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REVIEW ON DIFFERENTIAL SUBORDINATION

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ABSTRACT

The complex analysis is one of the beautiful and important subject in the mathematics because of its applications which are not only in several aspects of the analysis, but even in several areas of the mathematics and science in general. Recently, much interesting has been given to investigate the differential inequalities which are containing the functions with their derivatives, and that field has been developed in the last sixty years. In the theory of the complex functions, the characterization of the given analytic function can be determined by using the differential inequality, and also by the differential inequality, most of the geometric properties can be described in the theory of geometric functions. In addition, one of the important branch in the field of the complex analysis is the geometric function theory because this theory deals with the geometric properties of analytic functions. Now days, in the geometric function theory, many applications have been existed for the theory of differential subordinations which its origins going back to Miller and Mocanu [38]. Then, the extensions of that theory can be found in different fields, such like the harmonic functions theory, the partial differential equations, the integral operators theory and the meromorphic functions theory. Due to that, the present paper had done in order to provide a literature study which focused on the theory of differential subordination and its applications in the study of analytic and univalent functions. Furthermore, the properties of the first, the second and the third order differential subordination are studied. This investigation concern on the most important results that have been introduced in the theory of differential subordination by the previous studies. Then, the contribution of this review is to provide an overview on the developments in the theory of differential subordination since the appearance of that concept right up to the recent years. The review of the previous studies may open research issues in the future works, and will allow the future researchers to have a useful background on the essential aspects of this research field.

Keywords: *Analytic function, Admissible function, Best subordinant, Differential subordination, Differential superordination, Meromorphic function, Univalent function, Linear Operator, Multivalent Functions, Multiplier Transformation, Generalized Bessel Functions.*

1. INTRODUCTION

The study of the differential subordination and superordination for the analytic functions in the unit disk have been used to investigate some problems in the geometric function theory. The applications of the differential subordination and superordination encouraged many researchers to apply many attempts to the analytic and univalent functions by using this technique in order to provide new facts in that field. The theory of differential subordinations has many applications, and some of its extensions can be found in different fields such as the functions of several complex variables, the differential equations, the integral operators theory, the partial differential equations, the harmonic functions theory, the meromorphic functions theory and recently appeared the monographs in the differential subordinations and also in the differential superordinations. Furthermore, several studies have been contributed the theory of differential subordination, for example [1], [2] and [3]. Many other studies are also considered the topic of differential subordination of the analytic and univalent functions. Thus, the purpose of this paper is to present a literature study for the developments which have been introduced by the previous studies in the theory of differential subordination of the analytic and univalent functions. This paper is organized as follows.

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In section 2, some of the basic definitions which are concerning to the different class of univalent and analytic functions are presented. Then, some of the substantial developments in the subordination for the functions which are analytic have been given as well. Furthermore, the most important results and developments in the theory of differential subordination which included first order, second order, third order and fourth order differential subordination are given in section 3.

2. SUBORDINATION FOR ANALYTIC FUNCTIONS

In this section, the subordination of the analytic functions has been studied. Then, some of the basic definitions which are concern to several classes of analytic functions are also presented.

Let $f(z)$ be a complex function and let $z_0 \in \mathbb{C}$, then the function $f(z)$ is said to be differentiable at the point z_0, if it has the following derivative at the point z_0.

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Furthermore, if the function $f(z)$ is differentiable at each point of some neighborhood of the point z_0, then the function $f(z)$ is said to be analytic at the point z_0.

Let D be a given domain, and let $f(z)$ be a complex function defined on D, then the function $f(z)$ is said to be analytic in D if it has a derivative at every point of the domain D.

Suppose that $U = \{z \in \mathbb{C}; |z| < 1\}$ be an open unit disk. If the complex function $f(z)$ is analytic in U, then it said to be univalent in U, if $w = f(z)$, for different z in U.

Consequently, some other basic definitions for some special classes of analytic functions are given in this section. These definitions are concerning to some special classes of starlike and convex functions for the univalent and p-valent functions in the open unit disk. The definitions are given as follows.

Definition 1.2 [56] A domain D in $\mathbb{C}$ is said to be starlike with respect to a point w_0, if the line segment connecting any point in D to w_0 is contained in D.

Definition 2.2 [56] The function $f(z) \in S$ in U is said to be starlike with respect to w_0 if U mapped onto a domain starlike with respect to w_0.

Definition 3.2 [56] A function $f(z) \in S$ is said to be starlike of order $\alpha (0 \le \alpha < 1)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \le \alpha < 1; z \in U).$$

Also, the function $f(z) \in A_p$ is said to be p-valently starlike of order $\alpha (0 \le \alpha < p)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (p \in \mathbb{N};\ 0 \le \alpha < p;\ z \in U).$$

Definition 4.2 [56] A function $f(z) \in S$ is said to be convex of order $\alpha (0 \le \alpha < 1)$ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (0 \le \alpha < 1;\ z \in U).$$

Also, the function $f(z) \in A_p$ is said to be p-valently convex of order $\alpha (0 \le \alpha < p)$ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (p \in \mathbb{N};\ 0 \le \alpha < p;\ z \in U).$$

The generalization of the class of starlike functions has been investigated and introduced by [57] for the class of starlike functions. Next, some definitions are concerning into the class of close-to-convex functions which are given as follows.

