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# BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS BY USING LIE GROUP

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**ABSTRACT**

In this task, we applied Lie group method for solving boundary value problems (BVPs) of partial differential equations (PDEs) to obtained the fundamental solutions .

**Keywords:** *Lie group, Invariance ,BVPs, Exact Solution, Boundary Condition*

## 1. INTRODUCTION

The most common technique for finding the exact solution of the enormous variety of DEs comes from Lie group analysis of DEs . Many efficient methods for solving DEs like separation of variables , traveling wave solutions , self-similar solutions and exponential self-similar solutions .because the modern treatment of the classical Lie symmetry introduced by [20] .Moreover known as the classical symmetries method , this originally by Sophus Lie [1] over 120 years ago. His most serious work in this orientation [1],[2]. In this time the Lie symmetry method is exceedingly applied to study PDEs . Especially for their reductions to ODEs and constructing exact solution ,there are a major number of papers and many good books devoted such applications [3-7]. The theory of symmetries of DEs has been established intensely and has virtually grown . A huge amount of literature about the classical Lie symmetry theory its implementation and its expansion is obtainable in [8-10,4,11-13,14-17,18-20,21]. In real world implementation , mathematical models are typically based on PDEs with pertinent boundary and /or initial condition . Consequence one requirements to investigate boundary value problems (BVPs) and initial problems (Cauchy problems) . We observe that the Lie method has not been widely used for solving BVPs and initial problems . A summarized History first attempts to apply Lie symmetries for solving BVPs are discussed in the following papers as [22-25].

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### 2.1 Applications to Boundary Value Problem,[3]

In this section , we consider the problem of using invariance to solve BVPs posed for PDEs. The application of the Lie symmetries to BVPs for PDEs as follows : an invariant solution arising from an admitted point symmetry solves a given BVP provided that the symmetry leaves invariant all boundary conditions. In other words, that the domain of the BVP or equivalently its boundary as well as the conditions (boundary conditions) imposed on the boundary must be invariant. In the case of BVPs posed for linear PDEs , the BVP need not be completely invariant (incomplete invariance) .

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### 2.2 Formulation of Invariance BVPs for Scaler PDEs,[4]

Let a BVP for a 2th order scaler PDE that can be written in solved form :

$$G(x,u,\partial u,\partial^2 u)=u_t-g(x,u,\partial u,\partial^2 u)=0 \quad ...(2.1)$$

Where $g(x,u,\partial u,\partial^2 u)$ does not depend explicitly on $u_t$ defined on a domain $\Omega_x$ in x-space $\left[x=(x_1,x_2,\ldots,x_n)\right]$ with boundary conditions:

$$B_\alpha(x,u,\partial u,\partial^2 u)=0 \quad ...(2.2)$$

Prescribed on boundary surfaces :

$$\omega_\alpha(x)=0 \quad , \quad \alpha=1,2,\ldots,s \quad ...(2.3)$$

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We assume that the BVP ((2.1)-(2.3)) has a unique solution .Consider an infinitesimal generator of the form:

$$X = \zeta_i(x)\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} \quad ...(2.4)$$

Which defines a point symmetry acting on both (x,u)-space as well as on its projection to x-space.

**Definition (2.1),[4]**

The point symmetry X of the form (4.4) is admitted by the BVP((2.1)-(2.3)) if and only if

1- $X^k(G(x,u,\partial u,\partial^2 u))=0$ when $G(x,u,\partial u,\partial^2 u)=0$ ...(2.5)

2- $X\ \omega_\alpha(x)=0$ when $\omega_\alpha(x)=0$ ...(2.6)

3- $X^2 B_\alpha(x,u,\partial u,\partial^2 u)=0$ when $B_\alpha(x,u,\partial u,\partial^2 u)=0$ for $\alpha=1,2,\dots,s$

...(2.7)

**Theorem (2.1),[4]**

Suppose the BVP ((2.1)-(2.3)) admits the Lie group of point transformations with infinitesimal generator(2.4).Let $y=(y_1(x),y_2(x),\dots,y_{n-1}(x))$ ben-1functionally independent group invariants of (2.4) that depend only on x. Let v(x,u) be a group invariant of (2.4) such that $\frac{\partial v}{\partial u}\neq 0$ . Then the BVP ((2.1)-(2.3)) reduce to:

$$G(y,v,\partial v,\partial^2 v)=0 \quad ...(2.8)$$

Defined on some domain $\Omega_y$ in y-space with boundary conditions:

$$D(y,v,\partial v,\partial^2 v)=0 \quad ...(2.9)$$

Prescribed on boundary surfaces

$$V_\alpha(y)=0 \quad ...(2.10)$$

Forsome $G(y,v,\partial v,\partial^2 v), D_\alpha(y,v,\partial v,\partial^2 v), V_\alpha(y), \alpha=1,2,\dots,s$ .

Moreover in the BVP ((2.8)-(2.10)) .The surface $y_j(x)=0$ , $j=1,2,\dots,n-1$ , are invariants surface of the point symmetry (2.4) . the invariance condition ((2.5)-(2.7)) means that each boundary surface $\omega_\alpha(y)=0$ is an invariant surface $V_\alpha(y)=0$ of the projected point symmetry.

$$\zeta_j(x)\frac{\partial}{\partial x_i} \quad ...(2.11)$$

Given by the restoration of point symmetry (2.4) to x-space . From the invariance of the BVP under the point symmetry (2.4) , the number of independent variables in ((2.1)-(2.3)) is reduced by one. And the solution of the BVP ((2.1)-(2.3) is an invariant solution .

$$v=\phi(y_1,y_2,\dots,y_{n-1}) \quad ...(2.12)$$

Of the PDE (2.8) resulting from its invariance under point symmetry (2.8).In terms of the dependent variable u and independent variable x appearing in PDE (2.8) the corresponding invariant solution $u=\theta(x)$ of PDE (2.8) must satisfy

$$X(u-\theta(x))=0 \text{ when } u=\theta(x) \quad ...(2.13)$$

That is ,

$$\zeta_i(x)\frac{\partial\theta(x)}{\partial x_i}=\eta(x,\theta(x)) \quad ...(2.14)$$

**Theorem (2.2), [4]**

If the infinitesimal generator X , given by (2.4) is of the form:

$$X=\zeta_i(x)\frac{\partial}{\partial x_i}+f(x)u\frac{\partial}{\partial u} \quad ...(2.15)$$

Then the group invariant v(x,u) is of the form $v(x,u)=\frac{u}{g(x)}$ for some specific function g(x) and hence the invariant form related to invariant under X can be expressed in the separable form:

$$u=\theta(x)=g(x)\phi(y) \quad ...(2.16)$$

In terms of an arbitrary function $\phi(y)$ of $y=(y_1(x),y_2(x),\dots,y_{n-1}(x))$ .

