ASYMPTOTIC ESTIMATES OF THE SOLUTION OF A SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM WITH BOUNDARY JUMPS

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Abstract:
In this article the boundary value problem for ordinary simple differential equations of the third order with small parameter at the higher derivatives is considered. In the given work the author researches the singularly perturbed boundary value problems under the condition that real parts of the roots of additional distinctive equation have opposite signs. Boundary and initial functions are defined; their existence and uniqueness are proved. On the basis of the constructed boundary and initial functions analytical representation of the solution of a boundary value problem is found. On the basis of an analytical representation of a solution of a singularly Perturbed Boundary Value Problem for the differential equations of conditionally the steady type, we describe the character of growth of the derivatives of a solution of Perturbed Problem with $\varepsilon \to 0$. The class of boundary value problems with boundary jumps is singled out. Sizes of boundary jumps are determined. Asymptotic estimates of the solution of a boundary value problem are found. Estimates for the difference between solutions of the degenerate and perturbed value problems are obtained.

Keywords: *Asymptotic, Initial Function, Boundary Function, Boundary Value Problem, Additional Characteristic Equation, Perturbed and Degenerated Problems*

1. INTRODUCTION

Efficient asymptotic methods are developed for a fairly broad class of singularly perturbed problems. These methods enable one to construct uniform approximations with any required accuracy [1-7]. At the same time, for a fairly broad class of singularly perturbed boundary-value problems, the choice of a proper method for the construction of solutions or their asymptotic approximations without preliminary investigation appears to be quite complicated. The analysis shows that the boundary-value problems characterized by the presence of an initial jump can also be regarded as problems of this kind. First studies devoted to the phenomena of the initial jumps of solutions of nonlinear singularly perturbed initial value problems with unbounded initial values aspiring of the small parameter to zero are the works of Vishik and Lyusternik [8], and of Kasymov [9]. These studies were continued in [10,11].

The phenomenon of the jump in applied problems is an important factor, which should be taken into account when the replacement of the perturbed problem by more simplified degenerate problem. The magnitude of the jump makes it possible to determine the range of the simplified problem applicability. For example, new rationale of Painlevé paradox and the origin of jump phenomenon were proposed in the works of Neumark and Smirnov [12]. Mathematically, the jump phenomenon was investigated in [13-15]. In [16-18], the asymptotic behavior of solutions of singularly perturbed boundary-value problems with initial jumps was investigated, in the case when the additional characteristic equation had only roots with negative real parts. This case is called stable.

In this paper we consider a singularly perturbed boundary-value problem with more general requirement that the additional characteristic equation has 3 different roots, furthermore $\mu_1 = 0$, $\mathrm{Re}\mu_2 < 0, \mathrm{Re}\mu_3 > 0$. Following [19], we name this case as conditionally stable.

Our aim is to investigate the asymptotic behavior of the solution of the boundary value problem (1), (2) when the small parameter tends to zero.

In the present paper on the basis of an analytic representation of a problem solution (1), (2), we describe the character of derivatives growth of the problem solution (1), (2), when the small parameter tends to zero, select a class of boundary – value problems with boundary jumps, the values of boundary jumps are determined and we obtain

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asymptotic estimates for a solution of problem (1), (2) and for the differences between solutions of degenerate and original problems.

2. STATEMENT OF THE PROBLEM

We consider a boundary value problem for an ordinary differential equation of the third order with small parameter at the highest derivatives:

$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A(t)y'' + B(t)y' + C(t)y = F(t)$ (1)

with boundary conditions

$L_1 y \equiv \alpha_{10}y(0,\varepsilon) + \alpha_{11}y'(0,\varepsilon) + \beta_{10}y(1,\varepsilon) = a_1,$

$L_2 y \equiv \alpha_{21}y'(0,\varepsilon) + \beta_{20}y(1,\varepsilon) = a_2,$ (2)

$L_3 y \equiv \alpha_{30}y(0,\varepsilon) + \beta_{30}y(1,\varepsilon) = a_3,$

where $\varepsilon > 0$ is a small parameter, a_i, α_{ij} and β_{ij} are constants.