Definition 5.2 [56] A function $f(z) \in A$ is said to be close-to-convex of order $\alpha (0 \le \alpha < 1)$ if there is a convex function g such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \quad (z \in U).$$

Definition 6.2 [56] A function $f(z) \in A_p$ is said to be p-valent close-to-convex of order $\alpha (0 \le \alpha < p)$ if there is a p-valent convex function $g(z)$ such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \quad (z \in U).$$

Meanwhile, some classes such as the classes of uniformly p-valent starlike, uniformly p-valent convex and uniformly p-valent close-to-convex functions of order $\alpha (-p \le \alpha < p)$, have been studied by some authors. The classes of uniformly p-valent starlike and uniformly p-valent convex have been studied and introduced by [58,59,60] and [61]. However, for the starlike functions, there is another generalization class which is the class of $\Phi -$like functions. This class is introduced by [62] as follows.

Definition 7.2 [62] Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. The function $f \in A$ is called $\Phi -$like if

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$$\operatorname{Re}\left\{\frac{zf'(z)}{\Phi(f(z))}\right\}>0,\quad (z\in U).$$

Furthermore, the study by [62] showed that the analytic function $f\in A$ is univalent if and only if f is $\Phi-$like for some Φ.

Moreover, the study in [63] have also investigated the class of $\Phi-$like functions which is given as follows.

Definition 8.2 [63] Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0)=0$, $\Phi'(0)=1$, and $\Phi(\omega)\neq 0$ for $\omega\in f(U)\setminus\{0\}$. Let q be a fixed analytic function in U, and $q(0)=1$. The function $f\in A$ is called $\Phi-$like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))}\prec q(z),\ (z\in U).$$

Consequently, some definitions and important theorems which are concerning into the class of $\Phi-$like analytic functions can be found in [64], with subordination theorem that valid for all the normalized univalent analytic functions in U.

For the analytic functions, there are many studies have considered the differential subordination defined by means of linear operators such as the Dziok-Srivastava linear operator and multiplier transformation. Others, by determining the appropriate classes of admissible functions, and some others by investigate the subordination properties for some classes of the meromorphic functions such as [13] obtained the subordination for the class of meromorphic functions. They provided the following generalization.

Theorem 1.3 [13] Let $q(z)$ be univalent and $q(z)\neq 0$ in Δ and

i. $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in Δ and,

ii. $R\left[1+\dfrac{zq''(z)}{q'(z)}-\dfrac{zq'(z)}{q(z)}-\dfrac{q(z)}{\gamma}\right]>0$ for $z\in\Delta,\gamma\neq 0$. If

$f(z)\in\sum$ and

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\prec q(z)-\gamma\frac{zq'(z)}{q(z)},$$

then

$$-\frac{zf'(z)}{f(z)}\prec q(z)\text{ and } q(z)\text{ is the best dominant.}$$

The above generalization, is the generalization of the following theorems which are given by [4].

Theorem 1.1 [13] Let $\alpha<0$. If $f\in MC\left(\dfrac{\alpha(3-2\alpha)}{2(1-\alpha)}\right)$, then $f\in MS^{*}(\alpha)$.

Theorem 1.2 [13] Let $\alpha<0$ and $\gamma\geq 0$. If $f\in\sum_{\gamma}^{*}MC\left(\dfrac{2\alpha-2\alpha^{2}+\gamma\alpha}{2(1-\alpha)}\right)$, then $f\in MS^{*}(\alpha)$.

Then, they have concluded the following corollary by letting $q(z)=\dfrac{1+(1-2\alpha)z}{1-z}$ in Theorem 1.3 [13].

Corollary 3.1 [13] Let $\alpha<0$ and $\gamma\neq 0$. If $f(z)\in\sum$ and

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\prec$$

$$\frac{1+2[1-\gamma+(\alpha-1)\gamma]z+(1-2\alpha)^{2}z^{2}}{1-2\alpha z-(1-2\alpha)z^{2}},\text{ then}$$

$$-R\frac{zf'(z)}{f(z)}>\alpha.$$

Furthermore, the study by [6] introduced some new class $\sum_{n}(\alpha)$ of meromorphic functions which is defined by a multiplier transformation, then some properties of this class have been investigated. Meanwhile, in [5] some subordinations theorems involving certain integral operators for the analytic functions in U. Then, the papers by [65] and [66] had considered the subordination and superordination that preserving the properties of the integral operators with some conditions on the parameters δ,α,β and γ. In [7] the subordination and superordination that have preserving the properties of the general integral operators I, was obtained and defined as follows.

$$I(f)(z):=\left(\frac{\beta+\gamma}{z^{\gamma}}\int_{0}^{z}t^{\delta-1}f^{\alpha}(f)dt\right)^{1/\beta}\quad\text{where}$$

$(f\in A;\alpha,\gamma,\delta\in □;\ \beta\in □\setminus\{0\};\alpha+\delta=\beta+\gamma;\operatorname{Re}\{\beta+\gamma\}>0).$

Moreover, several new classes of meromorphic functions defined by using the meromorphic analogue of the Choi–Saigo–Srivastava operator for the analytic functions have been introduced by [8]. Then various inclusion properties of the classes $\mathrm{MS}_{a,b}^{\mu}(\eta,\phi)$, $\mathrm{MK}_{a,b}^{\mu}(\eta,\phi)$, and $\mathrm{MC}_{a,b}^{\mu}(\eta,\phi)$, associated with the operator $I_{\mu}(a,b)$ are investigated. Some new sufficient conditions for the class of starlike functions have been found by [9]

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with some examples of an integral operator which preserves the meromorphic starlike functions. However, several applications of Hadamard product to multivalently analytic and multivalently meromorphic functions are given by [10]. Some subclasses of the class of the meromorphic functions and multiplier transformation have been studied by [11]. Then, they have investigated various properties of this class, and some of the results are extended from the previous results of [25]. In [12] some necessary and sufficient conditions for the spirallikeness and the convex spirallikeness of the suitably normalized meromorphic p-valent function have been introduced. Then, by using the linear operator, they gave some applications for their results in order to obtain the convolution condition for the class of meromorphic functions. More studies on the Meromorphic multivalent functions can be found in [70-88].