In [3], the BVP ((2.1)-(2.3)) which admits an r-parameter Lie group of point transformations with infinitesimal generators of the form:

$$X_i=\zeta_{ij}(x)\frac{\partial}{\partial x_j}+\eta_i(x,u)\frac{\partial}{\partial u},\quad i=1,2,\dots,r$$

...(2.17)

Then the unique solution $u=\theta(x)$ of the BVP ((2.1)-(2.3)) is an invariant solution satisfying :

$$X_i(u-\theta(x))=0 \text{ when } u=\theta(x),\quad i=1,2,\dots,r$$

...(2.18)

The invariance of a BVP under a multiparameter Lie group of point transformations is given by the following theorem:

**Theorem (2.3), [4]**

Suppose the BVP ((2.1)-(2.3)) admits an r-parameter Lie group of point transformations with infinitesimal generators of the form:

$$X_i=\zeta_{ij}(x)\frac{\partial}{\partial x_j}+\eta_i(x,u)\frac{\partial}{\partial u},\quad i=1,2,\dots,r$$

...(2.19)

Let R be the rank of the $r \times n$ matrix

$$E(x)=\begin{bmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \zeta_{r1} & \zeta_{r2} & \cdots & \zeta_{rn} \end{bmatrix} \qquad ...(2.20)$$

q = n-R , and let $z_1(x), z_2(x), \ldots, z_q(x)$ be a complete set of functionally independent invariants of (2.19) satisfying :

$$\zeta_{ij}(x)\frac{\partial z_\gamma}{\partial x_j}=0, i=1,2,\ldots,r, \gamma=1,2,\ldots,q \qquad ...(2.21)$$

Let

$$v=\frac{u}{g(x)} \qquad ...(2.22)$$

Be an invariant of (2.19) satisfying:

$$X_i v = 0 \ , \ i=1,2,\ldots,r \qquad ...(2.23)$$

Then the BVP ((2.1)-(2.3)) reduces to BVP with q =n-R independent variables $z=(z_1(x), z_2(x), \ldots, z_q(x))$ and dependent variable v given by (2.22). The solution of the BVP((2.1)-(2.3)) is an invariant solution that can be expressed in terms of a separable form:

$$u = g(x)\phi(z) \qquad ...(2.24)$$

Where the function $\phi(z)$ is to be determined [3].

## 3. APPLICATIONS

In this portion, we inserted some examples for solving BVPS for PDEs.

**Example (1):** consider the PDE given in the form:

$$\frac{\partial u(t,x)}{\partial t}=\frac{\partial^2 u(t,x)}{\partial x^2}-u(t,x) \qquad ...(1)$$

With boundary condition

$$u(0,t)=u(+\infty,t)=0 \quad , \ t>0 \qquad ...(2)$$

And initial condition

$$u(x,0)=u_0(x) \quad , \ 0<x<+\infty \qquad ...(3)$$

At first let the vector field in the style:

$$X=\zeta(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u} \qquad ...(4)$$

To solve equation (1) we must find the 2st prolongation of (4) given as:

$$X^{[2]}=X+\zeta_t\frac{\partial}{\partial u_t}+\zeta_x\frac{\partial}{\partial u_x}+\zeta_{tt}\frac{\partial}{\partial u_{tt}}+\zeta_{tx}\frac{\partial}{\partial u_{tx}}+\zeta_{xx}\frac{\partial}{\partial u_{xx}} \qquad ...(5)$$

Now , applying formula given in (5) to equation (1) as:

$$X^{[2]}\left(u_t-u_{xx}+u\right)\Big|_{(1)=0}=0 \qquad ...(6)$$

Then

$$\zeta_t-\zeta_{xx}+\eta=0 \qquad ...(7)$$

Then the determining equation given by :

$$\eta_t+u_t\left(\eta_u-\tau_t\right)-u_t^2\tau_u-u_x\zeta_t-u_xu_t\zeta_u-\eta_{xx}-2u_x\eta_{xu}-u_{xx}\eta_u-u_x^2\eta_{uu}+2u_{xx}\zeta_x+u_x\zeta_{xx}+u_x^2\zeta_{xu}$$
$$+\tau_u\left(u_tu_{xx}+2u_xu_{xt}\right)+u_x^3\zeta_{uu}+2u_{xt}\tau_x+u_t\tau_{xx}+2u_xu_t\tau_{xu}+u_x^2u_t\tau_{uu}+3u_xu_{xx}\zeta_u+u=0 \qquad ...(8)$$

Immediately , replacing $u_t$ by $u_{xx}-u$ we obtain:

$$\eta_t+\left(u_{xx}-u\right)\left(\eta_u-\tau_t\right)-\left(u_{xx}-u\right)^2\tau_u-u_x\zeta_t-\left(u_{xx}-u\right)u_x\zeta_u-\eta_{xx}-2u_x\eta_{xu}-u_{xx}\eta_u$$
$$-u_x^2\eta_{uu}+2u_{xx}\zeta_x+u_x\zeta_{xx}+u_x^2\zeta_{xu}+\tau_u\left(\left(u_{xx}-u\right)u_{xx}+2u_xu_{xt}\right)+u_x^3\zeta_{uu}+2u_{xt}\tau_x$$
$$+\left(u_{xx}-u\right)\tau_{xx}+2\left(u_{xx}-u\right)u_x\tau_{xu}+\left(u_{xx}-u\right)u_x^2\tau_{uu}+3u_xu_{xx}\zeta_u+u=0 \qquad ...(9)$$

Then

$$\eta_t+\left(-\tau_t+u\tau_u+2\zeta_x+\tau_{xx}\right)u_{xx}+\left(-\zeta_t+u\zeta_u-2\eta_{xu}+\zeta_{xx}-2\tau_{xu}\right)u_x+\left(-\eta_{uu}+\zeta_{xu}-u\tau_{uu}\right)u_x^2$$
$$+\left(2\zeta_u+2\tau_{xu}\right)u_xu_{xx}+2u_xu_{xt}\tau_u+u_x^3\zeta_{uu}+2u_{xt}\tau_x+u_x^2u_{xx}\tau_{uu}-u^2\tau_u-\eta_{xx}+u\left(\tau_t-\eta_u-\tau_{xx}+1\right)=0 \qquad ...(10)$$

Solve (10) by separation of the coefficient of (10) we find the following:

$$u_{xx}:-\tau_t+u\tau_u+2\zeta_x+\tau_{xx}=0 \qquad ...(11)$$

$$u_x:-\zeta_t+u\zeta_u-2\eta_{xu}+\zeta_{xx}-2\tau_{xu}=0 \qquad ...(12)$$

$$u_x^2:-\eta_{uu}+\zeta_{xu}-u\tau_{uu}=0 \qquad ...(13)$$

$$u_xu_{xx}:2\zeta_u+2\tau_{xu}=0 \qquad ...(14)$$

$$u_xu_{xt}:2\tau_u=0 \qquad ...(15)$$

$$u_x^3:\zeta_{uu}=0 \qquad ...(16)$$

$$u_{xt}:2\tau_x=0 \qquad ...(17)$$

$$u_x^2u_{xx}:\tau_{uu}=0 \qquad ...(18)$$

$$1:\eta_t-u^2\tau_u-\eta_{xx}+u\left(\tau_t-\eta_u-\tau_{xx}+1\right)=0 \qquad ...(19)$$

