THE BACKSTEPPING METHOD FOR STABILIZING TIME FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT

In this article the nonlinear time fractional order partial differential equation (NTFPDE) subject to a boundary controller at the boundary is considered. The semi-discretized backstepping control technique is used for stabilize the partial differential equation with fractional order $0 < q \leq 1$. To the author best knowledge, this is the first time in the literature that the backstepping method is being used for stabilizing NTFPDE. Illustrative example is given to demonstrate the effectiveness of the proposed control scheme. Simulation results show that the proposed design not only can stabilize the NTFPDE but performs better than an integer order as well.

Keywords: *Backstepping Method*, *Fractional Lyapunov Function, Fractional Derivative*, *Boundary Control*, *Fractional Euler's Method*.

1. INTRODUCTION

In recent years, the number of scientific and engineering problems containing fractional derivative and control is already large and has gain a huge amount of attention. The concept of fractional calculus has interacted with the control community deeply due to the fractional order controller is proved to be given a more freedom in the design [1,2].

The backstepping control is one category of control approaches that has gain a considerable attention in the case of controlling parametric nonlinear strict feedback systems.

Due to the huge advantages the backstepping technique gives in integer order, such as global stability, good tracking and transient performance. The technique has been extensively studied in many areas. A number of result using this technique can be cited as robotics [3-6], neural networks [7-10], and secure communications [11-14] and several other research works can be found in the literature [15-20].

However, it has been very few research in the literature that are succeeded to apply the backstepping method on the case of the fractional order system. For instance, for the first time, Efe has tried to extend the backstepping technique to fractional order systems in [21]. Next, Sahab has implemented a generalization backstepping method in order to find an approximation error of the fractional differential equation regarding two new hyperchaotic system of fractional order [22]. In [23], the author has used the backstepping method to described and designed a controller for a fractional order chaotic system control issue.

Earlier this century, to invertebrates a new method to deal with partial differential equations (PDEs), the backstepping approach was developed. The development of a continuum backstepping approach for stabilizing parabolic linear PDEs was first introduced by Smyshlyaev and Krstic in 2004, [24]. While backstepping design for linearized Navier–Stokes equations have been introduced by Vazquez and Krstic in 2007, [25]. The extension of backstepping approach to the second-order hyperbolic PDEs is given by Krstic *et al.* in 2008, [26], [27]. Then, in 2008 a new adaptive designs for boundary control has been developed by Krstic and Smyshlyaev, for the linear parabolic PDEs with unknown parameters [28], [29]. Also, Krstic and Smyshlyaev in 2008 [30] developed the backstepping design for the first-order hyperbolic PDEs and presented a design for linear time invariant ordinary differential equations (ODEs) with time delays, these recovers the classical predictor designs for the finite spectrum assignment. In 2008, Vazquez and Krstic [31], [32] introduced for the first time boundary control designs of nonlinear PDEs, focusing on a certain class of parabolic PDEs with nonlinear functions and Volterra series nonlinear operators. In 2008, Krstic [33] employed an infinite-dimensional backstepping transformation, in connection with Lyapunov function, these results in infinite dimensional systems consisting of ODE plant state and delay state. Krstic in 2009, introduced an approach to design a least square estimator with the use of unfiltered regress. Then he presented the first last squares based adaptive nonlinear control design which yields completely to a Lyapunov function [34]. The next step for Krstic was to introduced an approach for compensating input delay of arbitrary length in nonlinear control system which is a nonlinear version of the smith predictor and it's various predictor based modifications for linear plants. This method deals with the infinite dimensionality of the actuator dynamics [35]. In 2010, Krstic considered the closed loop system with a time varying Lyapunov functional equation and he established the exponential stability [36]. The challenge is the selection of a state for a transport PDE, which has a non-constant propagation speed, and is the basis of the stability analysis. In 2010, Smyshlyaev and Krstic introduces a comprehensive methodology for adaptive control design of parabolic PDEs with unknown functional parameters, including reaction-convection-diffusion systems ubiquitous in chemical, thermal, biomedical, aerospace and energy systems [37]. In 2013, Bekiaris and Krstic consider nonlinear systems with time delays that depend on the delayed state, i.e., the delay is defined implicitly as a nonlinear function of the state at a past time, which depends on the delay parameter itself, [38]. Krstic and Bekiaris in 2013 [39], review several representative but with general results on nonlinear control in the infinite-dimensional setting. Firstly, they present certain designs for nonlinear ODEs with constant time-varying or state-dependent input delays that arise in numerous applications of networks control. Secondly, they present a design for nonlinear ODEs with a wave (string) PDE at its input, which is motivated by the drilling dynamics in petroleum engineering. Third, present a design for systems of two coupled nonlinear first-order hyperbolic PDEs, which is motivated by slugging flow dynamics in petroleum production in off- shore facilities. Bernard and Krstic in 2014 [40] address the problem of adaptive output feedback stabilization of general first-order hyperbolic partial integro differential equations (PIDE), where such systems are also referred to as PDEs with non-local (in space) terms, apply control at one boundary, take measurements on the other boundary, and allow the system's functional coefficients to be unknown.

