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MODIFIED APPROACH FOR SOLVING RANDOM ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper deals with the derivation of a modified approach for solving initial value problems of the *n*-th order random ordinary differential equations by means of using the variational iteration method and numerical integration methods. In addition, the convergence of the obtained sequence of approximate solutions to the exact solution has been proved. Also, some illustrative examples are presented as a numerical simulation in order to illustrate the accuracy and applicability of the proposed approach.

Keywords: *Random Ordinary Differential Equations, Variational Iteration Method, Numerical Integration Method, Trapezoidal Rule.*

1. INTRODUCTION

Random ordinary differential equations (RODEs) are considered to be an ordinary differential equation (ODE) that includes a stochastic process in their vector field. They seem to have a shadow existence as a shadow of stochastic ordinary differential equations (SODEs) with Itô process, which have so many well-known real life applications. In particular, RODEs play a fundamental role in the theory of random dynamical systems and/or modern control theory, [1,2].

Random ordinary differential equations, unlike SODEs, can be analyzed path wise with deterministic calculus, which require further analytical treatments beyond that of classical theory of ODEs, [3]. Specifically, since the deriving process in a RODE has at most Hölder continuous sample path, then the solution as a sample path is continuously differentiable, but the sample path of the derivative is no more than Hölder continuous in time, [4]. The obtained vector field resulting after introducing the deriving stochastic process is at most of Hölder continuous in time, and no matter how smooth the vector field is in its original variables. Therefore, the RODE solutions do not have sufficient smoothness and after that to have in the usual sense Taylor expansions.

In older mathematical physics and engineering literatures, simple kinds of RODEs are investigated with the vector field being chosen depending on random variables rather than depending on stochastic processes. Such RODEs are still of great importance in the uncertainty qualification community, which are a special case of stochastic models and considered by Xiaoying Han and Peter E. Kloden, and will not be treated separately, [4]. So many applications of real life problems in biology, physics and engineering involving rate of change that depends on the interaction of the basic particles, changes, populations, etc. in addition to the stochastic or random effects, which will produce models that formulated as RODEs, such that the solution of the differential equation which are measured experimentally are in fact not predicted, [3].

Also, we may note that, there is a great amount of articles concerned with SDEs that appeared in recent years, such as [5-9]. Furthermore, since many SDEs have no explicit known analytical solution, so it is necessary to derive numerical methods to numerical approximations of the exact solutions. Cortés J. C. et al. in 2007 [10] proposed a numerical solution approach based on the difference scheme Euler's method problems and then they propose in 2011 [11] an improved Euler's method to solve such type of equations. Khudair A.

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R. et al. in 2011 [12, 13] used the Adomian decomposition method and variational iteration method (VIM) to solve certain types of second order RODEs. In 2013, Farnaoosh R. et al. [14] present the numerical solution of ODEs with Gaussian random coefficients and time-varying using Euler-Maruyama and Milston methods. Runge-Kutta method had been used by Nouri K. and Ranjbar H. in 2015 [15] to find the numerical solution of initial value problems of RODEs. The mean and variance of the approximate solutions of the second-order RODEs using homotopy analysis method have been proposed by Khudair A. R. et al. in 2016 [16]. Tchier F. et al. in 2017 [17] study a family of RDEs with boundary conditions using random fixed point theorem.

Consequently to the above discussion, closed form solution of RODEs seems to be very difficult to evaluate and hence an accurate and reliable numerical and/or approximate methods are necessary to solve such type of problems. Thus, in this paper, one of the well-known approximate-numerical methods will be used to solve RODEs by introducing a modified approach which consists of combining the VIM and numerical integration methods to approximate the integral of the correction functional. This approach will treat the difficulties that arise upon using the Wiener process related to the integral operator appeared in the VIM.

2. PRELIMINARIES

In this section and for completeness purpose, some fundamental and basic concepts related to the present work of this paper are presented, where more elementary concepts will not be given. As a notation, we will consider $(\Omega,\mathcal{F},P)$ as the probability space, which comprises the sample space Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω (called events) and a probability measure P on F.

We start with the following basic definitions:

Definition 1 [1, 8]:

"A random variable is a real valued function $X(\omega)$, $\omega \in \Omega$, which is measurable with respect to the probability measure P".

Definition 2 [2]:

"A stochastic process is a family of random variables $X_t(\omega)$ (or briefly X_t) of two variables. Let $t \in [t_0, T] \subset [0,\infty)$, $\omega \in \Omega$ on a common probability space $(\Omega,\mathcal{F},P)$, which assumes real values and is P measurable as a function of ω for each fixed t. The parameter t is interpreted as time. $X_t(.)$ represents a random variable on the above probability space Ω, while $X.(\omega)$ is called a sample path or trajectory of the stochastic process".