Assume that:

(a) $A(t), B(t), C(t) \in C^3([0,1])$,

$F(t) \in C^1([0,1])$;

(b) The inequalities $B(t) \neq 0$, $t \in [0,1]$ and $\tilde{\alpha} = \alpha_{10}\alpha_{21}\beta_{30} - \alpha_{30}\alpha_{21}\beta_{10} + \alpha_{11}\alpha_{30}\beta_{20} \neq 0$ are valid;

(c) Additional characteristic equation $\mu^3 + A(t)\mu^2 + B(t)\mu = 0$ has roots: $\mu_1 = 0$, $\operatorname{Re}\mu_2 < 0$, $\operatorname{Re}\mu_3 > 0$.

Our aim is to establish the asymptotic estimates of solutions of singularly perturbed boundary value problem (1), (2), to prove the existence of the phenomenon of boundary jumps, determine the order of the boundary of the jump at the points $t = 0$ and $t = 1$, investigate the asymptotic behavior of the solution of the boundary value problem (1), (2) as $\varepsilon \to 0$.

The study will be conducted by a certain rule. In the first stage, on the basis of auxiliary functions we will construct the solution of the problem (1), (2). In the next step there will be investigated the asymptotic behavior of the solution of the problem (1), (2), phenomena of boundary jumps.

FUNDAMENTAL SOLUTION SYSTEM

Along with Eq. (1), we consider the corresponding homogeneous perturbed equation

$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A(t)y'' + B(t)y' + C(t)y = 0$. (3)

Lemma 1. If *conditions (a) and (b) are satisfied, then, for sufficiently small* $\varepsilon > 0$, *the fundamental solution system* $y_i(t,\varepsilon)$, $i = \overline{1,3}$ *of the singularly perturbed Eq. (3) permits the asymptotic representations:*

$$y_1^{(q)}(t,\varepsilon) = u_1^{(q)}(t) + O(\varepsilon),\ y_2^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu_2(x)dx\right)[u_2(t)\mu_1^q(t) + O(\varepsilon)], \quad (4)$$

$$y_3^{(q)}(t,\varepsilon) = \frac{1}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1 \mu_3(x)dx\right)[u_3(t)\mu_2^q(t) + O(\varepsilon)], \quad q = 0,1,2$$

as $\varepsilon \to 0$, *where*

$$u_1(t) = \exp\left(-\int_0^t \frac{C(x)}{B(x)}dx\right),$$

$$u_k(t) = \exp\left(-\int_0^t \frac{q_k(x)}{p_k(x)}dx\right) \neq 0, \quad t \in [0,1],\ k = 2,3,$$

$$p_k(t) = \mu_k(t)(A(t) + 2\mu_k(t)) \neq 0,$$

$$q_k(t) = C(t) + A(t)\mu_k'(t) + 3\mu_k(t)\mu_k'(t),\ k = 2,3.$$

Proof. The proof of the lemma is readily obtained from the well-known theorems of Schlesinger [16], Birkhoff [17] and Nurgabyl [18] (for example, see [19, pp.29-34]).

For the Wronskian's determinant $W(t,\varepsilon)$ the fundamental solution system $y_i(t,\varepsilon)$, $i = \overline{1,3}$ of Eq. (3), for sufficiently small $\varepsilon > 0$, we obtain

$$W(t,\varepsilon) = \frac{1}{\varepsilon^3}e^{\frac{1}{\varepsilon}\int_0^t \mu_2(x)dx + \frac{1}{\varepsilon}\int_1^t \mu_3(x)dx} u_1(t)u_2(t)u_3(t)\mu_3(t)\mu_2(t)(\mu_3(t) - \mu_2(t))(1 + O(\varepsilon)) \neq 0. \quad (5)$$