We find the general solution of the above system :

$$\tau(t,x,u)=\frac{1}{2}c_1t^2+c_2t+c_3 \qquad ...(20)$$

$$\zeta(t,x,u)=\frac{1}{2}c_1 x\,t+\frac{1}{2}\,c_2 x+c_4 t+c_5 \qquad ...(21)$$

$$\eta(t,x,u)=\left[\frac{1}{8}\left(-4t^2-x^2-2t\right)c_1-c_2 t-\frac{1}{2}c_4 x+c_6\right]u+\alpha(t,x) \qquad ...(22)$$

The Lie symmetries of (1) given as:

$$X_1=\frac{1}{2}x\,t\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}+\frac{1}{8}\left(-4t^2-x^2-2t\right)u\frac{\partial}{\partial u} \qquad ...(23)$$

$$X_2=\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-t\,u\frac{\partial}{\partial u} \qquad ...(24)$$

$$X_3=\frac{\partial}{\partial t} \qquad ...(25)$$

$$X_4=t\frac{\partial}{\partial t}-\frac{1}{2}x\,u\frac{\partial}{\partial u} \qquad ...(26)$$

$$X_5=\frac{\partial}{\partial x} \qquad ...(27)$$

$$X_6=u\frac{\partial}{\partial u} \qquad ...(28)$$

$$X_\alpha=\alpha(x,t)\frac{\partial}{\partial u} \qquad ...(29)$$

Now , to find the fundamental solution of the heat equation (1) as follows:
Consider the equation (1) defined on the domain t > 0 , a<x<b, recall the PDE given in (1) admitted by infinitesimal generator :

$$X=\zeta(x,t)\frac{\partial}{\partial x}+\tau(t)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u} \qquad ...(30)$$

With equations (20-22) . Then the boundary curves of the domain are t =0 , x= d , x= e, the invariance of t = 0 leads to

$$\tau(0)=0 \qquad ...(31)$$

We obtain $c_3=0$ , if $d=-\infty$ $and$ $e=\infty$ , then there is no further parameter reduction resulting form invariance of the boundary curves . if $d\neq-\infty$ then the invariance of $x=d$ leads to $\zeta(d,t)=0$ , for any t > 0 .
Then

$$\zeta(d,t)=\frac{1}{2}c_1 d\,t+\frac{1}{2}c_2 d+c_4 t+c_5=0 \qquad ...(32)$$

From equation (32)

$$c_5=-\frac{1}{2}c_2 d \quad , \quad c_4=-\frac{1}{2}c_1 d \qquad ...(33)$$

Similarity , if $e=\infty$ then the invariance of $x=e$ yields :

$$c_5=-\frac{1}{2}c_2 e \quad , \quad c_4=-\frac{1}{2}c_1 e$$

Consequently ,if $d\neq-\infty$ , $e\neq\infty$ the result $c_1=c_2=c_4=c_5=0$ , and so there is no non travail Lie group of point transformations admitted by heat equation with boundary condition given by equation (1-3) , defined on the domain t>0 , d<x<e we find from (32) .

**I-**If $d=-\infty$ $and$ $e=\infty$ , then the 4- parameter Lie group of point transformations is admitted by the boundary of problem given by (1-3) on the domain t>0 , d<x<e .Hence the BVP (1-3) could admitted most 5-parameter $(c_1,c_2,c_4,c_5,c_6)$ Lie group of point transformations.

**II-**If $d=-\infty$ (without damage of generality $d=0$) and $e=\infty$ then the 2- parameter Lie group of point transformations is admitted by the boundary of BVP (1-3) . Hence the BVP could admitted at most a 3-parameter $(c_3,c_4,c_5)$ .Lie group of point transformations with infinitesimals given by:

$$\tau(t)=\frac{1}{2}c_1 t^2+c_2 t \qquad ...(34)$$

$$\zeta(x,t)=\frac{1}{2}c_1 t\,x+\frac{1}{2}c_2 x \qquad ...(35)$$

$$\eta(x,t,u)=\left[\frac{1}{8}\left(-4t^2-x^2-2t\right)c_1-c_2 t+c_6\right]u \qquad ...(36)$$

Now , derive the fundamental solution for the heat equation , when

$$u(x,0)=\delta(x-x_0) \qquad ...(37)$$

Where $\delta(x-x_0)$ is the Dirac delta function centered at $x_0$ , d<$x_0$ <e , for an infinite domain $(d=-\infty,e=-\infty)$ or a semi–infinite domain $(d=0,e=\infty)$

**A- Infinite domain** $(d,e)=(-\infty,\infty)$

We see the fundamental solution $u=u(x,t,x_0,t_0)$ of the Cauchy problem defined as follows :

$$u_t=u_{xx}-u \quad ,t>0 \quad , -\infty<x<\infty \qquad ...(38)$$

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With boundary condition

$$u(\pm\infty, t) = 0 \quad , t > 0$$
$$u(x, 0) = \delta(x - x_0) \qquad ...(39)$$

Where $\delta(x - x_0)$ is the Dirac means at $x_0$ for infinite domain without loss of generality , we can take $x_0 = 0$ . Then the boundary curves of the domain are leads to $t = 0$ , $x = 0$ ,the invariance of $t = 0$ leads to.

$$\tau(0) = 0 \qquad ...(40)$$

We obtain $c_3 = 0$ , by the same way the invariance of x=0 leads to.

$$\zeta(0, t) = 0 \qquad ...(41)$$

For any t > 0 , and hence $c_5 = -c_4 t$ . The Lie group of point transformations with the infinitesimals given as ( 20 -22) is admitted by the initial boundary value problem (38) provided that is :

$$\eta(x,0)\,u(x,0) = \zeta(x,0)\,\delta'(x) \quad , where\ u(x,0) = \delta(x) \qquad ...(42)$$

Then

$$\eta(x, 0)\ \delta(x) = \zeta(x, 0)\ \delta'(x)$$

Here used Theorem (2.2) ,the infinitesimals generator:

$$X = \zeta_i(x)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}$$

is of the form

$$X = \zeta_i(x)\frac{\partial}{\partial x_i} + f(x, u)\frac{\partial}{\partial u}$$

From properties of the Dirac delta function equation (42) is satisfied if :

$$\zeta(0, 0) = 0 \qquad ...(43)$$
$$\eta(0, 0) = -\zeta_x(0, 0) \qquad ...(44)$$

Thus , in the infinitesimals equation (20-22) we have :

$$c_5 = 0$$
$$\zeta_x = \frac{1}{2}c_1 t + \frac{1}{2}c_2 \qquad ...(45)$$

Then

$$\zeta_x(0, 0) = \frac{1}{2}c_2 \qquad ...(46)$$

From $\eta(0, 0) = \zeta_x(0, 0)$ we obtain:

$$c_6 = -\frac{1}{2}c_2 \qquad ...(47)$$

And , hence 3-parameter $(c_1, c_2, c_4)$ Lie group of point transformations is admitted by the BVP (1-3) . This sub algebra of the Lie algebra span by the generator leaves invariant the initial manifold , that is , the line t = $t_0$.