Definition 3 [18]:

"A stochastic process W_t, for all $t \in [0,\infty)$, is said to be a Wiener process or Brownian motion, if":

1. $P(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1$.
2. For $0 < t_0 < t_1 < \ldots < t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1} \ldots$, $W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary t and $h > 0$, $W_{t+h} - W_t$ has a Normal distribution with mean 0 and variance h.

For the probability space $(\Omega,\mathcal{F},P)$ suppose that $\omega : [0,T]\times\Omega \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued stochastic process with continuous sample paths. Also, let $f : \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^d$ be a continuous function, then a random ordinary differential equation (RODE) in $\mathbb{R}^d$ may be defined as [4]:

$$\frac{dx}{dt} = f(x,W_t(\omega)),\ x \in \mathbb{R}^d$$

which may be written as a non-autonomous ordinary differential equation (ODE):

$$\frac{dx}{dt} = F_\omega(t,x)$$

for almost every realization random event $\omega \in \Omega$.

For convenience, it will be assumed that the above RODE holds for all $\omega \in \Omega$, by restricting Ω to a subset of full probability if necessary and that f is often infinitely continuously differentiable function with respect to its variables, although m-times continuously differentiable with m sufficiently large would suffice. In particular, f is then locally Lipschitz in x, so the initial value problem:

$$\frac{dx}{dt} = g(x(t,\omega),W_t(\omega)),\ x(0,\omega) = x_0(\omega)$$

where the initial value x_0 is an $\mathbb{R}^d$-valued random variable that has a unique pathwise solution $x(t,\omega)$

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for every $\omega \in \Omega$, which will be assumed to exist on the finite time interval $[0,T]$ under consideration.

Now, the more general form of the RODE that will be considered and solved in this work will be taken to be of the form:

$$x^{(n)}(t;\omega) = f(t,x(t;\omega),x'(t;\omega),\ldots,x^{(n-1)}(t;\omega)) \quad \ldots(1)$$

with initial conditions:

$$x(t_0;\omega)=x_0,\ x'(t_0;\omega)= x_0',\ \ldots,\ x^{(n-1)}(t_0;\omega)= x_0^{n-1} \quad \ldots(2)$$

where $t \in [t_0,T]$, t_0, $T \in$ and $x(t;\omega)$ is a random process.

"Sufficient conditions that guarantee the existence and uniqueness of such solution are similar to those for ODEs. The situation is more complicated when the sample paths of the deriving noise W_t are only measurable in t, because the function $F_\omega(t,x)$ is only measurable in t and the existence and uniqueness of solutions must now be understood in the sense of Carathéodory. The solution of the RODE (1) is a stochastic process x_t on the interval $[0,T]$ and its sample paths $t \mapsto x_t(\omega)$ are continuously differentiable, but need not to be further differentiable of higher order, since the vector field $F_\omega(t,x)$ of the non-autonomous RODE is usually only at most continuous, but not differentiable in t, no matter how smooth the function f is in its variables".

Definition 4 [8]:

"A sequence of random variables $\{x_n(\omega)\}$, $n \in$ is said to be converges with probability one (denoted by P-w.p.1 or w.p.1) to $x(\omega)$ if":

$$p(\{\omega \in \Omega : \lim_{n\to\infty} x_n(\omega) = x(\omega)\}) = 1$$

which is also called almost sure convergence (for short a.s.).

Definition (1.8), [8]:

"A sequence of random variables $\{x_n(\omega)\}$, $n \in$, converges in probability to $x(\omega)$, if":

$$\lim_{n\to\infty} p(\{\omega\in\Omega : |x_n(\omega) - x(\omega)| \geq \varepsilon\}) = 0,\ \forall\ \varepsilon > 0$$

As it is mentioned above, the approximated iterative method that will be used to solve problem (1)-(2) is the VIM, which has the basic idea of considering the following general non-linear equation given in operator form [19], [20-25]:

$$L(x(t)) + N(x(t)) = g(t),\ t \in [t_0, T] \quad \ldots(3)$$

where L is a linear operator, N is a nonlinear operator and g is any given function which is called the non-homogeneous term. Now, rewrite equation (3) as follows:

$$L(x(t)) + N(x(t)) - g(t) = 0 \quad \ldots(4)$$

and let x_m be the m-th approximate solution of equation (4), then it follows that:

$$L(x_m(t)) + N(x_m(t)) - g(t) \neq 0 \quad \ldots(5)$$

The correction functional for (3) is then given by:

$$x_{m+1}(t) = x_m(t) + \int_{t_0}^{t} \lambda(s,t)\{L(x_m(s)) + N(\tilde{x}_m(s)) - g(s)\}\, ds \quad \ldots(6)$$

where λ is called the general Lagrange multiplier, which can be identified optimally through variational theory, the subscript m denoted the m-th approximation of the solution x and $\tilde{x}_m$ is considered as a restricted variation, i.e., $\delta\tilde{x}_m = 0$, where δ refers to the variation, [21-24, 26].