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3. CONSTRUCTION OF INITIAL FUNCTIONS

Following work [10], let us introduce the Cauchy function for the Eq. (3):

$$K(t,s,\varepsilon)=\frac{W(t,s,\varepsilon)}{W(s,\varepsilon)},\ s,t\in[0,1],\quad (6)$$

where $W(t,s,\varepsilon)$ is the 3 th-order determinant obtained from the determinant $W(s,\varepsilon)$ by replacing the 3th row of the fundamental solution system with $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$ of Eq. (3). From the explicit expression (6) of Cauchy function $K(t,s,\varepsilon)$, defined for $0\le s\le t\le 1$, it satisfies the homogeneous Eq. (3) with respect to the variable t and the initial conditions: $K^{(j)}(s,s,\varepsilon)=0,\quad j=\overline{0,1},\quad K''(s,s,\varepsilon)=1$ and it does not depend on the choice of the fundamental system of solutions of the equation (3).

Now, we introduce the following functions:

$$K_0(t,s,\varepsilon)=\frac{P_0(t,s,\varepsilon)}{W(s,\varepsilon)};\quad K_1(t,s,\varepsilon)=\frac{P_1(t,s,\varepsilon)}{W(s,\varepsilon)}\quad (7)$$

Here $K_0(s,s,\varepsilon)+K_1(s,s,\varepsilon)=K(t,s,\varepsilon)$, $P_0(t,s,\varepsilon),\ P_1(t,s,\varepsilon)$ is the 3 d-order determinants obtained from the determinant $W(s,\varepsilon)$ as a result of replacement of the 3-d row respectively:

$$(y_1(t,\varepsilon),\, y_2(t,\varepsilon),\ 0);\ (0,\ 0, y_3(t,\varepsilon)),$$

where $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$ is the fundamental system of solutions of Eq. (3).

Notice, that $K_0(t,s,\varepsilon),\ K_1(t,s,\varepsilon)$ are continuous functions together with its derivatives up to third order inclusively, and as a functions of the variable t satisfy the homogeneous equation (3): $L_\varepsilon K_0=0,\ L_\varepsilon K_1=0, 0<t,s<1$, and the initial conditions

$$K_0(s,s,\varepsilon)+K_1(s,s,\varepsilon)=0,\ K'_{0t}(s,s,\varepsilon)+K'_{1t}(s,s,\varepsilon)=0,\quad K''_{0t}(s,s,\varepsilon)+K''_{1t}(s,s,\varepsilon)=1.$$

The functions $K_0(t,s,\varepsilon),\ K_1(t,s,\varepsilon)$ will be called the initial functions of the problem (1), (2).

Lemma 2. *Let conditions (a) and (b) be satisfied. Then, from (7) using estimates (4) and (5) for initial functions $K_0^{(q)}(t,s,\varepsilon)$ and $K_1^{(q)}(t,s,\varepsilon)$ for sufficiently small ε, we obtain the following asymptotic representations:*

$$K_0^{(q)}(t,s,\varepsilon)=\varepsilon^2\left(\frac{u_1^{(q)}(t)}{u_1(s)B(s)}-\frac{1}{\varepsilon^q}\frac{u_2(t)\mu_2^q(t)\exp\left(\frac{1}{\varepsilon}\int_s^t\mu_2(x)dx\right)}{u_2(s)\mu_2(s)(\mu_3(s)-\mu_2(s))}+O\left(\varepsilon+\frac{\varepsilon}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_s^t\mu_2(x)dx}\right)\right),\quad (8)$$

$$K_1^{(q)}(t,s,\varepsilon)=\varepsilon^2\left(\frac{1}{\varepsilon^q}\frac{u_3(t)\mu_3^q(t)}{u_3(s)\mu_3(s)(\mu_3(s)-\mu_2(s))}e^{-\frac{1}{\varepsilon}\int_t^s\mu_3(x)dx}+O\left(\varepsilon+\frac{\varepsilon}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_t^s\mu_3(x)dx}\right)\right),$$