Meanwhile, the study by [14] investigated the convolution properties with inclusion and related properties for the general classes of p-valent functions. The classes which are introduced, are an extension of the classes of starlike , α-convex , quasi-convex, convex, and close-to-convex functions. Some subordination preserving properties associated with the operator $J_{s,b}^{\lambda,\mu}$ have been provided by [15]. Then, several sandwich-type results involving this operator are given. Then, they proved that, if $f, g \in A$ and $\mu > 0$ with

$$R\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\rho \quad \left(z\in U;\ \varphi(z) := \frac{J_{s,b}^{\lambda,\mu+1}g(z)}{z}\right),$$

where $\rho := \dfrac{1+\mu^2-\left|1-\mu^2\right|}{4\mu}$. Then, the subordination $\dfrac{J_{s,b}^{\lambda,\mu+1}f(z)}{z} \prec \dfrac{J_{s,b}^{\lambda,\mu+1}g(z)}{z}$ implies that $\dfrac{J_{s,b}^{\lambda,\mu}f(z)}{z} \prec \dfrac{J_{s,b}^{\lambda,\mu}g(z)}{z}$. Furthermore, the function $\dfrac{J_{s,b}^{\lambda,\mu}g(z)}{z}$ is the best dominant.

Meanwhile, [16] generalized the classes $\Omega_{p,q,s}(\alpha_1;A,B)$ and $\Omega^{+}_{p,q,s}(\alpha_1;A,B)$ which was studied by [17] as follows.

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} = \{1,2,...\}),$$ which

are analytic and p-valent in the punctured disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0<|z|<1\} = U\setminus\{0\}$. By using the operator $H_{p,q,s}(\alpha_1)$, the function $f(z)\in\sum_p$ is in the class $\Omega_{p,q,s}(\alpha_1;A,B)$ if the function $f(z)$ satisfies the following subordination condition:

$$\frac{1}{p-\lambda}\left(\frac{(H_{p,q,s}(\alpha_1+1)f(z))'}{(H_{p,q,s}(\alpha_1)f(z))'}-1\right) \prec -\frac{(A-B)z}{\alpha_1(1+Bz)},$$

where

$(z\in U; -1\le B<A\le 1; \alpha_1 \in \mathbb{C}\setminus\{0\}; p,q,s\in\mathbb{N};$
$0\le\lambda<p),$

or equivalently, by using

$z(H_{p,q,s}(\alpha_1)f(z))' =$
$\alpha_1 H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1+p)H_{p,q,s}(\alpha_1)f(z)$ if

$$\left|\frac{1+\dfrac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'}+p}{B\left(1+\dfrac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'}\right)+[pB+(A-B)(p-\lambda)]}\right|<1.$$

Furthermore, they introduced the second class $\Omega^{+}_{p,q,s}(\alpha_1;A,B)$ which is defined as follows.

The function $f(z)\in\Omega^{+}_{p,q,s}(\alpha_1;A,B)$ whenever $f(z)$ is of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty}|a_k|z^{k-p} \quad (p\in\mathbb{N}).$$

A family of integral operators defined on the space of meromorphic functions have been introduced by [18]. Then, they have also defined several subclasses of meromorphic functions. However, they have introduced various inclusion relationships and integral-preserving properties for the meromorphic function classes.

The reference [19] investigated a certain class of meromorphic functions which are defined by the linear operator given by Liu and Srivastava [20], and proved the extensions of (Theorem 1.1 and Theorem 1.4,[19]) which are given by [21] and [22] respectively. By involving the linear operator $L_p(a,c)$, the extensions of Theorems 1.1 and 1.4 are given respectively as follows.

Theorem 2.3 [19] Let $q(z)\neq 0$ be univalent in Δ and $\dfrac{zq'(z)}{q(z)}$ be starlike. If $f\in\sum_p$ satisfies

$$(a+1)\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)}-\alpha a\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} \prec$$

$$1-(a+p)(\alpha-1)+\frac{zq'(z)}{q(z)}, \alpha\in C$$, then

$\frac{L_p(a+1,c)f(z)}{\left[L_p(a,c)f(z)\right]^{\alpha}} \prec q(z)$, and $q(z)$ is the best dominant.

Theorem 2.5 [19]
Let $a \neq -1$ and $\gamma \neq 0$. Let $q(z)$ satisfies the conditions of [Lemma 1.6, 19] with $\beta > \gamma$ and $\alpha > 1+a-\gamma$. If $f \in \sum_p$ satisfies

$$(1-\gamma)\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}+\gamma\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} \prec \frac{1}{a+1}\left[\gamma+(1+a-\gamma)q(z)+\gamma\frac{zq'(z)}{q(z)}\right]$$ then

$\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} \prec q(z)$, and $q(z)$ is the best dominant.