However, to the best of the author knowledge, there are not many attempts concerning the boundary feedback stabilization of an unstable time fractional-order diffusion system. The boundary stabilization for one dimensional fractional diffusion wave equation, based on numerical solution techniques, has been studied in [41,42]. In those studies, the focus was to use the fractional order boundary controller and derive the boundary control of a caputo fractional wave equation. In addition, in 1D system of the heat conduction process. Fourier law and the connection between anomalous diffusion are not satisfied [43]. It is confirmed that many real-world life systems can be well characterized by utilizing the notions of fractional order [44, 45], this is the reason why the fractional-order models are superior in comparison with the integer-order models.

In this paper, we propose the backstepping method for stabilizing NTFPDE. To the best of our knowledge, this is the first time in the literature that the backstepping is being used for stabilizing NTFPDE. The semi-discretized fractional-order backstepping approach will be introduced to find the boundary controller function which stabilizes the NTFPDE by transformation it into an equivalent stable closed loop. We describe fractional derivative by using Caputo definition for different order q with $q \in (0,1]$. Then our attempt is to design the feedback control law analytically using the fractional order backstepping. Illustrative example is presented to demonstrate the approach efficiency. The main aim of tis contribution is to derive a systematic method of constructing Mittag–Leffler stable closed-loop systems for NTFPDE and a global convergence is built into them.

The rest of this article is organized as follows: Some definitions and related theorems for fractional order calculus is listed in section two, and we illustrate in section three the main results of backstepping approach to stabilize NTFPDE based on fractional Lyapunov function. Finally, section four provides an example and the result is illustrated the availability of our proposed method. The conclusions is devoted in section five.

2. FRACTIONAL CALCULUS

In this section, we introduce the definitions of fractional derivative and some related theorems which are used further in this paper.

Definition 1 [46].
The Riemann-Liouville fraction-al integral operator of order $q \geq 0$, of a function $f \in C_\mu,\ \mu \geq -1$ is defined as

$$J^q f(x) = \frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1} f(s)ds,\ q>0,\ x>0 \qquad (1)$$

with the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$ (2)

Definition 2 [46].
The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D^q f(x) = J^{m-q}D^q \frac{1}{\Gamma(m-q)}\int_0^x (x-s)^{m-q-1} f^{(m)}(s)ds$$

for $m-1<q<m,\ m\in \mathrm{N},\ x>0$ (3)

Theorem 1 [47].
Assume that both $f\ (u)$ and $u\ (x)$ are q times differentiable with u and x respectively. The chain rule of fractional derivative can be described as the following equation

$$\frac{\partial^q f(u(x))}{\partial x^q} = \Gamma(2-q)u^{q-1}\frac{\partial^q f(u)}{\partial u^q}\frac{\partial^q u(x)}{\partial x^q} \qquad (4)$$

Theorem 2 (Mittage-Leffler stability) [48].
Let $u(t)=0$ be the equilibrium point of the fractional order system $D^q u = f(u,t), u\in\Omega$, where Ω is a neighborhood region of the origin. Assume that there exists a fractional Lyapunov function $V(t,u(t)):[0,\infty)\times\Re^n \to \Re$ and K-class functions $\xi_i,\ i=1,2,3$ satisfying

1. $\xi_1(\|u\|) \leq V(t,u(t)) \leq \xi_2(\|u\|);$ (5)
2. $D^q V(t,u(t)) \leq -\xi_3(\|u\|).$ (6)

Then the fractional order system is asymptotically Mittage-Leffler stable. Moreover, if $\Omega=\Re^n$, the fractional order system is globally asymptotically Mittage- Leffler stable.