(9)

Proof. By expanding $P_0{}^{(q)}(t,s,\varepsilon),\ q=0,1,2$ in entries of the 3th row, we obtain

$$P_0{}^{(q)}(t,s,\varepsilon)=y_1^{(q)}(t,\varepsilon)\widetilde{W}_{31}(s,\varepsilon)-y_2^{(q)}(t,\varepsilon)\widetilde{W}_{32}(s,\varepsilon).\quad (10)$$

From (4), the minors $\widetilde{W}_{31}(s,\varepsilon),\ \widetilde{W}_{32}(s,\varepsilon)$ for sufficiently small $\varepsilon>0$ can be represented in the following form:

$$\widetilde{W}_{31}(s,\varepsilon)=\frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int_0^s\mu_2(x)dx-\frac{1}{\varepsilon}\int_s^1\mu_3(x)dx\right)u_2(s)u_3(s)(\mu_3(s)-\mu_2(s))(1+O(\varepsilon)),$$

$$\widetilde{W}_{32}(s,\varepsilon)=\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\int_s^1\mu_3(x)dx\right)u_1(s)u_3(s)\mu_3(s)(\mu_3(s)-\mu_2(s))(1+O(\varepsilon)).\quad (11)$$

Then, from (10) and (4), relation (11) acquires the following form:

$$
\begin{aligned}
P_0^{(q)}(t,s,\varepsilon) &= \frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int_0^s \mu_2(x)dx - \frac{1}{\varepsilon}\int_s^1 \mu_3(x)dx\right)\times \\
&\times\Big[u_{10}^{(q)}(t)u_{20}(s)u_{30}(s)\big(\mu_3(s)-\mu_2(s)\big)- \\
&-\frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_s^t \mu_2(x)dx\right)u_{10}(s)u_{30}(s)\mu_3(s)u_{20}(t)\mu_2{}^q(t)+ \\
&+O\left(\varepsilon+\frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_s^t \mu_2(x)dx\right)\right)\Big].
\end{aligned} \tag{12}
$$

From (12) with regard to (5) and (7), we obtain the desired estimates (8):

$$
K_0^{(q)}(t,s,\varepsilon)=\varepsilon^2\left(\frac{u_1^{(q)}(t)}{u_1(s)B(s)}-\frac{1}{\varepsilon^q}\frac{u_2(t)\mu_2^q(t)\exp\left(\frac{1}{\varepsilon}\int_s^t \mu_2(x)dx\right)}{u_2(s)\mu_2(s)(\mu_3(s)-\mu_2(s))}+O\left(\varepsilon+\frac{\varepsilon}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_s^t \mu_2(x)dx}\right)\right),
$$

Similarly we get the following asymptotic representation for $P_1(t,s,\varepsilon)$:

$$
\begin{aligned}
P_1^{(q)}(t,s,\varepsilon) &= \frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int_0^s \mu_2(x)dx - \frac{1}{\varepsilon}\int_s^1 \mu_3(x)dx\right)\times \\
&\times\left[u_{10}(s)u_{20}(s)u_{30}(t)\mu_3{}^q(t)\mu_2(s)\frac{1}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^s \mu_3(x)dx\right)+\right. \\
&\left.+O\left(\varepsilon+\frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_s^t \mu_3(x)dx\right)\right)\right].
\end{aligned} \tag{13}
$$

From (13) with regard to (5) and (7), we obtain the desired estimates (9).

CONSTRUCTION OF BOUNDARY FUNCTIONS.