The subordination problem for the meromorphic functions which defined by the linear operator D^n has been investigated by Liu and Owa [23]. In particular, they determined a class of admissible functions such that,

$$\left|h\left(\frac{D^n f(z)}{D^{n-1} f(z)},\frac{D^{n+1} f(z)}{D^n f(z)},\frac{D^{n+2} f(z)}{D^{n+1} f(z)}\right)\right|<1 \Rightarrow \left|\frac{D^n f(z)}{D^{n-1} f(z)}\right|<1.$$

Meanwhile, for the multiplier transformation on the class of univalent meromorphic functions, there are also several studies have been discussed this case such as Cho and Kim [24] introduced several classes of meromorphic functions defined by using a meromorphic analogue of the Choi–Saigo–Srivastava operator for the generalized hypergeometric function and investigate various inclusion properties of these classes with some applications involving these and other classes of integral operators are also considered. Then, Cho et al. [25] provided some subclasses of meromorphic functions and investigated their inclusion relationships and argument properties. Meanwhile, Uralegaddi and Somanatha [26] established a new criteria for the meromorphic starlike univalent functions of the form

$$f(z)=\frac{a-1}{z}+\sum_{k=0}^{\infty}a_k z^k,\ (a-1\neq 0).$$

As a motivation from the previous works by investigation the multiplier transformation on the class of univalent functions, Ali et al. [27] defined the multiplier transformation $I_p(n,\lambda)$ on the class $f \in \sum_p$ of meromorphic functions by the infinite series

$$I_p(n,\lambda)f(z) > \frac{1}{z^p}+\sum_{k=1-p}^{\infty}\left(\frac{k+\lambda}{\lambda-p}\right)^n a_k z^k,(\lambda>p),$$

Can be notice that, the operator $I_p(n,\lambda)$ satisfies the identity

$$z\left[I_p(n,\lambda)f(z)\right]'=(\lambda-p)I_p(n+1,\lambda)f(z)-\lambda I_p(n,\lambda)f(z).$$

This identity played an important role in obtaining the information about the functions which defined by using of the multiplier transformation. Several studies that have considered the meromorphic functions can be found in [28-33].

3. DIFFERENTIAL SUBORDINATION

The theory of the differential subordination for the functions which are analytic in the unit disk has been studied by several authors to investigated some problems in the geometric function theory. In 1935, Goluzin [34] considered the first order differential subordination $zq'(z) \prec h(z)$ and showed that, if h is convex, then

$p(z) \prec q(z)=\int_0^z h(t)t^{-1}dt$ and this q is the best dominant. The Goluzin's result has been improved by [35]. Then, Robinson [36] considered the differential subordination $p(z)+zq'(z) \prec h(z)$ and proved that, if $q(z)=z^{-1}\int_0^z h(z)dz$ and h are univalent, then q is the best dominant for $|z|<1/5$.

Furthermore, Hallenbeck and Rusheweyh [37] considered the following differential subordination

$p(z)+\frac{zq'(z)}{\gamma} \prec h(z) \quad (\gamma \neq 0, Re\gamma \geq 0)$, and they showed that if h is a convex, then $p(z) \prec q(z)=\gamma z^{-\gamma}\int_0^z h(t)t^{\gamma-1}dt$, and this is the best dominant.

Consequently, Miller and Mocanu [38], had initiated the theory of the differential subordination by their works, which was developed by some other studies such like [39] and [40]. However, Miller and Mocanu [41], introduced the dual problem of differential superordination while, Bulboaca [42] investigated the differential for the subordination and superordination.

Several studies have been used the theory of first and second order differential subordination to solve some problems in that field. (see [43]–[48]). Many significant works on the theory of first and second differential subordination have been provided by Miller and Mocanu [49, p.69-70] which are given as follows.

With the theory of the differential equations, the differential subordination of the form

$$A(z)p'(z)+B(z)p(z)\prec h(z),$$

is called "a first-order linear differential subordination" which can be written in the form

$$zp'(z)+P(z)p(z)\prec h(z). \quad (1)$$

If $P(z)\equiv 0$ and h is convex, then we have the following differential subordination

$$zp'(z)\prec h(z) \quad (2)$$

which was considered by [34]. Then, if $P(z)\equiv 1$ and h is univalent, then we have the differential subordination

$$zp'(z)+p(z)\prec h(z) \quad (3)$$

which was also considered by [36]. The function h which maps U onto the right half-plane is a convex function. Then, they proved in the following theorem that, this case can be extended to be hold for any convex function.

Theorem 3.1a [49, p70] Let h be convex in U and let $P:\mathrm{U}\to\mathrm{C}$, with $ReP(z)>0$. If p is analytic in U, then

$$p(z)+P(z)\cdot zp'(z)\prec h(z) \Rightarrow p(z)\prec h(z). \quad (4)$$

Meanwhile, the following theorem has been proved by Hallenbeck and Ruscheweyh in [37] and again proved by Miller and Mocanu [49].

Theorem 3.1b [49, p.71] Let h be convex in U with $h(0)=a$, $\gamma\neq 0$ and $Re\gamma\geq 0$. If $p\in \mathrm{M}[a,n]$ and

$$p(z)+\frac{zp'(z)}{\gamma}\prec h(z) \quad (5)$$

then $p(z)\prec q(z)\prec h(z)$, where

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{(\gamma/n)-1}dt \quad (6)$$

and the function q is convex and the best (a,n)-dominant.

Note that, if $\gamma=1$, then this theorem reduces to Robinson's differential subordination (3) with superordinate convex function h. In the following theorem, they showed that, the convex function h of (Theorem 3.1b) can be replaced with the function of the form $\left[(1+z)/(1-z)\right]^{\alpha}$. For $\alpha>1$, this function is not convex, but is starlike with respect to 1.