Definition 3 [49].
A smooth function $V(t,u(t)):[0,\infty)\times\Re^n\to\Re$ is called a control fractional Lyapunov function for the fractional order system $D^q u = f(u,U), u\in\Re^n$, $f(0,0)=0$ with the control law $U=\alpha(u)$ if there exist three K-class functions $\xi_i,\ i=1,2,3$ such that

1. $\xi_1(\|u\|) \leq V(t,u(t)) \leq \xi_2(\|u\|);$.
2. $D^q V(t,u(t)) \leq -\xi_3(\|u\|).$

Lemma 1 [49].
Let $u(t)\in\Re$ be a real continuously differentiable function. Then for any

$$D^q u^r(t) \leq r u^{(r-1)}(t) D^q u(t) \qquad (7)$$

where $0<q\leq 1$ is the fractional order.

Lemma 2 [49].
For the fractional order system $D^q u = f(u,U), u\in\Re$, $0<q\leq 1,\ f(0,0)=0$ with the control law $U=\alpha(u)$ is asymptotically Mittage-Leffler stable if for $r=2^n, n\in\mathrm{N}$, there exist a K-class functions ξ, such that

$$u^{r-1}D^q u = u^{r-1} f(u,\alpha(u)) \leq -\xi(\|u\|) \qquad (8)$$

3. MAIN RESULTS

Consider the following nonlinear time fractional order partial differential equation

$${}^cD_t^q u(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (9)$$

where ${}^cD_t^q$ is the fractional derivative of $u(x,t)$ of order q with respect to t in the sense of Caputo and the fractional order q belong to (0,1], $u\in L^2(\Omega)$, $\Omega=(0,1)\times[0,\mathrm{T}],\ \mathrm{T}>0$, and f is a nonlinear function of u such that $f\in C^\infty(\Re)$. With initial condition

$$u(x,0)=g(x),\quad 0<x<1 \qquad (10)$$

The boundary condition at $x=0$ is homogenous Dirichlet

$$u(0,t)=0,\quad t\geq 0 \qquad (11)$$

and the boundary condition at other end

$$u(1,t)=U(t) \qquad (12)$$

where $U(t): C[0,1]\to C[0,1]$ is the unknown non-linear feedback control function to be design to achieve stabilization.

The backstepping design technique is applied to obtain the boundary control function of equation (9). The design procedure is divided into the following stages:

In the first stage the nonlinear time fractional order partial differential equation (9) will be semi-discretized into an equivalent nonlinear system of fractional order as follows:

Fix $N \in \mathbb{N}$ and $h = \frac{1}{N+1}$ as the step size of discretization of system (9)-(12) over the interval of the space variable $x \in (0,1)$. Also, let $u_i(t) = u(ih,t)$ for all $i = 0,1,\ldots,N+1$ where it is assumed that $u_0(t)$ is the boundary condition at $x = 0$ and $u_{N+1}(t)$ is the control function at $x = 1$, hence using the central differencing for discretizing $u_{xx}(x,t)$, we have

$$u_0(t) = 0 \tag{13}$$

$${}^cD_t^q u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f(u_i(t)),\ i = 1,2,\ldots,N \tag{14}$$

$$u_{N+1} = U(t) \tag{15}$$

We can write the nonlinear semi-discretized system of fractional differential equations as:

$$\begin{aligned} {}^cD_t^q u_1 &= \frac{1}{h^2}u_2 - \frac{2}{h^2}u_1 + f(u_1) \\ {}^cD_t^q u_2 &= \frac{1}{h^2}u_3 - \frac{2}{h^2}u_2 + \frac{1}{h^2}u_1 + f(u_2) \\ &\vdots \\ {}^cD_t^q u_n &= \frac{1}{h^2}U - \frac{2}{h^2}u_n + \frac{1}{h^2}u_{n-1} + f(u_n) \end{aligned} \tag{16}$$

In the second stage we will design the need controller according to the idea of backstepping. The backstepping design procedure requires n steps, and the virtual control α_i and the controller U will be constructed. The design procedure is elaborated in the following.