Definition. Functions $\Phi_k(t,\varepsilon),\ k=\overline{1,3}$ are referred to as boundary functions of the perturbed problem (1) (2) if they satisfy the homogeneous Eq. (3) and boundary conditions

$$
L_i\Phi_k=\begin{cases}1,\ i=k,\ k=1,2,3,\\ 0,\ i\neq k,\ i=1,2,3;\ k=1,2,3.\end{cases} \tag{14}
$$

Consider the determinant of the third order

$$
J(\varepsilon)=\begin{vmatrix} L_1y_1 & L_1y_2 & L_1y_3\\ L_2y_1 & L_2y_2 & L_2y_3\\ L_3y_1 & L_3y_2 & L_3y_3\end{vmatrix},
$$

where the elements of L_iy_k in view of (6) can be represented in the form

$$
L_iy_1=L_iu_1+O(\varepsilon)\ i=1,2,3;\qquad L_iy_2=\frac{\alpha_{i1}}{\varepsilon}u_2(0)\mu_2(0)\big(1+O(\varepsilon)\big),i=1,2;
$$
$$
L_3y_2=\alpha_{30}u_2(0)\big(1+O(\varepsilon)\big);\ \ L_iy_3=\beta_{30}u_3(1)+O(\varepsilon),i=1,2,3.
$$

Expanding the determinant $J(\varepsilon)$ in elements of the 3-d row and taking into account (4), at $\varepsilon\to 0$ we obtain

$$
J(\varepsilon)=-\frac{1}{\varepsilon}\tilde{\alpha}\,\mu_2(0)u_2(0)u_3(1)(1+O(\varepsilon))\neq 0, \tag{15}
$$

where $\tilde{\alpha}=\alpha_{10}\alpha_{21}\beta_{30}-\alpha_{30}\alpha_{21}\beta_{10}+\alpha_{11}\alpha_{30}\beta_{20}$.

Theorem 1. *Let conditions (a), (b), and (c) be satisfied. Then, for sufficiently small $\varepsilon > 0$, the boundary functions $\Phi_k(t,\varepsilon),\, k=\overline{1,3}$ exist on the interval [0,1], are unique, and are given as follows:*

$$\Phi_k(t,\varepsilon)=\frac{J_k(t,\varepsilon)}{J(\varepsilon)}, \quad k=\overline{1,3}, \tag{16}$$

where $J_k(t,\varepsilon)$ is the determinant obtained from $J(\varepsilon)$ by substituting the k^{th} row with the fundamental solution system $y_i(t,\varepsilon),\, i=\overline{1,3}$ of Eq. (3).

Proof. It follows immediately from (11) that the functions $\Phi_k(t,\varepsilon),\, k=\overline{1,3}$ is a solution of Eq. (3), satisfy boundary conditions (9), and does not depend on the choice of a fundamental system of solutions of Eq. (4). Thus, the functions (12) are boundary functions of the boundary value problem (1), (2).

Lemma 3. *If conditions* (*a*), (*b*) and (*c*) *are satisfied, then, for sufficiently small ε, the boundary functions $\Phi_i^{(q)}(t,\varepsilon)$ satisfy the following estimates on* $[0,1]$*:*