Now, in order to solve equation (6) by means of the VIM, the Lagrange multiplier λ must first be determined, which may be identified via integration by parts. Then the successive approximations x_m, for all $m = 0, 1, \ldots$, of the exact solution x will be obtained readily upon applying equation (6) with the a pre-evaluated Lagrange multiplier and starting with any selected function x_0 as the initial guess solution that satisfies the initial and/or boundary conditions. Then several approximations x_m, for all $m = 0, 1, \ldots$ will follows immediately and consequently the exact solution may be arrived since for the VIM we can prove that:

$$x(t) = \lim_{m\to\infty} x_m(t) \quad \ldots(7)$$

3. VARIATIONAL ITERATION METHOD FOR SOLVING RODEs

The VIM may be modified and improved to solve the RODE (1) with initial conditions (2). For this purpose, rewrite the RODE (1) in an operator form as:

$$L(x(t;\omega)) + N(x(t;\omega)) = g(t;\omega),\ t \in [t_0, T] \quad \ldots(8)$$

where L is a linear operator, N is a nonlinear operator, g is a known analytic function and x is the unknown function to be determined and hence the sequence of approximate solutions using the VIM will take the form:

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$$x_{m+1}(t;\omega) = x_m(t;\omega) + \int_{t_0}^{t} \lambda(s,t)\{L(x_m(s;\omega)) + N(\tilde{x}_m(s;\omega)) - g(s;\omega)\}\,ds \quad \ldots(9)$$

It is notable that the Wiener appeared in (9) implies that the integral operator will be difficult to evaluate, and therefore integration formulas may be used as it is seen in the next section. Application of the VIM for the RODE (1) may be achieved and then the general Lagrange multiplier is derived as in the next theorem:

Theorem 1:

If the RODE (1) has a unique solution and $x_m(t;\omega) \in C^n[t_0,T]$. Then the sequence of approximate solutions using the VIM is given by:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \frac{(-1)^n}{(n-1)!}\int_{t_0}^{t} (s-t)^{n-1}\left\{x_m^{(n)}(s;\omega) - f(s, x_m(s;\omega), x_m'(s;\omega),..., x_m^{(n-1)}(s;\omega))\right\} ds \quad \ldots(10)$$

where $x_m(t;\omega)$ is the m^{th} approximate solution.

Proof:

The proof will be achieved by using mathematical induction.

Since from (1), we have:

$$x^{(n)}(t;\omega) = f(t, x(t;\omega), x'(t;\omega), \ldots, x^{(n-1)}(t;\omega))$$

which may be rewritten as:

$$x^{(n)}(t;\omega) - f(t, x(t;\omega), x'(t;\omega),\ldots, x^{(n-1)}(t;\omega)) = 0 \quad \ldots(11)$$

Multiply eq.(11) be the general Lagrange multiplier $\lambda(s,t)$, yields to:

$$\lambda(s,t)\{x^{(n)}(t;\omega) - f(t, x(t;\omega), x'(t;\omega), \ldots, x^{(n-1)}(t;\omega))\} = 0 \quad \ldots(12)$$

Now, integrating both sides of eq.(12) will give:

$$\int_{t_0}^{t} \lambda(s,t)\{x^{(n)}(s;\omega) - f(s, x(s;\omega), x'(s;\omega), \ldots, x^{(n-1)}(s;\omega))\}\,ds = 0 \quad \ldots(13)$$

then, the correction functional for eq.(1) will reads as follows:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \int_{t_0}^{t} \lambda(s,t)\left\{x_m^{(n)}(s;\omega) - f(s, x_m(s;\omega), x_m'(s;\omega),..., x_m^{(n-1)}(s;\omega))\right\} ds \quad \ldots(14)$$

In this case, the value of λ cannot be evaluated easily from eq.(14), which will give a nonlinear functional. Therefore, the approximation of the correction function can be expressed in terms of the restricted variation $\tilde{x}_m(t;\omega)$ as follows:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \int_{t_0}^{t} \lambda(s,t)\left\{x_m^{(n)}(s;\omega) - f(s, \tilde{x}_m(s;\omega), \tilde{x}_m'(s;\omega),..., \tilde{x}_m^{(n-1)}(s;\omega))\right\} ds \quad \ldots(15)$$