The coordinate transformation of error variables can be expressed as

$$w_i = u_i - \alpha_{i-1},\ i = 1,2,\ldots,n \tag{17}$$

where $\alpha_0 = 0$ and $w_{n+1} = 0$

Step 1: we start with first equation of (16). Design a suitable stabilizing function α_1 to stabilize $w_1(t)$. Select the first fractional Lyapunov function

$$v_1 = \frac{1}{2}w_1^2 \tag{18}$$

Then the q-th order time derivative of v_1 is given by

$$D_t^q v_1 \le -k_1 w_1^2 + \frac{1}{h^2}w_1 w_2 + w_1\left(\frac{1}{h^2}\alpha_1 - \frac{2}{h^2}w_1 + k_1 w_1\right) \tag{19}$$

The virtual control law α_1 is designed as

$$\alpha_1 = (2 - k_1 h^2)u_1 - h^2 f(u_1) \tag{20}$$

where $k_1 > 0$ is a design parameter. w_2 is to be governed to zero. Then the resulting q-th order derivative is

$$D_t^q v_1 \le -k_1 w_1^2 + \frac{1}{h^2}w_1 w_2,\ k_1 > 0 \tag{21}$$

Step 2: Study the second equation of eq.(16) by considering α_2 as a virtual control variable. The control objective is to make $w_2 \to 0$ as $t \to \infty$. Define a second fractional control Lyapunov function as

$$v_2 = \frac{1}{2}w_2^2 \tag{22}$$

and its q-th order time derivative is given by

$$\begin{aligned} D_t^q v_2 \le{}& -k_1 w_1^2 - k_2 w_2^2 + \frac{1}{h^2}w_2 w_3 + w_2\Big(\frac{1}{h^2}\alpha_2 \\ & -\frac{2}{h^2}u_2 + \frac{1}{h^2}u_1 + f(u_2) + k_2 w_2 + \frac{1}{h^2}w_1 \\ & -\Gamma(2-q)u_1^{q-1}D_{u1}^q\alpha_1 D_t^q u_1 \end{aligned} \tag{23}$$

By selecting

$$\begin{aligned} \alpha_2 = h^2\Big(&-k_2 w_2 - \frac{1}{h^2}w_1 + \frac{2}{h^2}u_2 - \frac{1}{h^2}u_1 - \\ & f(u_2) + \Gamma(2-q)u_1^{q-1}D_{u1}^q\alpha_1 D_t^q u_1 \end{aligned} \tag{24}$$

where $k_2 > 0$, is the design parameter, w_3 is to be governed to zero. Thus we have

$$D_t^q v_2 \le -\sum_{i=1}^{2} k_i w_i^2 + \frac{1}{h^2}w_2 w_3 \tag{25}$$

Step *i* ($i = 3,\ldots,n-1$) study the *ith* equation of eq.(16) with the virtual control variable α_i. The control fractional Lyapunov function is chosen as

$$v_i = v_{i-1} + \frac{1}{2}w_i^2, \tag{26}$$

Its q-th time derivative is given by

$$\begin{aligned} D_t^q v_i \le{}& -\sum_{j=1}^{i} k_j w_j^2 + \frac{1}{h^2}w_i w_{i+1} + w_i\Big(\frac{1}{h^2}\alpha_i + k_i w_i \\ & + \frac{1}{h^2}w_{i-1} - \frac{2}{h^2}u_{i-1} + \frac{1}{h^2}u_{i-2} + f(u_i) - \\ & \sum_{j=1}^{i-1}\Gamma(2-q)u_j^{q-1}D_{u_j}^q\alpha_{i-1}D_t^q u_j\Big) \end{aligned} \tag{27}$$

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By slecting

$$\alpha_i = h^2(-k_i w_i - \frac{1}{h^2} w_{i-1} + \frac{2}{h^2} u_{i-1} - \frac{1}{h^2} u_{i-2} - f(u_i) + \sum_{j=1}^{i-1} \Gamma(2-q) u_j^{q-1} D_{u_j}^q \alpha_{i-1} D_t^q u_j) \quad (28)$$

where $k_i > 0$, is the design parameter, w_{i+1} is to be governed to zero. Then the resulting q-th order derivative of v_i is

$$D_t^q v_i \le -\sum_{j=1}^{i} k_j w_j^2 + \frac{1}{h^2} w_i w_{i+1} \quad (29)$$

At this point, one can conclude that w_i converge to zero asymptotically.