$$\Phi_1^{(q)}(t,\varepsilon)=u_1^q(t)\frac{\alpha_{21}\beta_{30}}{\tilde\alpha}-\frac{\varepsilon}{\varepsilon^q}\frac{u_2(t)\mu_2^q(t)}{u_2(0)\mu_2(0)}e^{\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx}\cdot\frac{\beta_{30}L_2u_1-\beta_{20}L_3u_1}{\tilde\alpha}+$$
$$-\frac{1}{\varepsilon^q}\frac{u_3(t)\mu_3^q(t)}{u_3(1)}e^{-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx}\frac{\alpha_{21}L_3u_1}{\tilde\alpha}+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx}+\frac{\varepsilon}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx}\right)$$
$$\Phi_2^{(q)}(t,\varepsilon)=-u_1^q(t)\frac{\alpha_{11}\beta_{30}}{\tilde\alpha}-\frac{\varepsilon}{\varepsilon^q}\frac{u_2(t)\mu_2^q(t)}{u_2(0)\mu_2(0)}e^{\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx}\cdot\frac{\beta_{30}L_1u_1-\beta_{10}L_3u_1}{\tilde\alpha}-$$
$$-\frac{1}{\varepsilon^q}\frac{u_3(t)\mu_3^q(t)}{u_3(1)}e^{-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx}\frac{\alpha_{11}L_3u_1}{\tilde\alpha}+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx}+\frac{\varepsilon}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx}\right), \tag{17}$$
$$\Phi_3^{(q)}(t,\varepsilon)=u_1^q(t)\frac{\alpha_{11}\beta_{20}-\alpha_{21}\beta_{10}}{\tilde\alpha}-\frac{\varepsilon}{\varepsilon^q}\frac{u_2(t)\mu_2^q(t)}{u_2(0)\mu_2(0)}e^{\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx}\cdot\frac{\beta_{20}L_1u_1-\beta_{10}L_2u_1}{\tilde\alpha}+$$
$$+\frac{1}{\varepsilon^q}\frac{u_3(t)\mu_3^q(t)}{u_3(1)}e^{-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx}\frac{\alpha_{21}L_1u_1-\alpha_{11}L_2u_1}{\tilde\alpha}+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^q}e^{\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx}+\frac{\varepsilon}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx}\right).$$

Proof. We expand the determinant $J_i^{(q)}(t,\varepsilon)$ ($i=1,2,3;\ \ q=0,1,2$), in elements of i th row formed by the fundamental system of solutions $y_k^{(q)}(t,\varepsilon)$ $(k=\overline{1,3},\ q=\overline{0,2})$, of Eq. (3):

$$J_1^{(q)}(t,\varepsilon)=\begin{vmatrix} y_1{}^{(q)}(t,\varepsilon) & y_2{}^{(q)}(t,\varepsilon) & y_3{}^{(q)}(t,\varepsilon)\\ y_1(1,\varepsilon) & y_2(1,\varepsilon) & y_3(1,\varepsilon)\\ y_1'(1,\varepsilon) & y_2'(1,\varepsilon) & y_3'(1,\varepsilon)\end{vmatrix}=$$
$$=-\frac{1}{\varepsilon}\{\frac{1}{\varepsilon^q}u_{10}(1)u_{20}(t)u_{30}(1)\mu_3(1)\mu_2{}^q(t)\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)+ \tag{18}$$
$$+O\left(\varepsilon+\frac{\varepsilon}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)\right)\},$$

$$J_2^{(q)}(t,\varepsilon)=\begin{vmatrix} y_1'(0,\varepsilon) & y_2'(0,\varepsilon) & y_3'(0,\varepsilon) \\ y_1^{(q)}(t,\varepsilon) & y_2^{(q)}(t,\varepsilon) & y_3^{(q)}(t,\varepsilon) \\ y_1'(1,\varepsilon) & y_2'(1,\varepsilon) & y_3'(1,\varepsilon)\end{vmatrix}=$$

$$=\frac{1}{\varepsilon^2}\Big[-u_{10}^{(q)}(t)u_{30}(1)u_{20}(0)\mu_2(0)\mu_3(1)+ \tag{19}$$

$$+u_{10}'(0)u_{30}(1)\mu_3(1)u_{20}(t)\mu_2^{\,q}(t)\cdot\frac{\varepsilon}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)+$$

$$+u_{10}'(1)u_{20}(0)\mu_2(0)u_{30}(t)\mu_3^{\,q}(t)\cdot\frac{\varepsilon}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx\right)+$$

$$+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)+\frac{\varepsilon^2}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx\right)\right)\Big],$$

$$J_3^{(q)}(t,\varepsilon)=\begin{vmatrix} y_1'(0,\varepsilon) & y_2'(0,\varepsilon) & y_3'(0,\varepsilon) \\ y_1(1,\varepsilon) & y_2(1,\varepsilon) & y_3(1,\varepsilon) \\ y_1^{(q)}(t,\varepsilon) & y_2^{(q)}(t,\varepsilon) & y_3^{(q)}(t,\varepsilon)\end{vmatrix}=$$