And its restriction on t = $t_0$ convers the initial condition $u(0,t) = u(+\infty,t) = 0$ , $t > 0$ this sub algebra 3-dimensional algebra span by

$$X_1 = \frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} + \frac{1}{8}\left(-4t^2 - x^2 - 2t\right)u\frac{\partial}{\partial u} \qquad ...(48)$$

$$X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \left(t + \frac{1}{2}\right)u\frac{\partial}{\partial u} \qquad ...(49)$$

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}xu\frac{\partial}{\partial u} \qquad ...(50)$$

Corresponding to the infinitesimals :

$$\zeta = \frac{1}{2}c_1 xt + \frac{1}{2}c_2 x + c_4 t \qquad ...(51)$$

$$\tau = \frac{1}{2}c_1 t^2 + c_2 t \qquad ...(52)$$

$$\eta = \left[\frac{1}{8}\left(-4t^2 - x^2 - 2t\right)c_1 - c_2 t - \frac{1}{2}c_4 x - \frac{1}{2}c_2\right]u \qquad ...(53)$$

Let $u = \theta(x,t)$ be an invariant solution resulting from invariant under the infinitesimal generator $X_i$ then:

$$X_1 = \frac{1}{2}\left[tX_2 + \frac{1}{2}xX_3\right] = \frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} + \frac{1}{8}\left(-4t^2 - x^2 - 2t\right)u\frac{\partial}{\partial u} \qquad ...(54)$$

Hence an invariant solution corresponding to $X_2$ and $X_3$ is also an invariant solution corresponding to $X_1$ .

$u = \theta(x, t)$ an invariant solution corresponding to $X_1$ and $X_2$ then :

$$X_2\left(u - \theta(x, t)\right) = 0 \qquad ...(55)$$

By using the following equation given as:

$$Q = \eta - \zeta u_x - \tau u_t = 0 \qquad ...(56)$$

Then

$$Q = -\left(t + \frac{1}{2}\right)u - \frac{1}{2}xu_x - tu_t = 0 \qquad ...(57)$$

$$\frac{dx}{\frac{1}{2}x} = \frac{dt}{t} = \frac{du}{-\left(t + \frac{1}{2}\right)u} \qquad ...(58)$$

Now , we solve the characteristic equation (58) give the result as:

$$r_1 = \frac{x}{\sqrt{t}} \quad ...(59)$$

$$u = v\, t^{-\frac{1}{2}} e^{-t} \quad ...(60)$$

From (59) and (60) we obtain:

$$u = \theta(x,t) = \theta_1(r_1)t^{-\frac{1}{2}} e^{-t} \quad , where \quad r_1 = \frac{x}{\sqrt{t}} \quad ...(61)$$

By using (56) we find the following:

$$Q = -\frac{1}{2} x\, u - t\, u_x - 0.u_t = 0 \quad ...(62)$$

Now, we solve the characteristic equation we obtain:

$$\frac{dx}{t} = \frac{dt}{0} = \frac{du}{-\frac{1}{2}x\,u} \quad ...(63)$$

$$r_2 = t \quad ...(64)$$

$$u = v\, e^{-\frac{x^2}{4t}} \quad ...(65)$$

Then , from (64) and (65) we obtain the following:

$$u = \theta_2(r_2)e^{-\frac{x^2}{4t}} \quad , r_2 = t \quad ...(66)$$

From the uniqueness of the solution of the BVP (1-3) we get:

$$\theta_1(r_1)t^{-\frac{1}{2}} e^{-t} = \theta_2(r_2)e^{-\frac{x^2}{4t}} \quad ...(67)$$

$$\rightarrow \theta_1(r_1)r_2^{-\frac{1}{2}} e^{-r_2} = \theta_2(r_2)e^{-\frac{r^2}{4t}} \quad ...(68)$$

$$\rightarrow \theta_1(r_1)e^{\frac{r^2}{4t}} = \theta_2(r_2)\sqrt{r_2}\, e^{r_2} = c \quad ...(69)$$

Hence the solution of the BVP (1-3) on the domain $t > 0$ , $-\infty < x < \infty$ , with boundary condition :

$$u(-\infty,t) = 0,\ t > 0 \quad ...(70)$$

$$u(x,0) = \delta(x) \quad ...(71)$$

$$\rightarrow u = \frac{c}{\sqrt{t}} e^{-\left(\frac{x^2}{4t}+t\right)} \quad ...(72)$$

By using initial condition (71) we get $c = \frac{1}{\sqrt{4\pi}}$ ,

from $X_1$ we obtain automatically :

$$X_1\left(u - \frac{e^{-\left(\frac{x^2}{4t}+t\right)}}{\sqrt{4\pi t}}\right) = 0 \quad ...(73)$$

**B- semi- infinite Domain** $(d,e) = (0,\infty)$ consider the BVP (1) on the domain $t > 0$ , $x > 0$ with boundary condition given as:

$$u(0,t) = 0 \quad , \quad t > 0 \quad ...(74)$$

$$u(x,0) = \delta(x - x_0) \quad , 0 < x_0 < \infty \quad ...(75)$$

The 3-parameter Lie group of point transformations with infinitesimals give (51-53) is admitted by PDE (1) . Then the boundary curves $t = 0$ and $x = 0$ , and the boundary condition (2) , the invariance of the initial condition (3) leads to the restriction .

$$f(x,0)u(x,0) = \zeta(x,0)\delta'(x - x_0) \quad ...(76)$$

Where

$$u(x,0) = \delta(x - x_0) \quad ...(77)$$

$$\rightarrow f(x,0)\delta(x - x_0) = \zeta(x,0)\delta'(x - x_0) \quad ...(78)$$

Hence

$$\zeta(x_0,0) = 0$$

$$f(x_0,0) = -\zeta_x(x_0,0) \quad ...(79)$$

Consequently , in the infinitesimals equation (51-53) we must have from (79) :

$$c_2 = 0 \quad , \quad c_6 = \frac{1}{8}x_0^2 c_1 \quad ...(80)$$

Thus , the BVP of (1-3) admits the point symmetry :

$$X = \frac{1}{2}t\,x\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} + \left[\frac{1}{8}\left(-4t^2 - x^2 - 2t\right) + \frac{1}{8}x_0^2\right]u\frac{\partial}{\partial u} \quad ...(81)$$