Step *n*: In the last step n, the actual control U appears and is at our disposal. The aim is that design a suitable control law to make $w_n \to 0$ as $t \to \infty$, select the fractional Lyapunov function as

$$v_n = v_{n-1} + \frac{1}{2} w_n^2, \quad (30)$$

Then we can obtain the q-th order time derivative as

$$D_t^q v_n \le -\sum_{j=1}^{i} k_j w_j^2 + w_n(\frac{1}{h^2} U + k_n w_n + \frac{1}{h^2} w_{n-1} - \frac{2}{h^2} u_n + \frac{1}{h^2} u_{n-1} + f(u_n) - \sum_{j=1}^{n-1} \Gamma(2-q) u_j^{q-1} D_{u_j}^q \alpha_{n-1} D_t^q u_j) \quad (31)$$

The controller $U(t)$ is given by

$$U = h^2((\frac{2}{h^2} - k_n) u_n - \frac{2}{h^2} u_{n-1} - f(u_n) + \frac{1}{h^2} \alpha_{n-2} - k_n \alpha_{n-1} + \sum_{j=1}^{n-1} \Gamma(2-q) u_j^{q-1} D_{u_j}^q \alpha_{n-1} D_t^q u_j) \quad (32)$$

where $k_n > 0$, is the design parameter. Then the resulting q-th order derivative of v_n is

$$D^q v_n \le -\sum_{i=1}^{n} k_i w_i^2, \quad (33)$$

In this sage, it is convenient to consider, according to lemma (2) that the closed-loop system is stable regarding to the classical Lyapunov stability. Then, two cases is considered in our work.

1. When $w \ne 0$, we know $D^q V_n < 0$. There exists a K-class function ξ_1 such that $D^q V_n \le -\xi_1(\|\overline{w}\|), \overline{w} = [w_1, \ldots, w_n]^T$
2. When $w = 0$, we know $D^q V_n \le 0$. Accordind to the fractional comparsion principle [48], we know that $D^q V_n \le D^q k \Rightarrow V_n \le k$,

where $k = V_n(t=0)$ is apositive constant.

According to the first case in theorem (2), the closed loop system it's defined to be asympotically Mittage-Leffler stable.

In the third stage substitute $U(t)$ evaluated by equation (32) back into system (16), for $i = N$, a system of N nonlinear fractional order differential equations is obtained. The solution of resulting system may be solved by using any method for solving a nonlinear system of fractional order.

4. SIMULATION RESULT

Consider the nonlinear time fractional order partial differential equation:

$${}^c D_t^q u(x,t) = u_{xx}(x,t) + u^2(x,t), \; 0 < q \le 1 \quad (34)$$

$$u(x,0) = e^x \sin(\pi x), \quad 0 < x < 1 \quad (35)$$

$$u(0,t) = 0, \quad u(1,t) = U(t), \quad t \ge 0 \quad (36)$$

The open loop system (34)-(36) with $u(1,t) = 0$ is unstable. Using the central differencing discretize-ation with N=3 for the space variable will give

$$\begin{aligned} {}^c D_t^q u_1 &= 16u_2 - 32u_1 + u_1^2 \\ {}^c D_t^q u_2 &= 16u_3 - 32u_2 + 16u_1 + u_2^2 \\ {}^c D_t^q u_3 &= 16U - 32u_3 + 16u_2 + u_3^2 \end{aligned} \quad (37)$$

Step1. Let $w_1 = u_1, w_2 = u_2 - \alpha_1$, the first Lyapunov function $v_1 = \frac{1}{2} w_1^2$, $D_t^q v_1 \le w_1(16w_2 - 32w_1 + u_1^2 + 16\alpha_1)$

If choose $\alpha_1 = (2 - \frac{k_1}{16}) u_1 - \frac{1}{16} u_1^2$, $k_1 > 0$, w_2 is to be governed to zero.

Step 2. The second Lyapunov function $v_2 = \frac{1}{2} w_1^2 + \frac{1}{2} w_2^2$, let $w_3 = u_3 - \alpha_2$, we have

$$D_t^q v_2 \le w_2(32u_1 - 32u_2 + u_2^2 + 16w_3 + 16\alpha_2 - D_t^q \alpha_1)$$

If choose

$$\alpha_2 = \frac{1}{16}(-32u_1 + (32 - k_2)u_2 - u_2^2 + k_2\alpha_1 + D_t^q \alpha_1), k_2 > 0$$

w_3 is to be governed to zero.