$$=\frac{1}{\varepsilon}\Big[u_{10}^{(q)}(t)u_{30}(1)u_{20}(0)\mu_2(0)- \tag{20}$$

$$-u_{10}(1)u_{30}(t)\mu_2(0)u_{20}(0)\mu_3^{\,q}(t)\cdot\frac{1}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx\right)-$$

$$-u_{10}'(0)u_{20}(t)u_{30}(1)\mu_2^{\,q}(t)\cdot\frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)+$$

$$+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu_2(x)dx\right)+\frac{\varepsilon}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1\mu_3(x)dx\right)\right)\Big].$$

By using (15), (16), (18), (19) and (20), we obtain relation (17) for the function $\Phi_i^{(q)}(t,\varepsilon)$.

4. ANALYTIC REPRESENTATION AND ASYMPTOTIC ESTIMATES FOR A SOLUTION OF THE BOUNDARY VALUE PROBLEM

Theorem 2. *Suppose that conditions (a), (b) and (c) are satisfied. Then, for sufficiently small* $\varepsilon>0$, *a solution* $y(t,\varepsilon)$ *of the boundary-value problem (1), (2) exists, is unique, and has the form*

$$y(t,\varepsilon)=a_1\Phi_1(t,\varepsilon)+a_2\Phi_2(t,\varepsilon)+a_3\Phi_3(t,\varepsilon)+$$

$$+\Phi_1(t,\varepsilon)\left[\sum_{j=0}^{1}\alpha_{1j}\frac{1}{\varepsilon^2}\int_0^1K_1^{(j)}(0,s,\varepsilon)F(s)ds-\beta_{10}\frac{1}{\varepsilon^2}\int_0^1K_0(1,s,\varepsilon)F(s)ds\right]+$$

$$+\Phi_2(t,\varepsilon)\left[\alpha_{21}\frac{1}{\varepsilon^2}\int_0^1 K_1'(0,s,\varepsilon)F(s)ds-\beta_{20}\frac{1}{\varepsilon^2}\int_0^1 K_0(1,s,\varepsilon)F(s)ds\right]+$$

$$+\Phi_3(t,\varepsilon)\left[\alpha_{30}\frac{1}{\varepsilon^2}\int_0^1 K_1(0,s,\varepsilon)F(s)ds-\beta_{30}\frac{1}{\varepsilon^2}\int_0^1 K_0(1,s,\varepsilon)F(s)ds\right]+$$

$$+\frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)F(s)ds+\frac{1}{\varepsilon^2}\int_1^t K_1(t,s,\varepsilon)F(s)ds. \tag{21}$$

Proof. We seek a solution $y(t,\varepsilon)$ of the boundary-value problem (1), (2) in the from

$$y(t,\varepsilon)=c_1\Phi_1(t,\varepsilon)+c_2\Phi_2(t,\varepsilon)+c_3\Phi_3(t,\varepsilon)+\frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)F(s)ds+\frac{1}{\varepsilon^2}\int_1^t K_1(t,s,\varepsilon)F(s)ds. \tag{22}$$

where c_i are unknown constants. By direct verification, we can establish that the function $y(t,\varepsilon)$ defined by (22) is a solution of Eq. (1). To determine c_i, we substitute (22) in the boundary conditions (2). Then, taking into account the boundary conditions (9), we uniquely obtain

$$c_1=a_1+\sum_{j=0}^{1}\alpha_{1j}\frac{1}{\varepsilon^2}\int_0^1 K_1^{(j)}(0,s,\varepsilon)F(s)ds-\beta_{10}\frac{1}{\varepsilon^2}\int_0^1 K_0(1,s,\varepsilon)F(s)ds,$$

$$c_2=a_2+\alpha_{21}\frac{1}{\varepsilon^2}\int_0^1 K_1'(0,s,\varepsilon)F(s)ds-\beta_{20}\frac{1}{\varepsilon^2}\int_0^1 K_0(1,s,\varepsilon)F(s)ds, \tag{23}$$

$$c_1=a_1+\alpha_{30}\frac{1}{\varepsilon^2}\int_0^1 K_1(0,s,\varepsilon)F(s)ds-\beta_{30}\frac{1}{\varepsilon^2}\int_0^1 K_0(1,s,\varepsilon)F(s)ds.$$