Theorem 3.1c [49, p.73]Let n be a positive integer, $\lambda>0$ and let $\beta_0=\beta_0(\lambda,n)$ be the root of the equation

$$\beta\pi=3\pi/2-tan^{-1}(n\lambda\beta) \quad (7)$$

Furthermore, let $\alpha=\alpha(\beta,\lambda,n)=\beta+(2\pi)tan^{-1}(n\lambda\beta)$, for $0<\beta\leq\beta_0$.

If $p\in\mathrm{M}[1,n]$, then

$$p(z)+\lambda zp'(z)\prec\left[\frac{1+z}{1-z}\right]^{\alpha} \Rightarrow p(z)\prec\left[\frac{1+z}{1-z}\right]^{\beta} \quad (8).$$

Miller and Mocanu [49] have also provided some subordination properties which are given as follows.

Let $q(z)$ be univalent in the unit disc U and θ with φ be an analytic in a domain D containing $q(U)$ with $\varphi(w)\neq 0$ when $w\in q(U)$, $Q(z)=zq'(z)\varphi(q(z))$, $h(z)=\theta(q(z))+Q(z)$, suppose that $Q(z)$ is starlike univalent in U and $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}>0$ for $z\in U$, if p is analytic in U with $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(q(z))+zq'(z)\varphi(q(z)),$$

then $p(z)\prec q(z)$, and $q(z)$ is the best dominant .

Meanwhile, Miller and Mocanu [49] provided much informations and descriptions for the second order differential subordination. Some of these informations have been summarized as follows.

Let $\psi:\mathrm{C}^3\times\mathrm{U}\to\mathrm{C}$ and let h be univalent in U.. If p is analytic in U.and satisfies the "second order differential subordination"

$$\psi\left(p(z),zp'(z),z^2p''(z);z\right)\prec h(z), \quad (9)$$

then p is called the solution of the differential subordination.

The function q is called the dominant of the solutions of the differential subordination or simply a dominant, if $p\prec q$ for all p satisfying (9). A dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants q of (9) is said to be the best dominant of (9). For the more restrictive condition, $p\in\mathrm{M}[a,n]$, then p will be called an (a,n)-solution, q an (a,n)-dominant, and $\tilde{q}$ the best (a,n)-dominant. Now, let Ω be a set in C and suppose that (9) replaced by

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \text{ for } z \in \mathrm{U} \qquad (9a)$$

Although this is a "differential inclusion" and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ may not be analytic in U., then, can also refer to (9a) as a second order differential subordination.

Recently, some authors had discussed the second order differential subordination for multivalent function such as Juma et al. [50] gave some results for that case on the analytic and multivalent functions in the open unite disk U involving the linear operator $F^m_{\lambda,p}(f*g)(z)$. These results are obtained by investigating appropriate classes of admissible functions which are given as follows.

Definition 2.1 [50] Let Ω be a set in $\mathrm{C}, q \in Q_0 \cap H[0,p]$. The class of admissible functions $\Psi_n[\Omega,\mathrm{q}]$ consists of those functions $\phi: \mathrm{C}^3 \times \mathrm{U} \to \mathrm{C}$ that satisfy the admissibility condition:

$\phi(u,v,w;z,\xi) \notin \Omega$ whenever $u = q(\xi)$,

$$v = \frac{k\xi q'(\xi) + \dfrac{p(1-\lambda)}{\lambda} q(\xi)}{\dfrac{p}{\lambda}}, \text{ and}$$

$$Re\left\{\frac{p^2w + 2p^2(1-\lambda)v - 3p^2(1-\lambda)^2 u}{\lambda pv - p\lambda(1-\lambda)u}\right\} \geq kRe\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)}\right\}$$

where $z \in \mathrm{U}, \xi \in \partial\mathrm{U} \setminus E(q), \lambda > 0$ and $k \geq p$.

Theorem 2.1 [50] Let $\phi \in \Psi_n[\Omega,\mathrm{q}]$. If $f \in \mathrm{A}(p)$ satisfies

$$\phi\left(F^m_{\lambda,p}(f*g)(z), F^{m+1}_{\lambda,p}(f*g)(z), F^{m+2}_{\lambda,p}(f*g)(z)\right) \subset \Omega, \text{ where } \lambda > 0, m \in N_0 = \{0,1,2,\ldots\}, z \in \mathrm{U}.$$

Then $F^m_{\lambda,p}(f*g)(z) \prec q(z) \quad (z \in \mathrm{U})$.

In this case, $\Omega = h(\mathrm{U})$, where h is a conformal mapping of U.onto Ω and the class is written as $\Psi_n[\Omega,\mathrm{q}]$. The following results are an immediate consequence of Theorem 2.1.

Theorem 2.2 [50] Let $\phi \in \Psi_n[h,\mathrm{q}]$. If $f \in \mathrm{A}(p)$ satisfies

$$\phi\begin{pmatrix} F^m_{\lambda,p}(f*g)(z), F^{m+1}_{\lambda,p}(f*g)(z), \\ F^{m+2}_{\lambda,p}(f*g)(z); z \end{pmatrix} \prec h(z) \quad (10)$$

where $\lambda > 0, m \in N_0, z \in \mathrm{U}$.

Then $F^m_{\lambda,p}(f*g)(z) \prec q(z), \ (z \in \mathrm{U})$.

Theorem 2.3 [50] Let h and q be univalent in U.with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi: \mathrm{C}^3 \times \mathrm{U} \to \mathrm{C}$ satisfy one of the following conditions

i. $\phi \in \Psi_n[h, \mathrm{q}_\rho]$ for some $\rho \in (0,1)$,

ii. There exist $\rho_0 \in (0,1)$ such that $\phi \in \Psi_n[h_\rho, \mathrm{q}_\rho]$ for all $\rho \in (\rho_0, 1)$. If $f \in \mathrm{A}(p)$ and satisfies (10), then $F^m_{\lambda,p}(f*g)(z) \prec q(z)$.