The corresponding invariant solution has the invariant form :

$$Q = \eta - \zeta\, u_x - \tau\, u_t = 0 \quad ...(82)$$

$$\rightarrow \frac{1}{8}\left(-4t^2 - x^2 - 2t\right) + \frac{1}{8}x_0^2 - \frac{1}{2}t\,x\,u_x - \frac{1}{2}t^2 u_t = 0 \quad ...(83)$$

$$\rightarrow \frac{dx}{\frac{1}{2}t\,x} = \frac{dt}{\frac{1}{2}t^2} = \frac{du}{\left(\frac{1}{8}\left(-4t^2 - x^2 - 2t\right) + \frac{1}{8}x_0^2\right)u} \quad ...(84)$$

From solve (83) we obtain :

$$r_1 = \frac{x}{t} \quad ...(85)$$

$$u = \frac{e^{-\left(t+\frac{1}{4t}\left(x_0^2 - x^2\right)\right)}}{\sqrt{t}} \cdot \frac{1}{v} \quad ...(86)$$

Then

$$u = \theta(x,t) = \theta_1(r_1)\frac{e^{-\left(t+\frac{1}{4t}\left(x_0^2 - x^2\right)\right)}}{\sqrt{t}} , where\ r_1 = \frac{x}{t} \quad ...(87)$$

Where $\theta_1(r_1)$ arbitrarily function of the similarity variable $r_1 = \frac{x}{t}$ .After that substituting the invariant from (87) in (1) , we find that $\theta_1(r_1)$ satisfies the ODE:

$$u = \theta(x,t) = \theta(\zeta)\frac{e^{-\left(t+\frac{1}{4t}\left(x_0^2-x^2\right)\right)}}{\sqrt{t}} \ , \ where\ \zeta = \frac{x}{t}$$

...(88)

Now , substituting (88) in (1) we result:

$$\theta''(\zeta) = \left[-4t^2 + \frac{1}{4}x_0^2\right]\theta(\zeta)$$

...(89)

By solving equation (89) we obtain:

$$u = \theta(x,t) = c_1 \sin\left(\frac{1}{2}\sqrt{16t^2 - x_0^2 x}\right) + c_2 \cos\left(\frac{1}{2}\sqrt{16t^2 - x_0^2 x}\right)$$

...(90)

For constant $c_1$ *and* $c_2$ . From the boundary condition (74) of B we find $c_1 \tan t = c_2$, from the condition (75) of B we get $c_2 = 1$ then $c_1 = \frac{1}{\tan t}$. Then well- known solution of the BVP (1) an boundary condition (74-75):

$$u = \theta(x,t) = \frac{1}{\tan t}\sin\left(\frac{1}{2}\sqrt{16t^2 - x_0^2 x}\right) + \cos\left(\frac{1}{2}\sqrt{16t^2 - x_0^2 x}\right)$$

...(91)

**Example (2):** consider the PDE with Transient condition in semi-infinite solid with constant surface temperature given as :

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + x\frac{\partial u(x,t)}{\partial x} \quad ...(1)$$

With initial and boundary conditions:

$$u\big|_{t=0} = u_0 \ , \ u\big|_{x=0} = u_1 \quad , \quad u\big|_{x\to 0} = u_0$$

...(2)

To solve (1) write the vector field as follows:

$$X = \zeta(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$$

...(3)

After that , we need the 2-prolongation of X as:

$$X^{[2]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}$$

...(4)

Now , apply the formula give in (4) to equation (1) as:

$$X^{[2]}\left(u_t - u_{xx} - x u_x\right)\Big|_{(1)=0} = 0$$

...(5)

We obtain the determining equation as follows:

$$\zeta_t - x\zeta_x - \zeta_{xx} - \zeta u_x = 0$$

...(6)

By using definition of $\zeta_t, \zeta_x$ *and* $\zeta_{xx}$ we get:

$$\eta_t + u_t(\eta_u - \tau_t) - u_t^2\tau_u - u_x\zeta_t - u_x u_t\zeta_u - x\left[\eta_x + u_x(\eta_u - \zeta_x) - u_x^2\zeta_u - u_t\tau_x - u_x u_t\tau_u\right]$$
$$-\eta_{xx} - 2u_x\eta_{xu} - u_{xx}\eta_u - u_x^2\eta_{uu} + 2u_{xx}\zeta_x + u_x\zeta_{xx} + u_x^2\zeta_{xu} + \tau_u\left(u_t u_{xx} + 2u_x u_{xt}\right) + u_x^3\zeta_{uu}$$
$$+2u_{xt}\tau_x + u_t\tau_{xx} + 2u_x u_t\tau_{xu} + u_x^2 u_t\tau_{uu} + 3u_x u_{xx}\zeta_u - \zeta u_x = 0$$

...(7)

Immediately , replacing $u_t$ *by* $\left(u_{xx} + x u_x\right)$ we find:

$$\eta_t + (u_{xx} + xu_x)(\eta_u - \tau_t) - (u_{xx} + xu_x)^2\tau_u - u_x\zeta_t - (u_{xx} + xu_x)u_x\zeta_u$$
$$-x\left[\eta_x + u_x(\eta_u - \zeta_x) - u_x^2\zeta_u - (u_{xx} + xu_x)\tau_x - (u_{xx} + xu_x)u_x\tau_u\right] - \eta_{xx} - 2u_x\eta_{xu} - u_{xx}\eta_u$$
$$-u_x^2\eta_{uu} + 2u_{xx}\zeta_x + u_x\zeta_{xx} + u_x^2\zeta_{xu} + \tau_u\left((u_{xx} + xu_x)u_{xx} + 2u_x u_{xt}\right) + u_x^3\zeta_{uu} + 2u_{xt}\tau_x$$
$$+(u_{xx} + xu_x)\tau_{xx} + 2(u_{xx} + xu_x)u_x\tau_{xu} + (u_{xx} + xu_x)u_x^2\tau_{uu} + 3u_x u_{xx}\zeta_u - \zeta u_x = 0$$

...(8)

$$\eta_t - \eta_{xx} - x\eta_x + \left(-\tau_t + x\tau_x + 2\zeta_x + \tau_{xx}\right)u_{xx} + \left(-x\tau_t + x\zeta_x + x^2\tau_x - \zeta_t - 2\eta_{xu} + \zeta_{xx} + x\tau_{xx} - \zeta\right)u_x$$
$$+\left(-\eta_{uu} + \zeta_{xu} + 2x\tau_{xu}\right)u_x^2 + \left(\zeta_{uu} + x\tau_{uu}\right)u_x^3 + \left(2\tau_{xu} + 2\zeta_u\right)u_x u_{xx} + 2\tau_u u_x u_{xt} + \tau_{uu}u_x^2 u_{xx}$$
$$+2\tau_x u_{xt} = 0$$

...(9)

By separation of the coefficient $u_{xx}, u_x$ *and ect* we obtain :

$$u_{xx} : \tau_{xx} + 2\zeta_x - \tau_t + x\tau_x = 0 \quad ...(10)$$

$$u_x : x\zeta_x - x\tau_t + x^2\tau_x - \zeta_t - 2\eta_{xu} + \zeta_{xx} + x\tau_{xx} - \zeta = 0$$