Thus, by taking the first variation of eq.(15) with respect to the independent variable $x_m(t;\omega)$ and noticing that $\delta x_m(t_0;\omega) = 0$, yields to:

$$\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \delta\int_{t_0}^{t} \lambda(s,t)\left\{x_m^{(n)}(s;\omega) - f(s, \tilde{x}_m(s;\omega), \tilde{x}_m'(s;\omega),..., \tilde{x}_m^{(n-1)}(s;\omega))\right\} ds \quad \ldots(16)$$

where $\tilde{x}_m(t;\omega)$ is considered as a restricted variation, which means that $\delta\tilde{x}_m(t;\omega) = 0$ and consequently eq.(16) with $n = 1$ will be reduced to:

$$\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \delta\int_{t_0}^{t} \lambda(s,t)x_m'(s;\omega)\,ds \quad \ldots(17)$$

Hence, using the method of integration by parts on eq.(17) will give the following formula:

$$\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \lambda(s,t)\delta x_m(s;\omega)\big|_{s=t} - \int_{t_0}^{t} \lambda'(s,t)\delta x_m(s;\omega)\,ds$$

and then:

$$\delta x_{m+1}(t;\omega) = (1+\lambda(s,t))\delta x_m(s;\omega)\big|_{s=t} - \int_{t_0}^{t} \lambda'(s,t)\delta x_m(s;\omega)\,ds = 0$$

As a result, the following necessary and stationary conditions are obtained:

$$\lambda'(s,t) = 0,\ (1+\lambda(s,t))\big|_{s=t} = 0$$

which may be solved to give the Lagrange multiplier:

$$\lambda(s,t)\big|_{s=t} = -1$$

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Similarly, when $n = 2$

$$\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \delta \int_{t_0}^{t} \lambda(s,t) x_m''(s;\omega)\, ds$$

and also integrating by part twice, will give:

$$\begin{aligned}\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \lambda(s,t)\delta x_m'(s;\omega)\big|_{s=t} - \\ \lambda'(s,t)\delta x_m(s;\omega)\big|_{s=t} + \\ \int_{t_0}^{t} \lambda''(s,t)\delta x_m(s;\omega)\, ds \\ = (1-\lambda'(s,t))\delta x_m(s;\omega)\big|_{s=t} + \\ \lambda(s,t)\delta x_m'(s;\omega)\big|_{s=t} + \\ \int_{t_0}^{t} \lambda''(s,t)\delta x_m(s;\omega)\, ds\end{aligned}$$

and so using the ideas of the variational theory, the following necessary condition is obtained

$$\lambda''(s,t) = 0 \qquad \ldots(18)$$

with natural boundary conditions:

$$\lambda(s,t)\big|_{s=t} = 0\,,\ (1-\lambda'(s,t))\big|_{s=t} = 0 \qquad \ldots(19)$$

and solving eqs.(18) and (19) will yields the solution as the Lagrange multiplier:

$$\lambda(s,\, t) = s - t$$

Also, if $n = 3$, then eq.(16) will take the form:

$$\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \delta \int_{t_0}^{t} \lambda(s,t) x_m'''(s;\omega)\, ds$$

and so integration by parts three times will implies to the following initial value problem:

$$\lambda'''(s,\ t) = 0,\ \ \lambda(s,t)\big|_{s=t} = 0\,,\ \ \lambda'(s,t)\big|_{s=t} = 0\,,$$
$$(1+\lambda''(s,t))\big|_{s=t} = 0$$

which may be solved to give the Lagrange multiplier:

$$\lambda(s,\, t) = \frac{-(s-t)^2}{2}$$

Now, if $n = 4$, then:

$$\delta x_{m+1}(t;\omega) = \delta x_m(t;\omega) + \delta \int_{t_0}^{t} \lambda(s,t) x_m^{(4)}(s;\omega)\, ds$$

and hence upon carrying integration by parts four times will yields to the following initial value problem:

$$\lambda^{(4)}(s,\ t) = 0,\ \ \lambda(s,t)\big|_{s=t} = 0\,,\ \ \lambda'(s,t)\big|_{s=t} = 0\,,$$
$$\lambda''(s,t)\big|_{s=t} = 0\,,\ (1-\lambda'''(s,t))\big|_{s=t} = 0$$

which may be solved also to give the Lagrange multiplier:

$$\lambda(s,\, t) = \frac{(s-t)^3}{6}$$

By mathematical induction, the general form of the Lagrange multiplier related to the first variation given by eq.(16) will take the form:

$$\lambda(s,\, t) = \frac{(-1)^n}{(n-1)!}(s-t)^{n-1},\ n \in \square \qquad \ldots(20)$$

and substituting the value of λ from eq.(20) into the correction functional (14) will results the following iteration formula:

$$\begin{aligned}x_{m+1}(t;\omega) = x_m(t;\omega) + \\ \frac{(-1)^n}{(n-1)!}\int_{t_0}^{t} (s-t)^{n-1} \Big\{ x_m^{(n)}(s;\omega) - \\ f(s, x_m(s;\omega), x_m'(s;\omega), \ldots, \\ x_m^{(n-1)}(s;\omega)\Big\}\, ds \quad \blacksquare\end{aligned}$$

The convergence of the obtained sequence of the obtained sequence of approximate solution may be proved as in the next section.

4. CONVERGENCE ANALYSIS

The VIM which provides an analytical approximate solution is applied to various nonlinear problems [21-24, 26]. In this section, we shall study the convergence of the approximate solutions using the VIM (10) to the exact solution of the RODE (1) using an alternative approach which is presented in [27], applied to the VIM (10).

This approach is based upon rewriting the VIM (10) in operator form given by eq.(9) as:

$$\begin{aligned}x_{m+1}(t;\omega) = x_m(t;\omega) + \\ \frac{(-1)^n}{(n-1)!}\int_{t_0}^{t} (s-t)^{n-1} \big\{ L(x_m(s;\omega)) + \\ N(x_m(s;\omega)) - g(s;\omega)\big\}\, ds \qquad \ldots(21)\end{aligned}$$

Hence, the convergence analysis will be proceeds depending on Banach fixed point theorem and for this purpose, define the following operator:

$$\begin{aligned}A(x(t;\omega) = \frac{(-1)^n}{(n-1)!}\int_{t_0}^{t} (s-t)^{n-1} \big\{ L(x_m(s;\omega)) + \\ N(x_m(s;\omega)) - g(s;\omega)\big\}\, ds \qquad \ldots(22)\end{aligned}$$

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Also, define the following new components $v_k(t;\omega)$, for all $k = 0, 1, \ldots$

$$\left.\begin{aligned} v_0(t;\omega) &= x_0(t;\omega) = x_0 \\ v_1(t;\omega) &= A(x_0(t;\omega)) = A(v_0(t;\omega)) \\ v_2(t;\omega) &= A(v_0(t;\omega) + v_1(t;\omega)) \\ &\vdots \\ v_{k+1}(t;\omega) &= A(v_0(t;\omega) + v_1(t;\omega) + \ldots + v_k(t;\omega)) \end{aligned}\right\} \quad \ldots(23)$$

Hence, for the convergence of the VIM, one must have:

$$x(t;\omega) = \lim_{k\to\infty} x_k(t;\omega)$$

$$= \sum_{k=0}^{\infty} v_k(t;\omega)$$

Therefore, as a result, the solution of problem (8) can be obtained using the following series:

$$x(t;\omega) = \sum_{k=0}^{\infty} v_k(t;\omega) \quad \ldots(24)$$

The zeroth (initial) approximation $v_0(t;\omega)$ can be freely chosen if it satisfies the initial and boundary condition of the considered problem. The success of the method depends on the proper selection of the initial approximation $v_0(t;\omega)$. However, using the initial values $x^{(k)}(t;\omega) = c_k$, $k = 0$ 1, …, n − 1are preferably used for the selective zeroth approximation $v_0(t;\omega)$ as will be seen later, and for simplicity select the initial approximation as:

$$v_0(t;\omega) = \sum_{k=0}^{\infty} \frac{c_k}{k!}(t^k;\omega) \quad \ldots(25)$$

Thus, the exact solution may be approximated by the truncated series up to the m terms as:

$$x(t;\omega) = \sum_{k=0}^{m} v_k(t;\omega)$$

The convergence of the VIM, according to the above alternative approach, when applied to problem (8) in which the sufficient condition of the method and the error estimate are presented. The convergence of the obtained analytical approximate solution of the RODE (1) given by the correction functional (14) may be obtained depending on the following theorems which are proposed in [27], but applied for eq.(8).