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where

$$D_t^q\alpha_1=(\frac{32-k_1}{16}-\frac{\Gamma(2-q)}{8\Gamma(3-q)}u_1)(16u_2-32u_1+u_1^2),k_1>0$$

Step 3. The third Lyapunov function

$v_3=\frac{1}{2}w_1^2+\frac{1}{2}w_2^2+\frac{1}{2}w_3^2$, we have

$$D_t^q v_3\le w_3(16U-32u_3+16u_2+u_3^2+16w_2+k_3w_3-D_t^q\alpha_2),\quad k_3>0$$

The design parameters are chosen as

$$k_1=k_2=k_3=32 \quad (38)$$

The nonlinear controller $U(t)$ becomes

$$U(t)=\frac{1}{16}(-64u_2-u_3^2+(\frac{8}{(2-q)^2}-\frac{2}{2-q})u_1u_2+(\frac{14}{2-q}-\frac{16}{(2-q)^2}-7)u_1^2+(-2+\frac{2}{2-q})u_2^2+(\frac{-1}{8(2-q)}-\frac{3}{4(2-q)^2(3-q)})u_1^2u_2+(\frac{-1}{2(2-q)}+\frac{1}{2(2-q)^2}+\frac{3}{2(2-q)^2}\frac{1}{(3-q)})u_1^3-\frac{3}{64(2-q)^2(3-q)}u_1^4-\frac{2}{2-q}u_2u_3-\frac{1}{8(2-q)}u_2^3-\frac{2}{2-q}u_1u_3-\frac{1}{8(2-q)}u_1u_2^2) \quad (39)$$

Hence, we have $D_t^q v_3\le -k_1w_1^2-k_2w_2^2-k_3w_3^2$

$\rightarrow D_t^q v_3\le -32w_1^2-32w_2^2-32w_3^2$

Substitute equation (39) into (37), we have

$$^cD_t^q u_1=16u_2-32u_1+u_1^2$$
$$^cD_t^q u_2=16u_3-32u_2+16u_1+u_2^2$$
$$^cD_t^q u_3=-32u_3-48u_2+(\frac{8}{(2-q)^2}-\frac{2}{2-q})u_1u_2+(\frac{14}{2-q}-\frac{16}{(2-q)^2}-7)u_1^2+(-2+\frac{2}{2-q})u_2^2+(\frac{-1}{8(2-q)}-\frac{3\Gamma(2-q)}{4(2-q)\Gamma(4-q)})u_1^2u_2+(\frac{-1}{2(2-q)}+\frac{1}{2(2-q)^2}+\frac{3\Gamma(2-q)}{2(2-q)\Gamma(4-q)})u_1^3-\frac{3\Gamma(2-q)}{64(2-q)\Gamma(4-q)}u_1^4-\frac{2}{2-q}u_2u_3-\frac{1}{8(2-q)}u_2^3-\frac{2}{2-q}u_1u_3-\frac{1}{8(2-q)}u_1u_2^2 \quad (40)$$

Numerical simulation have carried out using fractional Euler's method with time step size is set to 0.01. The initial state is (0.0176, 0.0452, 0.087).

Figure (1) and Figure (4) illustrate the solution of $u_1(t),u_2(t),u_3(t)$ for different values of $t\in[0,1]$, while the controlled function $U(t)$ is presented in figures (2& 5) and figures (3 & 6) illustrate the solution of system (40) with the initial condition $u(x,0)=e^x\sin(\pi x)$ which is equivalent to the solution of the original nonlinear time fractional order partial differential equation (34) with order q =0.75and q = 1 respectively.