...(11)

$$u_x^2 : \zeta_{xu} + 2x\tau_{xu} - \eta_{uu} = 0$$

...(12)

$$u_x^3 : \zeta_{uu} + x\tau_{uu} = 0$$

...(13)

$$u_x u_{xx} : 2\tau_{xu} + 2\zeta_u = 0$$

...(14)

$$u_x u_{xt} : 2\tau_u = 0$$

...(15)

$$u_x^2 u_{xx} : \tau_{uu} = 0 \quad ...(16)$$

$$u_{xt} : 2\tau_x = 0 \quad ...(17)$$

$$1 : \eta_t - \eta_{xx} - x\eta_x = 0 \quad ...(18)$$

Then the general solution of above system is:

$$\tau(x,t,u) = c_1 + c_2 e^{2t} + c_3 e^{-2t} \quad ...(19)$$

$$\zeta(x,t,u) = c_2 x e^{2t} - c_3 x e^{-2t} - c_4 e^{t} + c_6 e^{-t} \quad ...(20)$$

$$\eta(x,t,u) = \left(\left(-c_2 x^2 - c_2\right) e^{2t} + c_4 x e^{t} + c_5\right) u + \alpha(x,t) \quad ...(21)$$

Where $\alpha(x,t)$ is arbitrary function then:

$$X_1 = \frac{\partial}{\partial t} \quad ...(22)$$

$$X_2 = \left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \left(x^2+1\right)u\frac{\partial}{\partial u}\right)e^{2t} \quad ...(23)$$

$$X_3 = \left(-x\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)e^{-2t} \quad ...(24)$$

$$X_4 = \left(-\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}\right)e^{t} \quad ...(25)$$

$$X_5 = u\frac{\partial}{\partial u} \quad ...(26)$$

$$X_6 = \frac{\partial}{\partial x}e^{-t} \quad ...(27)$$

$$X_\alpha = \alpha(x,t)\frac{\partial}{\partial u} \quad ...(28)$$

Now, solution the IBVP:

We consider the general symmetry operator of the form:

$$X = k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 + k_5X_5 + k_6X_6 \quad ...(29)$$

of PDE (1) and for the operator that preserves the boundary and boundary conditions (2) the invariance of the boundaries x=0 , t=0 or equivalently:

$$\left[X\left(x-0\right)\right]_{x=0} = 0 \quad ...(30)$$

$$\left[X\left(t-0\right)\right]_{t=0} = 0 \quad ...(31)$$

From equation (30) we find :

$$k_4 = k_5 = 0 \quad ...(32)$$

From (31) we obtain :

$$k_1 = k_2 - k_3 \quad ...(33)$$

Hence

$$X = k_1X_1 + k_2X_2 + k_3X_3 + k_5X_5 \quad ...(34)$$

In addition to the restriction imposed (34) the invariance of the initial and boundary conditions :

$$\left[X\left(u(x,t)-u_0\right)\right]_{t=0} = 0 \quad , \; u(x,0) = u_0 \quad ...(35)$$

$$\left[X\left(u(x,t)-u_1\right)\right]_{x=0} = 0 \quad , \; u(0,t) = u_1 \quad ...(36)$$

From equations (35) and (36) we result $k_2 = k_5 = 0$ , hence the IVP of (1) and (2) is invariant under symmetry we have :

$$X = k_1X_1 + k_3X_3 \quad ...(37)$$

$$\rightarrow X = -x\frac{\partial}{\partial x} + \left(e^{-2t}+1\right)\frac{\partial}{\partial t} \quad ...(38)$$

Where $k_1 = k_3 = 1$ the invariant solution of the problem is constructed by utilizing the transformations through similarity variables for x . solving the characteristic system for $XI = 0$ gives:

$$Q = \eta - \zeta u_x - \tau u_t = 0 \quad ...(39)$$

Then

$$Q = 0 + xu_x - \left(e^{-2t}+1\right)u_t = 0 \quad ...(40)$$

$$\rightarrow \frac{dx}{x} = \frac{dt}{\left(e^{-2t}+1\right)} \quad ...(41)$$

From equation (41) we obtain:

$$\zeta(t,x) = x^2\left(e^{-2t}+1\right)e^{2t} \quad ...(42)$$

$$v(\zeta) = u \quad ...(43)$$

Now , substituting of similarity variables in (1) satisfies that corresponding similarity solution of PDE (1) of the form given in (43) where $v(\zeta)$ satisfies the ODE given as:

$$\frac{d^2v}{d\zeta^2} + \frac{\left(x^2 + e^{2t}\left(e^{-2t}+1\right)\right)}{2\zeta e^{2t}\left(e^{-2t}+1\right)}\frac{dv}{d\zeta} = 0 \quad ...(44)$$

The above equation (44) can be solved by substitution $w = \frac{dv}{d\zeta}$ we get:

$$v(\zeta) = c_1 + c_2\zeta^{\frac{3e^{2t}\left(e^{-2t}+1\right)+x^2}{2e^{2t}\left(e^{-2t}+1\right)}} \quad ...(45)$$

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From equations (42),(43) and (45) we obtain the exact solution of PDE (1) that the invariant under (38)

$$u(x,t)=c_1+c_2\left(\frac{x}{t}\right)^{\frac{3e^{2t}\left(e^{-2t}+1\right)+x^2}{2e^{2t}\left(e^{-2t}+1\right)}} \quad ...(46)$$

Imposing the initial and boundary condition determines when $x=0, u=u_1$

$$u_1=c_1 \quad ...(47)$$

Now, when $t=0$ , $u=u_0$ we find :

$$c_2=\left(u_2-u_1\right)erf^{-1}\left(\frac{3\left(e^{-2t}+1\right)+x^2}{2\sqrt{2}\left(e^{-2t}+1\right)}\right) \quad ...(48)$$

Then we write the general solution for PDE (1) as:

$$u(t,x)=u_1+\left(u_2-u_1\right)erf^{-1}\left(\frac{3\left(e^{-2t}+1\right)+x^2}{2\sqrt{2}\left(e^{-2t}+1\right)}\right)\left(\frac{x}{t}\right)^{\frac{3e^{2t}\left(e^{-2t}+1\right)+x^2}{2e^{2t}\left(e^{-2t}+1\right)}} \quad ...(49)$$

**Example (3):** consider the nonlinear PDE given as:

$$\frac{\partial u(x,t)}{\partial t}=u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}+\left(\frac{\partial u(x,t)}{\partial x}\right)^2 \quad ...(1)$$

With initial – boundary conditions

$$u\big|_{t=0}=u_1 \quad , \quad u\big|_{x=0}=u_2 \quad , \quad u\big|_{x\to\infty}=u_1$$

$$u\big|_{t=0}=1 \quad , -k\frac{\partial u}{\partial x}\bigg|_{x=0}=q_0'' \quad , u\big|_{x\to\infty}=1 \quad ...(2)$$