Theorem 2:

"Let A, defined in (22), be an operator from a Hilbert space H to H. The series solution $x(t;\omega) = \sum_{k=0}^{\infty} v_k(t;\omega)$, defined in eq.(24) converges if there exists $0 < \gamma < 1$, such that:"

"$\|A(v_0(t;\omega) + v_1(t;\omega) + \ldots + v_{k+1}(t;\omega))\| \le \gamma\|A(v_0(t;\omega) + v_1(t;\omega) + \ldots + v_k(t;\omega))\|$"

that is, $\|v_{k+1}(t;\omega)\| \le \gamma\|v_k(t;\omega)\|$, for all $k = 0, 1, \ldots$

Theorem 3:

"If the series solution $\sum_{k=0}^{\infty} v_k(t;\omega)$ defined by eq.(24) converges, then it is an exact solution of the nonlinear problem (8)".

Theorem 4:

"Assume that the series solution $\sum_{k=0}^{\infty} v_k(t;\omega)$ defined in (24) is convergent to the solution $x(t;\omega)$. If the truncated series $\sum_{k=0}^{j} v_k(t;\omega)$ is used as an approximation to the solution $x(t;\omega)$ of problem (8), then the maximum error $E_j(t;\omega)$ is estimated as:"

$$E_j(t;\omega) \le \frac{1}{1-\gamma}\gamma^{j+1} \| v_0(t;\omega) \|$$

5. THE MODIFIED APPROACH

As it is expected, the Wiener process appeared in the RODE (1) will add more difficulties to the simulation of the approximate solution using the VIM (10). To avoid this difficulty, numerical integration methods will be used, say trapezoidal rule, which is not used previously to approximate the integral operator appeared in the VIM (10), which seems to be new, up to our knowledge. This will yields to:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \frac{(-1)^n}{(n-1)!}\int_{t_0}^{t} (s-t)^{n-1} \left\{ x_m^{(n)}(s;\omega) - f(s, x_m(s;\omega), x_m'(s;\omega), \ldots, x_m^{(n-1)}(s;\omega)) \right\} ds$$

and for simplicity of calculations, let:

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$$G(s,t,x_m(s;\omega)) = (s-t)^{n-1}\left\{x_m^{(n)}(s;\omega) - f(s,x_m(s;\omega),x_m'(s;\omega),\ldots, x_m^{(n-1)}(s;\omega)\right\}$$

We get:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \frac{(-1)^n}{(n-1)!}\int_{t_0}^{t} G(s,t,x_m(s;\omega))\,ds$$

and upon applying the trapezoidal rule for all $t_i \in (t_0,T]$, $i = 1, 2, \ldots, k$, then:

$$x_{m+1}(t_i;\omega) = x_m(t_i;\omega) + \frac{(-1)^n}{(n-1)!}\frac{h}{2}\left\{G(t_0,t_i,x_m(t_0;\omega)) + 2G(t_1,t_i,x_m(t_1;\omega)) + \ldots + 2G(t_{k-1},t_i,x_m(t_{k-1};\omega)) + G(t_k,t_i,x_m(t_k;\omega))\right\} \quad \ldots(26)$$

where $h = \frac{T-t_0}{k}$, $t_i \in (t_0,T]$ is the discretization step size and $k \in \square$ is the number of discretization points of the time interval $[t_0,T]$.

The sequence of approximate-numerical solution (26) will justify the proposed approach followed in this work for solving RODEs (1).

6. NUMERICAL SIMULATION

In this section, some numerical examples will be simulated and solved using the above proposed approach, in which two examples will be considered, the first one for linear RODE while the second example for nonlinear RODE.

Example 1:

To solve the linear RODE:

$$\frac{dx}{dt} = -x + \sin(W_t(\omega)),\ x(t_0,\omega) = 1,\ t \in [0,1]$$

in which the exact solution for comparison purpose is given by [4]:

$$x(t;\omega) = e^{-(t-t_0)} + e^{-t}\int_{t_0}^{t} e^{s}\sin(W_s(\omega))\,ds$$

Figure 1, presents the signal simulation of the discretized Brownian motion of the interval [0, 1] with number of discritizations $k = 100$, so that $h = 1/100$.

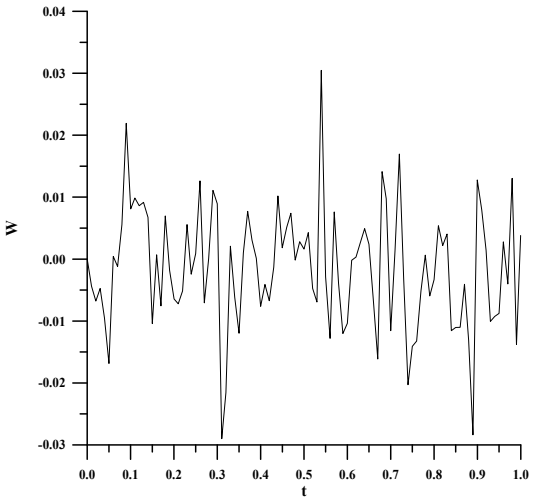

Figure 1. The discretized Brownian bath for 100 generation.