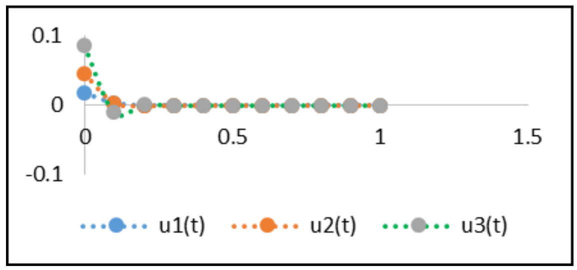

Figure 1: The solutions $u_1(t),u_2(t),u_3(t)$ with q =0.75

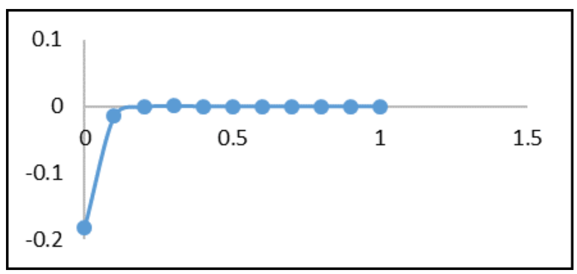

Figure 2: The controller U (t) with q =0.75

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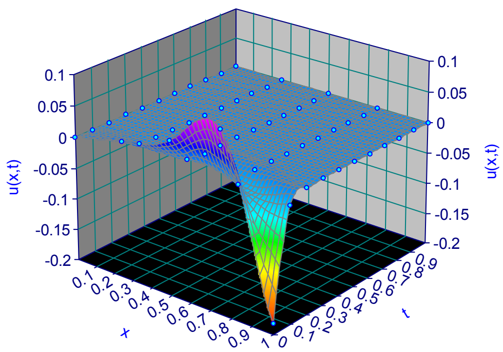

Figure 3: Closed-loop response with controller (q =0.75)

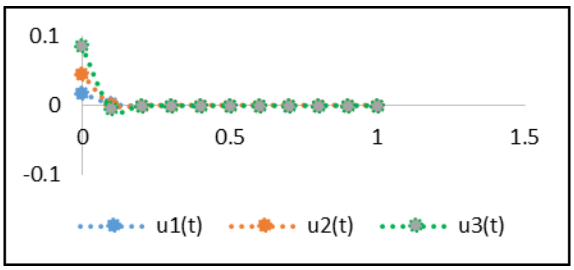

Figure 4: The solutions $u_1(t), u_2(t), u_3(t)$ with q =1

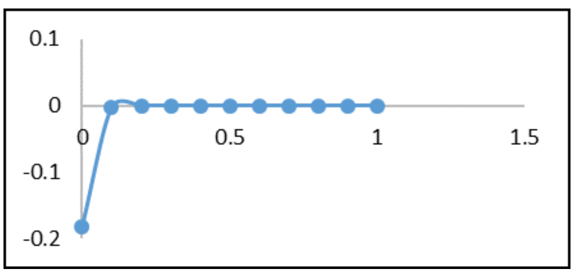

Figure 5: The controller U (t) with q =1

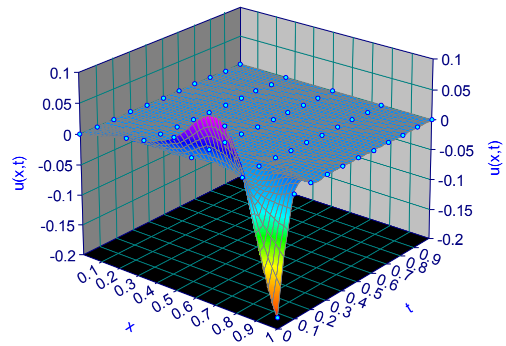

Figure 6: Closed-loop response with controller (q =1)

In order to give a good comparison, the simulation results are given in tabulated form. Table 1 and Table 2 present the obtained result of computer simulation for $u_1(t), u_2(t), u_3(t)$ of system (40) and the boundary control $U(t)$, for $q = 0.75$, and $q = 1$ respectively. While Table 3 present the simulation result for q=1 by using the procedure proposed in [50]

From the results, it is seen that

1. $u_1(t), u_2(t), u_3(t)$ converges in a finite time which is equivalent to the solution of the original NTFPDE (34), it is seen that the proposed technique is feasible for stabilizing NTFPDE. Also our technique performs better than the proposed procedure in [50].
2. We need to calculate α_i symbolically using the recursive relationship (28) and then to evaluate it for several different functions $u(t)$ and for different nonlinear functions $f(u)$. The symbolic calculation becomes extremely demanding computationally for increasing values of N.