To solve problem (1) write the vector field as follows:

$$X=\zeta(t,x,u)\frac{\partial}{\partial x}+\tau(t,x,u)\frac{\partial}{\partial t}+\eta(t,x,u)\frac{\partial}{\partial u} \quad ...(3)$$

We need the 2nd prolongation of the form:

$$X^{[2]}=X+\zeta_t\frac{\partial}{\partial u_t}+\zeta_x\frac{\partial}{\partial u_x}+\zeta_{tt}\frac{\partial}{\partial u_{tt}}+\zeta_{tx}\frac{\partial}{\partial u_{tx}}+\zeta_{xx}\frac{\partial}{\partial u_{xx}} \quad ...(4)$$

Now, applying the prolongation give in equation (4) to equation (1) we find:

$$X^{[2]}\left(u_t-uu_{xx}-u_x^2\right)\Big|_{(1)=0}=0 \quad ...(5)$$

Then the determining equation gives:

$$\zeta_t+\zeta_{xx}u+2\zeta_x u_x+\eta u_{xx}=0 \quad ...(6)$$

Then

$$\eta_t+u_t\left(\eta_u-\tau_t\right)-u_t^2\tau_u-u_x\zeta_t-u_xu_t\zeta_u+u\begin{bmatrix}\eta_{xx}+2u_x\eta_{xu}+u_{xx}\eta_u+u_x^2\eta_{uu}-2u_{xx}\zeta_x-u_x\zeta_{xx}\\-u_x^2\zeta_{xu}-\tau_u\left(u_tu_{xx}+2u_xu_{xt}\right)-u_x^3\zeta_{uu}-2u_{xt}\tau_x\\-u_t\tau_{xx}-2u_xu_t\tau_{xu}-u_x^2u_t\tau_{uu}-3u_xu_{xx}\zeta_u\end{bmatrix}$$

$$+2u_x\left[\eta_x+u_x\left(\eta_u-\zeta_x\right)-u_x^2\zeta_u-u_t\tau_x-u_xu_t\tau_u\right]+\eta u_{xx}=0 \quad ...(7)$$

Yet , replacing $u_t$ *by* $\left(uu_{xx}+u_x^2\right)$ we obtain:

$$\eta_t+\left(uu_{xx}+u_x^2\right)\left(\eta_u-\tau_t\right)-\left(uu_{xx}+u_x^2\right)^2\tau_u-u_x\zeta_t-\left(uu_{xx}+u_x^2\right)u_x\zeta_u$$

$$+u\begin{bmatrix}\eta_{xx}+2u_x\eta_{xu}+u_{xx}\eta_u+u_x^2\eta_{uu}-2u_{xx}\zeta_x-u_x\zeta_{xx}-u_x^2\zeta_{xu}\\-\tau_u\left(\left(uu_{xx}+u_x^2\right)u_{xx}+2u_xu_{xt}\right)-u_x^3\zeta_{uu}-2u_{xt}\tau_x-\left(uu_{xx}+u_x^2\right)\tau_{xx}\\-2\left(uu_{xx}+u_x^2\right)u_x\tau_{xu}-\left(uu_{xx}+u_x^2\right)u_x^2\tau_{uu}-3u_xu_{xx}\zeta_u\end{bmatrix}$$

$$+2u_x\left[\eta_x+u_x\left(\eta_u-\zeta_x\right)-u_x^2\zeta_u-\left(uu_{xx}+u_x^2\right)\tau_x-\left(uu_{xx}+u_x^2\right)u_x\tau_u\right]+\eta u_{xx}=0 \quad ...(8)$$

Then

$$\left(u\left(\eta_u-\tau_t\right)+u\eta_u-2u\zeta_x-u^2\tau_{xx}+\eta\right)u_{xx}+\left(-2u^2\tau_u\right)u_{xx}^2+\left(-4u\zeta_u-2u^2\tau_{xu}-2u\tau_x\right)u_xu_{xx}$$

$$+\left(-3u\tau_u-u^2\tau_{uu}\right)u_x^2u_{xx}+\left(\eta_u-\tau_t+u\eta_{uu}-u\zeta_{xu}+u\tau_{xx}+2\left(\eta_u-\zeta_x\right)\right)u_x^2$$

$$+\left(-3\zeta_u-2u\tau_{xu}-2\tau_x-u\zeta_{uu}\right)u_x^3+\left(-\zeta_t+2u\eta_{xu}-u\zeta_{xx}+2\eta_x\right)u_x$$

$$+\left(-3\tau_u-u\tau_u\right)u_x^4-2u_{xx}u_x^3\tau_u-2u_xu_{xt}u\tau_u-2u_{xt}u\tau_x+\eta_t+u\eta_{xx}=0 \quad ...(9)$$

By separation the coefficient of variables $u_{xx}$ , $u_x$ and ect :

$$u_{xx}:u\left(2\eta_u-\tau_t-2\zeta_x\right)-u^2\tau_{xx}+\eta=0 \quad ...(10)$$

$$u_{xx}^2:-2u^2\tau_u=0 \quad ...(11)$$

$$u_xu_{xx}:-4\zeta_u-2u^2\tau_{xu}-3u\zeta_u-2u\tau_x=0 \quad ...(12)$$

$$u_x^2u_{xx}:-3u\tau_u-u^2\tau_{uu}=0 \quad ...(13)$$

$$u_x:-\zeta_t+2u\eta_{xu}-u\zeta_{xx}+2\eta_x=0 \quad ...(14)$$

$$u_x^2:\eta_u-\tau_t-u\eta_{uu}-u\zeta_{xu}+u\tau_{xx}+2\left(\eta_u-\zeta_x\right)=0 \quad ...(15)$$

$$u_x^3 : -3\zeta_u - 2u\tau_{xu} - 2\tau_x - u\zeta_{uu} = 0 \quad ...(16)$$

$$u_x^4 : -3\tau_u - u\tau_u = 0 \quad ...(17)$$

$$u_{xx}u_x^3 : 2\tau_u = 0 \quad ...(18)$$

$$u_x u_{xt} : -2\tau_u = 0 \quad ...(19)$$

$$u_{xt} : -2u\tau_x = 0 \quad ...(20)$$

$$1 : \eta_t + u\eta_{xx} = 0 \quad ...(21)$$

We obtain the general solution of above system is:

$$\tau = c_1 t + c_2 \quad ...(22)$$

$$\zeta = \frac{1}{2}(c_1 + c_3)x + c_4 \quad ...(23)$$

$$\eta = c_3 u \quad ...(24)$$

Then the Lie symmetries given as:

$$X_1 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \quad ...(25)$$

$$X_2 = \frac{\partial}{\partial t} \quad ...(26)$$

$$X_3 = \frac{1}{2}x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} \quad ...(27)$$

$$X_4 = \frac{\partial}{\partial x} \quad ...(28)$$

We consider the general symmetry operator of PDE (1) :

$$X = k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 \quad ...(29)$$

Now , solve PDE (1) by using the first boundary condition search for the operator that preserves the boundary and boundary condition (2):