Therefore, using the approximate numerical solution using the VIM given by eq.(10), one may get:

$$x_{m+1}(t;\omega) = x_m(t;\omega) - \int_0^t \left\{x_m'(s;\omega) + x_m(s;\omega) - \sin(W_s(\omega))\right\} ds\ ,\ t \in [0,1]$$

which may be simplified in connection with the trapezoidal rule to give for all $m = 0, 1, \ldots$:

$$x_{m+1}(t;\omega) = x_m(t;\omega) - \int_0^t x_m'(s;\omega)\,ds - \int_0^t x_m(s;\omega)\,ds + \int_0^t \sin(W_s(\omega))\,ds$$

$$= x_m(0;\omega) - \int_0^t x_m(s;\omega)\,ds + \frac{h}{2}\left\{\sin W_{t_0}(\omega) + 2\sin W_{t_1}(\omega) + \ldots + 2\sin W_{t_{k-1}}(\omega) + \sin W_{t_k}(\omega)\right\}$$

where h is the step size which is taken in this example with $k \in \square$ (say $k = 100$)and hence $h = \frac{T}{k}$, $t_i \in (0,1]$. The simulation process with Winner process will yields the results presented in Table 1, which are given for $t = 0, 0.1, \ldots, 1$. In addition, Figure 2 provides graphical illustration for the exact and the first five approximate solutions.

Table 1. The exact and the first five approximate solutions of example 1 using the modified VIM.

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t	*Exact Solution x*	*Approximate solution x_1*	*Approximate solution x_2*	*Approximate solution x_3*	*Approximate solution x_4*	*Approximate solution x_5*
0.0	1	1	1	1	1	1
0.1	0.90474	0.89988	0.90478	0.90451	0.90441	0.90431
0.2	0.81885	0.80011	0.82021	0.81897	0.81913	0.81922
0.3	0.74103	0.70021	0.74536	0.74101	0.74151	0.74164
0.4	0.67001	0.59967	0.67947	0.66858	0.66942	0.66912
0.5	0.60638	0.49979	0.62468	0.60372	0.60619	0.60581
0.6	0.54861	0.39974	0.57963	0.54348	0.54873	0.54794
0.7	0.4964	0.29973	0.54465	0.48734	0.49722	0.49569
0.8	0.44868	0.19923	0.51908	0.43334	0.45008	0.447
0.9	0.40537	0.0986	0.50346	0.38126	0.40806	0.40256
1.0	0.36667	1.20215×10^{-3}	0.49848	0.33241	0.37312	0.36432

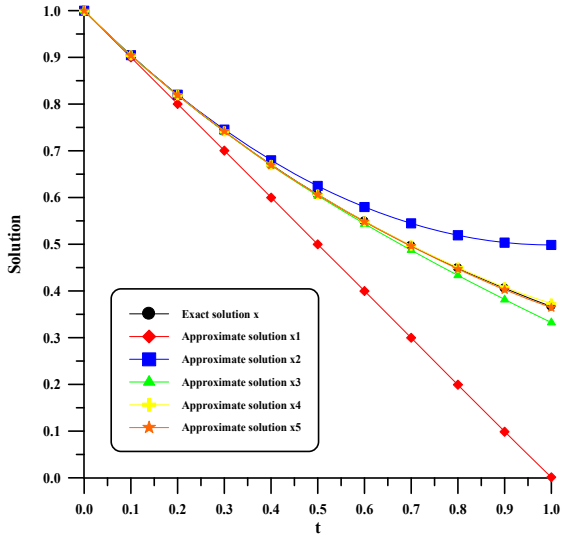

Figure 2. Comparison between the exact and the first five approximate solutions of example 1.

Example 2:

To solve the RODE:

$$\frac{dx}{dt} = -x^2 + \sinh(W_t(\omega)),\ x(t_0,\omega) = 1,\ t \in [0,1]$$

Therefore, using the approximate numerical solution using the VIM given by eq.(10), one may get:

$$x_{m+1}(t;\omega) = x_m(t;\omega) - \int_0^t \{x'_m(s;\omega) + x_m^2(s;\omega) - \sinh(W_s(\omega))\}\,ds,\ t \in [0,1]$$

which may be simplified in connection with the trapezoidal rule to give:

$$x_{m+1}(t;\omega) = x_m(t;\omega) - \int_0^t x'_m(s;\omega)\,ds - \int_0^t x_m^2(s;\omega)\,ds + \int_0^t \sinh(W_s(\omega))\,ds$$