Table 1:Simulation Result When q = 0.75

t	u_1	u_2	u_3	U
0	0.017601	0.045195	0.087	-0.182
0.1	0.3427×10^{-3}	0.3681×10^{-3}	-0.01	-0.015
0.2	-0.3066×10^{-4}	-0.3241×10^{-4}	0.92×10^{-4}	-0.1297×10^{-3}
0.3	0.7336×10^{-5}	-0.1185×10^{-4}	-0.2201×10^{-4}	0.474×10^{-4}
0.4	0.5356×10^{-6}	0.1178×10^{-5}	-0.1607×10^{-5}	-0.4712×10^{-5}
0.5	-0.844×10^{-7}	3.577×10^{-7}	0.1808×10^{-6}	-0.1431×10^{-6}
0.6	-9.771×10^{-9}	-6.704×10^{-3}	2.931×10^{-8}	2.682×10^{-7}
0.7	5.546×10^{-9}	-8.669×10^{-10}	-1.664×10^{-8}	3.468×10^{-9}
0.8	$-8.402\text{E}\times10^{-11}$	6.919×10^{-10}	2.521×10^{-10}	-2.768×10^{-9}
0.9	-3.976×10^{-11}	-2.718×10^{-11}	1.193×10^{-10}	1.087×10^{-10}
1.0	2.621×10^{-12}	-4.138×10^{-12}	-7.862×10^{-12}	1.655×10^{-11}

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5. CONCLUSIONS

In this article, the discretize backstepping method has been proposed. With this method, an effective controller can be designed for stabilizing NTFPDE with order $0 < q \leq 1$. The design procedure which consist of three steps is constructed such that the analytical form of boundary controller can always be constructed with appropriate choices of some design parameters. Simulation results show that the discretized fractional order backstepping technique is very effective and convenient but the calculation of the virtual control α becomes demanding computationally for decreasing values of h (step size of discretization) and its depending on the complexity of the nonlinear function $f(u)$.

For future work, one can assume more applications of the proposed procedure for other types of fractional partial differential equations such as fractional hyperbolic and fractional elliptic partial differential equations.

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Table 2:Simulation Result When q = 1

t	u_1	u_2	u_3	U
0	0.017601	0.045195	0.087	-0.182
0.1	0.2615×10^{-3}	0.6768×10^{-4}	-0.5069×10^{-3}	-0.271×10^{-3}
0.2	-0.1562×10^{-5}	-0.1977×10^{-5}	0.1055×10^{-4}	0.7908×10^{-5}
0.3	0.262×10^{-5}	1.084×10^{-7}	-0.662×10^{-5}	-4.336×10^{-7}
0.4	-5.838×10^{-8}	3.28×10^{-9}	2.01×10^{-7}	-1.311×10^{-8}
0.5	2.82×10^{-9}	-3.721×10^{-10}	6.84×10^{-9}	1.49×10^{-9}
0.6	3.35×10^{-10}	-3.982×10^{-12}	-6.802×10^{-10}	1.59×10^{-11}
0.7	1.18×10^{-11}	3.16×10^{-14}	-2.859×10^{-11}	-1.262×10^{-13}
0.8	-2.269×10^{-13}	2.88×10^{-14}	8.25×10^{-13}	-1.154×10^{-13}
0.9	1.21×10^{-14}	-2.096×10^{-15}	-3.337×10^{-14}	8.38×10^{-15}
1.0	0	0	1.09×10^{-15}	0

Table 3:Simulation Result When q = 1 Using Proposed Procedure In [50]

t	u_1	u_2	u_3	U
0	0.017601	0.045195	0.087	-8.175×10^{-4}
0.1	4.274×10^{-3}	4.844×10^{-3}	2.419×10^{-3}	3.803×10^{-6}
0.2	1.286×10^{-3}	1.791×10^{-3}	1.247×10^{-3}	2.645×10^{-7}
0.3	4.712×10^{-4}	6.658×10^{-4}	4.704×10^{-4}	3.453×10^{-8}
0.4	1.76×10^{-4}	2.489×10^{-4}	1.76×10^{-4}	4.811×10^{-9}
0.5	6.578×10^{-5}	9.302×10^{-5}	6.84×10^{-9}	6.72×10^{-10}
0.6	2.458×10^{-5}	3.477×10^{-5}	2.458×10^{-5}	9.388×10^{-11}
0.7	7.751×10^{-6}	1.083×10^{-5}	7.751×10^{-6}	8.947×10^{-12}
0.8	2.88×10^{-6}	4.072×10^{-6}	2.88×10^{-6}	1.288×10^{-12}
0.9	1.076×10^{-6}	1.522×10^{-6}	1.076×10^{-6}	1.799×10^{-13}
1.0	4.023×10^{-7}	05.689×10^{-7}	4.023×10^{-7}	2.513×10^{-14}