The next theorem gives the best dominant of the differential subordination (10).

Theorem 2.4 [50] Let h be univalent in U.and let $\phi: \mathrm{C}^3 \times \mathrm{U} \to \mathrm{C}$. Suppose that the differential equation $\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z)$ has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

i. $q \in Q_0$ and $\phi \in \Psi_n[h,\mathrm{q}]$.

ii. q be univalent in U. and $\phi \in \Psi_n[h, \mathrm{q}_\rho]$ for some $\rho \in (0,1)$,

iii. q be univalent in U and there exist $\rho_0 \in (0,1)$ such that $\phi \in \Psi_n[h_\rho, \mathrm{q}_\rho]$ for all $\rho \in (\rho_0, 1)$. If $f \in \mathrm{A}(p)$ and satisfies (10), then $F^m_{\lambda,p}(f*g)(z) \prec q(z)$ and q is the best dominant.

The results which have been done for the first and second order differential subordination can be extend to the case of third and higher order differential subordinations. The solutions for the higher order differential equations are difficult to be obtain and a few are known about that. By referring to [49] many details for the third order differential subordination which are summarized as follows.

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z); z\right) \prec h(z) \ (11).$$

Let h, q and $\psi\left(p(z), zp'(z), z^2p''(z), \ldots z^n p^{(n)}(z); z\right)$ be analytic in U.where $\psi: \mathrm{C}^{n+1} \times \mathrm{U} \to \mathrm{C}$. If p satisfies the n^{th} order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z), \ldots z^n p^{(n)}(z); z\right) \prec h(z) \quad (12),$$

then p is called a solution of the differential subordination. The univalent function q is called a

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dominant of the solutions of the differential subordination, or simply a dominant if $p \prec q$ for all p satisfying (12). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (12) is said to be the best dominant of (12).

A survey paper on the concept of subordination for the certain classes of univalent functions has been given by [67]. Then a short study on the concept of the differential superordination has been provided by [69]. Furthermore, few articles have been discussed the third order inequalities and subordination. Antonino and Miller [51] extended the theory of the second order differential subordination in the open unit disk U.which introduced by Miller and Mocanu [49] to the case of the third order. The properties of the p functions which satisfies the following third order differential subordination have been determined.

$$\left\{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in U\right\} \subset \Omega. \quad (13).$$

Furthermore, Jeyaraman et al. [52] investigated the third order differential subordination on the Schwarzian derivative. Some third order differential subordination results for the analytic functions in the open unit disk involving the generalized Bessel functions was investigated by Tang et al. [53]. Then, they used the operator $B_k^c f$ by means of normalized form of the generalized Bessel functions of the first kind, which is defined as follows.

$$z\left(B_{k+1}^c f(z)\right)' = kB_k^c f(z) - (k-1)B_{k+1}^c f(z), \quad (14)$$

where $b, c, p \in \mathrm{C}$ and $k = p + (b+1)/2 \in \mathrm{C} \setminus z_0^-\left(z_0^- = \{0, -1, -2, \ldots\}\right)$. The results are obtained by considering suitable classes of admissible functions which are defined as follows.

Definition 2.1 [53] Let Ω be a set in C, and $q \in \mathrm{Q}_0 \cap H_0$. The class of admissible functions $\Phi_B[\Omega, \mathrm{q}]$ consists of those functions $\phi : \mathrm{C}^4 \times \mathrm{U} \to \mathrm{C}$ that satisfy the following admissibility condition

$\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$, whenever $\alpha = q(\xi)$,

$$\beta = \frac{\xi q'(\xi) + (k-1)q(\xi)}{k},$$

$$Re\left\{\frac{k(k-1)\gamma - (k-1)(k-2)\alpha}{k\beta - (k-1)\alpha} - (2k-3)\right\} \geq kRe\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}$$

and

$$Re\left\{\frac{k(k-1)\left((1-k)\alpha + 3k\beta + (1-3k)\gamma + (k-2)\delta\right)}{\alpha + k(\beta - \alpha)}\right\} \geq k^2 Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\},$$

where $z \in \mathrm{U}, k \in \mathrm{C} \setminus \{0,1,2\}, \xi \in \partial\mathrm{U} \setminus E(q)$ and $k \geq 2$.

Theorem 2.1 [53] Let $\phi \in \Phi_B[\Omega, \mathrm{q}]$. If $f \in \mathrm{A}$ and $q \in \mathrm{Q}_0$ satisfy the following conditions:

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0, \left|\frac{B_k^c f(z)}{q'(\xi)}\right| \leq k, \text{ and} \quad (15)$$

$$\left\{\phi\left(B_{k+1}^c f(z), B_k^c f(z), B_{k-1}^c, B_{k-2}^c f(z); z\right) : z \in \mathrm{U}\right\} \subset \Omega \quad (16)$$

then $B_{k+1}^c f(z) \prec q(z)$, $(z \in \mathrm{U})$.

If $\Omega \neq \mathrm{C}$ is a simply connected domain, then $\Omega = h(\mathrm{U})$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_B[h(\mathrm{U}), \mathrm{q}]$ is written as $\Phi_B[h, \mathrm{q}]$. The following theorem is an immediate consequences of Theorem 2.1.