$$= x_m(0;\omega) - \int_0^t x_m^2(s;\omega)\,ds + \frac{h}{2}\{\sinh W_{t_0}(\omega) + 2\sinh W_{t_1}(\omega) + \ldots + 2\sinh W_{t_{k-1}}(\omega) + \sinh W_{t_k}(\omega)\},$$

$$m = 0, 1, \ldots$$

where h is the step size which is taken in this example with $k = 100$ and hence $h = \frac{T}{k}$, $t_i \in (0,1]$. The simulation process with Winner process will yields the results presented in Table 2, which are given for $t = 0, 0.1, \ldots, 1$. It may be seen that, the results of the approximate numerical solutions given in Table 2 are converge to certain solution, which is the exact solution, as it is seen also from Table 3, which give the absolute error between the successive approximate solutions up to the 6th iterative solution. Moreover, Figure 3 provides graphical illustration for the exact and the first four approximate solutions.

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Table 2. The first six approximate solutions of example 2 using the modified VIM.

t	Approximate solution x_1	Approximate solution x_2	Approximate solution x_3	Approximate solution x_4	Approximate solution x_5	Approximate solution x_6
0.0	1	1	1	1	1	1
0.1	0.89988	0.90969	0.90895	0.90888	0.90879	0.90869
0.2	0.80011	0.83752	0.83324	0.83371	0.83376	0.83384
0.3	0.70021	0.78131	0.76811	0.76996	0.7699	0.77004
0.4	0.59967	0.73822	0.70946	0.71406	0.71326	0.71316
0.5	0.49979	0.70807	0.65751	0.66764	0.66593	0.66605
0.6	0.39974	0.68769	0.60882	0.62746	0.62392	0.62434
0.7	0.29973	0.67538	0.56245	0.59316	0.58671	0.58773
0.8	0.19923	0.66853	0.5167	0.56318	0.55232	0.55415
0.9	0.0986	0.66559	0.47152	0.53775	0.52092	0.52403
1.0	–0.00152	0.66514	0.42713	0.51736	0.49291	0.4981

Table 3. The absolute error between the successive approximate solutions of example 2.

t	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	$\lvert x_5 - x_4\rvert$	$\lvert x_6 - x_5\rvert$
0.0	0	0	0	0	0
0.1	0.00981	0.00074	0.00007	0.00009	0.00001
0.2	0.03741	0.00428	0.00047	0.00005	0.00008
0.3	0.0811	0.0132	0.00185	0.00006	0.00014
0.4	0.13855	0.02876	0.0046	0.0008	0.0003
0.5	0.20828	0.05056	0.01013	0.00171	0.00012
0.6	0.28795	0.07887	0.01864	0.00354	0.00042
0.7	0.37565	0.11293	0.03071	0.00645	0.00102
0.8	0.4693	0.15183	0.04648	0.01086	0.00183
0.9	0.56699	0.19407	0.06623	0.01683	0.00311
1.0	0.66666	0.23801	0.09023	0.02445	0.00519

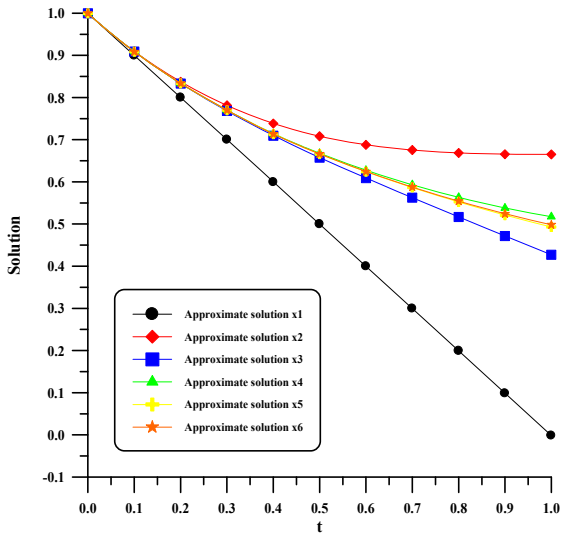

Figure 3. Comparison between first six approximate solutions of example 2.

7. CONCLUSIONS AND FUTURE WORK

The proposed approach followed in this paper is very efficient for solving stochastic and RODEs, which give reliable results for such type of problems consisting of Wiener process in comparison with the exact solution of example 1 (test example) or the absolute error between the successive approximate solutions of example 2. This approach may be used effectively to solve other type of problems which are given in operator form, which are so difficult to handle and then integrated.

In addition for future work, other numerical integration methods may be used instead of the trapezoidal rule, such as, Simpsons rule or Gauss quadrature integration methods.

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