Then the invariance of the boundaries x=0 , t=0 or identically :

$$[X(x-0)]_{x=0} = 0 \quad ...(30)$$

$$[X(t-0)]_{t=0} = 0 \quad ...(31)$$

From (30) we obtain:

$$k_2 = k_4 = 0 \quad ...(32)$$

Then X must have:

$$X = k_1 X_1 + k_2 X_2 \quad ...(33)$$

In addition to restriction imposed by equation (32) the invariance of initial and boundary conditions write:

$$\left[X\left(u(x,t) - u_1\right)\right]_{t=0} = 0 \quad , \; u(x,0) = u_1 \quad ...(34)$$

$$\left[X\left(u(x,t) - u_2\right)\right]_{x=0} = 0 \quad , \; u(0,t) = u_2 \quad ...(35)$$

From (34) we find $k_3 = 0$ . Hence the IBVP (1) and (2) is invariant under the symmetry :

Now, we have chosen $k_1 = 1$ , we write the generator:

$$X_1 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \quad ...(36)$$

The invariant solution of the problem is constructed by utilizing the transformations through similarity variables for X. Solving the characteristic system for $XI = 0$ , gives :

$$\frac{dx}{\frac{1}{2}x} = \frac{dt}{t} = \frac{du}{0} \quad ...(37)$$

By solving (37) we result :

$$\zeta(x,t) = \frac{x^2}{t} \quad ...(38)$$

$$v(\zeta) = u \quad ...(39)$$

After that , substituting of similarity variables equation (1) satisfies the corresponding similarity solution of PDE(1) is of the form $v(\zeta) = u$ where $v(\zeta)$ satisfies the ODE as:

$$\frac{d^2v}{d\zeta} + \frac{1}{v}\left(\frac{dv}{d\zeta}\right)^2 + \left(\frac{1}{2\zeta} + \frac{1}{4v}\right)\frac{dv}{d\zeta} = 0 \quad ...(40)$$

Then the general solution of above system given as:

$$v(\zeta) = \ln\left(2c_1\sqrt{\pi}\, erf\left(\frac{1}{2}\sqrt{\zeta}\right) + c_2\right) \quad ...(41)$$

Where erf denotes the error function . Hence the exact solution of PDE(1) that is invariant under $X = X_1 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$ is :

$$u(x,t) = \ln\left(2c_1\sqrt{\pi}\, erf\left(\frac{1}{2}\frac{x}{\sqrt{t}}\right) + c_2\right) \quad ...(42)$$

Imposing the initial and boundary conditions determines:

When $x = 0$ we obtain :

$$u_2 = c_2 \quad ...(43)$$

When $t = 0$ we find:

$$c_1 = \frac{1}{2\sqrt{\pi}}\left(e^{u_1} - u_2\right) \quad ...(44)$$

Taking the general symmetry operator given in equation (29) of PDE (1) , we determines the operator that leaves the boundary and the boundary conditions in second part of equation (2) invariant :
Now, the invariance of the boundaries x=0 , t=0 or equivalently :

$$\left[X\ (x - 0)\right]_{x=0} = 0 \quad ...(45)$$

$$\left[X\ (t - 0)\right]_{t=0} = 0 \quad ...(46)$$

From (45) and (46) yields:

$$k_2 = k_4 = 0 \quad ...(47)$$

Then X must be write:

$$X = k_1 X_1 + k_2 X_2 \quad ...(48)$$

Since

$$\left[X\ (u - 1)\right]_{t=0} = 0 \quad ...(49)$$

From (49) gives $k_3 u = 0$ , the invariance of the condition $u\big|_{t=0} = 1$ dose note impose any restriction on x .

Now , the invariance of the boundary condition is:

$$\left[-k\frac{\partial u}{\partial x} = q''_0\right]_{x=0} \quad ...(50)$$

The (50) requires the 1st prolongation:

$$\left[X^{[1]}\left(k\frac{\partial u}{\partial x} + q''_0\right)\right]_{x=0} \text{ ,on} -k\frac{\partial u}{\partial x} = q'' \quad ...(51)$$

Then

$$\eta_x + u_x\left(\eta_u - \zeta_x\right) - u_x^2\zeta_u - u_t\tau_x - u_x u_t \tau_u - \tau q'''_0 = 0 \quad ...(52)$$

From (52) we get :

$$k_1 = k_3 \quad ...(53)$$

Choosing $k_1 = k_3 = 1$ ,provides the symmetry:

$$X = X_1 + X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \quad ...(54)$$

That leaves the IBVP (1) and the second part of condition (2) invariant , to find the similarity transformations that will lead to the solution the characteristic system for $XI = 0$ .is solved this provides the similarity variables :

$$\frac{dx}{x} = \frac{dt}{t} = \frac{du}{u} \quad ...(55)$$

By solved equation (55) we find:

$$v(\zeta) = \frac{u}{x} \quad ...(56)$$

$$\zeta(x,t) = \frac{x}{t} \quad ...(57)$$

Now , substituting similarity variables in PDE (1) where $v(\zeta)$ satisfies the ODE.

$$\zeta^2 v(\zeta)\frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial v}{\partial \zeta}\left(2\zeta v(\zeta) + \zeta^2 + \frac{\partial v}{\partial \zeta}\right) + 2\zeta v(\zeta) + v^2(\zeta) = 0 \quad ...(58)$$

Then the general solution of above ODE equation (58) is:

$$v(\zeta) = c_1 e^{-\zeta} + c_2\left(erf\ \sqrt{\zeta} + \frac{1}{\zeta}\right) \quad ...(59)$$

Then

$$u(x,t) = c_1 e^{-\frac{x}{t}} + c_2\left(erf\sqrt{\frac{x}{t}} + \frac{t}{x}\right) \quad ...(60)$$

When t = 0 , $u = 1$ then $c_2 = 1$ .

When x=0 then $-k\frac{\partial u}{\partial x} = q''_0$ we obtain

$$c_1 = \frac{-q''_0}{k} + \frac{2}{\sqrt{\pi}} .$$

$$\rightarrow u(x,t) = \left(\frac{-q''_0}{k} + \frac{2}{\sqrt{\pi}}\right)e^{-\frac{x}{t}} + \left(erf\sqrt{\frac{x}{t}} + \frac{t}{x}\right) \quad ...(61)$$

## 4. CONCLUSION AND DISCUSSION

We established in this study of Lie group theory is applied to PDEs to determine symmetries. The one parameter Lie group which leaves the PDEs invariant, in example (1) and (2) we want to construct a solution to boundary and initial value problem for PDEs , in example (3) we use technique assumption for constant after substituting of initial boundary value problem to find the fundamental solution and introduced Derivative for condition , we solved it by using 1st prolongation at last we get the general solution.

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