Theorem 2.2 [53] Let $\phi \in \Phi_B[\Omega, \mathrm{q}]$. If $f \in \mathrm{A}$ and $q \in \mathrm{Q}_0$ satisfy the conditions (15) , then

$$\phi\left(B_{k+1}^c f(z), B_k^c f(z), B_{k-1}^c, B_{k-2}^c f(z); z\right) \prec h(z) \quad (17)$$

implies that $B_{k+1}^c f(z) \prec q(z)$, $(z \in \mathrm{U})$.

The next theorem yields the best dominant of the differential subordination (17).

Theorem 2.3 [53] Let h be univalent in U and let $\phi : \mathrm{C}^4 \times \mathrm{U} \to \mathrm{C}$ and ψ be given as follows.

$$\psi(r, s, t, u; z) = \phi(\alpha, \beta, \gamma, \delta; z) =$$

$$\phi\left(r, \frac{s + (k-1)r}{k}, \frac{t + (k-1)s + (k-1)(k-2)r}{k(k-1)}, \frac{u + 3(k-1)t + 3(k-1)(k-2)s + (k-1)(k-2)(k-3)r}{k(k-1)(k-2)}; z\right).$$

Suppose that the differential equation

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z) \quad (18)$$

has a solution $q(z)$ with $q(0) = 0$ and satisfies the condition (15). If $f \in \mathrm{A}, \phi \in \Phi_B\left[h, \mathrm{q}_\rho\right]$ and $\phi\left(B_{k+1}^c f(z), B_k^c f(z), B_{k-1}^c, B_{k-2}^c f(z); z\right)$ is analytic in U, then (17) implies that $B_{k+1}^c f(z) \prec q(z)$ and q is the best dominant.

Furthermore, some third order differential subordination results for the analytic functions in the open disk which are associated with the fractional derivative operator have been discussed by Farzana et al. [54]. These results are obtained by investigating suitable classes of admissible functions which are presented as follows.

Definition 2.1 [54] Let Ω be a set in C, $q \in Q_0 \cap \mathrm{H}[0, p]$. The class of admissible functions $\Phi_\Delta[\Omega, \mathrm{q}]$ consists of those functions $\phi : \mathrm{C}^4 \times \mathrm{U} \to \mathrm{C}$ that satisfy the admissible condition $\phi(a, b, c, d; z) \notin \Omega$ whenever

$$a = q(\zeta), \quad b = \frac{n\zeta q'(\zeta) - \mu q(\zeta)}{(p-\mu)},$$

$$\mathrm{R}\left\{\frac{(p-\mu)(p-\mu-1)c - \mu(\mu+1)a}{(p-\mu)b + \mu a} + 2\mu + 1\right\} \geq n\mathrm{R}\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}, \text{ and}$$

$$\mathrm{R}\left\{\frac{(p-\mu)(p-\mu-1)(p-\mu-2)d + 3\mu(p-\mu)(p-\mu-1)c - 2\mu a(\mu^2-1)}{(p-\mu)b + \mu a} + 3\mu(\mu+1)\right\} \geq n^2\mathrm{R}\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\}$$

where $(z \in U; \zeta \in \partial\mathrm{U} \setminus E(q); \mu \neq p, p \in \mathrm{N}, n \geq p)$.

Theorem 2.1[54] Let $\phi \in \Phi_\Delta[\Omega, \mathrm{q}]$. If $f \in \mathrm{A}_p$ and $q \in Q_0 \cap \mathrm{H}[0, p]$ with $\mathrm{R}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0$ and

$$\left|(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,v+1} f(z) + \mu\Delta_{z,p}^{\lambda,\mu,v} f(z)\right| \leq n\left|q'(\zeta)\right|,$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathrm{N}; z \in U;$ $\zeta \in \partial\mathrm{U} \setminus E(q), n \geq p, n \geq 2)$ and

$$\left\{\phi\begin{pmatrix}\Delta_{z,p}^{\lambda,\mu,v} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,v+1} f(z), \\ \Delta_{z,p}^{\lambda+2,\mu+2,v+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,v+3} f(z); z\end{pmatrix} z \in U\right\} \subset \Omega$$

then $\Delta_{z,p}^{\lambda,\mu,v} f(z) \prec q(z)$, $(z \in U)$.

If $\Omega \neq \mathrm{C}$ is a simply connected domain, then $\Omega = h(\mathrm{U})$ for some conformal mapping h of U onto Ω. In this case, the class $\Phi_\Delta\left[h(\mathrm{U}), \mathrm{q}\right]$ is written as $\Phi_\Delta[h, \mathrm{q}]$. The following theorem is an immediate consequences of Theorem 2.1.

Theorem 2.2 [54] Let $\phi \in \Phi\Delta[\Omega, \mathrm{q}]$. If $f \in \mathrm{A}_p$ and $q \in Q_0 \cap \mathrm{H}[0, p]$ with $\mathrm{R}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0$ and

$$\left|(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,v+1} f(z) + \mu\Delta_{z,p}^{\lambda,\mu,v} f(z)\right| \leq n\left|q'(\zeta)\right|,$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathrm{N}; z \in U;$ $\zeta \in \partial\mathrm{U} \setminus E(q), n \geq p, n \geq 2)$

and if

$$\phi\begin{pmatrix}\Delta_{z,p}^{\lambda,\mu,v} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,v+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,v+2} f(z), \\ \Delta_{z,p}^{\lambda+3,\mu+3,v+3} f(z); z\end{pmatrix}$$

is analytic in U, then

$$\phi\begin{pmatrix}\Delta_{z,p}^{\lambda,\mu,v} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,v+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,v+2} f(z), \\ \Delta_{z,p}^{\lambda+3,\mu+3,v+3} f(z); z\end{pmatrix}$$

$\prec h(z)$, implies $\Delta_{z,p}^{\lambda,\mu,v} f(z) \prec q(z)$, $(z \in U)$.

