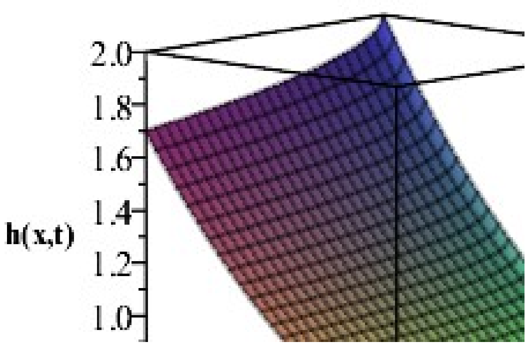

Figure 7. Plot of (27) at $\alpha = 0.5,\ \beta = 1.8,\ \xi = 0.01$.

Note: Figures 1 through 7, show the graphical representations of the solutions at different values of the associated parameters. All these are obtained via the maple software. It is remarked that the approach can be linked to other transformational techniques such as cosine transformation, slant transformation, Adomian decomposition, Haar transformation, Dobeshi-4 transform, Hankel transformation, hadamard transformation and so on with various aspect of applications including textural images, high way transportation, etc [36-40].

5. CONCLUDING REMARKS

In this paper, Adomian Conformable Decomposition Method (CADM) has been successfully implemented for the solutions of a time-space fractional intermediate scalar transportation model as posed by Sayed and Behiry and Raslan [17]. The results from the Illustrative applications considered showed the efficiency and effectiveness of the proposed technique, while the solutions expressed in infinite series form converged to their exact form of solution. The effects of the time and space fractional parameters were considered for the cases of pure advection and pure diffusion. The considered algorithm or proposed solution method has special feature with ease in overcoming the tedious nature posed by space-fractional models unlike the time-fractional models. Hence, our recommendation of this approach.

CONFLICT OF INTERESTS

No conflict of interest is declared by the authors.

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