Moreover, Ibrahim et al. [2] utilized the methods of the third order differential subordination results of Antonino and Miller [51] and Tang et al. [56], respectively. Certain suitable classes of admissible functions are considered as follows.

Definition 3.1 [2] Let Ω be a set in C, $c_\alpha, c_{\alpha+1}, c_{\alpha+2} \in \mathrm{C} \setminus \{0\}$ and $q \in \mathrm{Q}_0 \cap \mathrm{H}_0$. The class of admissible functions $\Phi_T[\Omega, \mathrm{q}]$ consists of those functions $\phi : \mathrm{C}^4 \times U \to \mathrm{C}$ that satisfy the following admissibility condition:

$\phi(v, w, x, y; z) \notin \Omega$ whenever $v = q(\zeta)$,

$$w = \frac{k\zeta q'(\zeta) + (c_\alpha - 1)q(\zeta)}{c_\alpha},$$

$$\mathrm{R}\left(\frac{c_\alpha c_{\alpha+1}x - (c_\alpha - 1)(c_{\alpha+1} - 1)v}{c_\alpha w - (c_\alpha - 1)v} - (c_\alpha + c_{\alpha+1} - 2)\right) \geq k\mathrm{R}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right) \text{ and}$$

$$\mathrm{R}\left[\frac{(c_\alpha c_{\alpha+1} c_{\alpha+2})y - (c_\alpha - 1)(c_{\alpha+1} - 1)(c_{\alpha+2} - 1)v}{c_\alpha w - (c_\alpha - 1)v} - (c_\alpha + c_{\alpha+1} + c_{\alpha+2})\left[\frac{c_\alpha c_{\alpha+1}x - (c_\alpha - 1)(c_{\alpha+1} - 1)v}{c_\alpha w - (c_\alpha - 1)v} - (c_\alpha + c_{\alpha+1} - 1)\right] - \left(c_{\alpha+2}(c_\alpha + c_{\alpha+1} - 1) + (c_\alpha - 1)(c_{\alpha+1} - 1)\right)\right] \geq k^2\mathrm{R}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U; \zeta \in \partial U \setminus E(q)$ and $k \geq 2$.

Theorem 3.2 [2] Let $\phi \in \Phi_T[\Omega, \mathrm{q}]$. If the function $f \in \mathrm{A}$ and $q \in \mathrm{Q}_0$ satisfy the following conditions:

$$\mathrm{R}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{T_{\alpha+1} f(z)}{q'(\zeta)}\right| \leq k \text{ and}$$

$$\left\{\phi\left(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z\right) : z \in U\right\} \subset \Omega \tag{19}$$

then $T_\alpha f(z) \prec q(z), (z \in U)$.

If $\Omega \neq \mathrm{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_T[h(U), \mathrm{q}]$ is written as $\Phi_T[h, \mathrm{q}]$. The following theorem is an immediate consequences of Theorem 3.2.

Theorem 3.4 [2] Let $\phi \in \Phi_T[\Omega, \mathrm{q}]$. If the function $f \in \mathrm{A}$ and $q \in \mathrm{Q}_0$ satisfy the condition (19) and

$$\phi\left(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z\right) \prec h(z)$$

then $T_\alpha f(z) \prec q(z), (z \in U)$.

The following result yields the best dominant of the differential subordination (20).

Theorem 3.6 [2] Let the function h be univalent in U and let $\phi : \mathrm{C}^4 \times U \to \mathrm{C}$ and ψ be given by

$$\psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) = \phi\left(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z\right).$$

Suppose that the differential equation

$$\psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z\right) = h(z), \tag{20}$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies condition (19). If the function $f \in \mathrm{A}$ satisfies condition (20) and $\phi\left(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z\right)$ is analytic in U, then $T_\alpha f(z) \prec q(z)$ and $q(z)$ is the best dominant.

A new concept which is called the fourth-order differential subordination and superordination associated with differential linear operator $\mathrm{I}_p(n, \lambda)$ in the open unit disk has been introduced by [68]. Some properties of this concept for the multivalent analytic functions have been also studied.

4. CONCLUSION

The theory of differential subordination is becoming an extensive field that bring a serious research interest. This reason which made such concept earning much attention recently from those who are specialized to brings new results in the field of geometric function theory. Different from the reviewing on the concept of the subordination given by [67] which was merely take a look to the few of the previous works without going through the details. In this study, we comprehensively discussed about the previous studies which were concerned on several types of classes of the analytic functions, meromorphic functions, multiplier transformation on the class of univalent meromorphic functions, uniformly p-valent starlike functions, uniformly p-valent convex functions and uniformly p-valent clos-to-convex functions with some other classes and their properties. Furthermore, this study had summarizes some of the essential definitions and most of the fundamental results and developments which have been introduced in the concept of the differential subordination on several classes of analytic functions, since the appearance of that concept right up to the current years. Meanwhile. This investigation concluded that many of the previous studies had discussed the first and the second order differential subordinations for several classes of analytic functions, while few of them had considered the third order differential subordination. Moreover, this research has also concluded that the concept of the differential subordination has been developed recently to introduce the new concept which is the concept of the fourth order differential subordination and superordination in the geometric function theory, which introduced in 2017 by [68]. Furthermore, Some of the results that have concern to the differential subordinations by using the integral operator have been included in this study. Considering to the research concept which has been discussed in this paper, the researchers could find a useful background information which can benefit and support their research on that concept.

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