Substituting (23) in (22), we get (21).Thus, a solution of problem (1), (2) exists and can be represented in the form (21). The uniqueness of a solution of problem (1), (2) can be proved by contradiction. Theorem 1 is proved.

Theorem 3. *Suppose that conditions (a), (b) and (c) are satisfied. Then, for sufficiently small* $\varepsilon>0$, *the following estimates for a solution* $y(t,\varepsilon)$ *of the boundary-value problem (1), (2) and its derivatives hold on the segment* $0\le t\le 1$ *:*

$$\left|y^{(q)}(t,\varepsilon)\right|\le C\Big[\left|\alpha_{21}\beta_{30}a_1-\alpha_{11}\beta_{30}a_2+(\alpha_{11}\beta_{20}-\alpha_{21}\beta_{10})a_3\right|+\max_{0\le t\le 1}|F(t)|+$$
$$+\frac{\varepsilon}{\varepsilon^q}\exp\left(-\frac{\nu t}{\varepsilon}\right)\left(1+\max_{0\le t\le 1}|F(t)|\right)+\frac{1}{\varepsilon^q}\exp\left(-\frac{\nu(1-t)}{\varepsilon}\right)\left(1+\max_{0\le t\le 1}|F(t)|\right)\Big] \tag{24}$$

Proof. The validity of estimates (24) follows immediately from Theorem 2 and Lemmas 2 and 3.

On the right-hand side of (16) for $q=0$, the coefficients of a_1, a_2, a_3 are of order $O(1)$. Consequently, the boundary conditions for the solution $\bar{y}(t)$ of the degenerate equation

$$L_0\bar{y}\equiv B(t)\bar{y}''+C(t)\bar{y}'=F(t), \tag{25}$$

can be obtained from the boundary condition (2) in the form:

$$H\bar{y}\equiv\tilde{\alpha}\bar{y}(0)=\tilde{a}\,, \tag{26}$$

where

$$\widetilde{\alpha} = \alpha_{10}\alpha_{21}\beta_{30} - \alpha_{30}\alpha_{21}\beta_{10} + \alpha_{11}\alpha_{30}\beta_{20},$$
$$\widetilde{a} = \alpha_{21}\beta_{30}a_1 - \alpha_{11}\beta_{30}a_2 + (\alpha_{11}\beta_{20} - \alpha_{21}\beta_{10})a_3,$$

which is one of the features of the problem being investigated.

In what follows, we show that Eq. (25) and the boundary conditions (26), in fact, define a degenerate problem whose solution is the limit of the solution of the perturbed boundary-value problem (1), (2) as $\varepsilon \to 0$. The solution of problem (25), (26) can be represented in the form

$$\bar{y}(t) = \widetilde{a}\,\frac{u_1(t)}{\widetilde{\alpha}} + \int_1^t \frac{u_1(t)F(s)}{u_1(s)B(s)}ds,$$

$$\bar{y}'(t) = \widetilde{a}\,\frac{u_1'(t)}{\widetilde{\alpha}} + \int_1^t \frac{u_1'(t)F(s)}{u_1(s)B(s)}ds + \frac{F(t)}{B(t)}.$$

Theorem 4. *Suppose that conditions (a), (b) and (c) are satisfied. Then, for sufficiently small $\varepsilon > 0$, the following estimate is true for the difference between the solution $y(t,\varepsilon)$ of problem (1), (2) and the solution $\bar{y}(t)$ of problem (17), (18) on the segment $0 \le t \le 1$:*

$$\left|y^{(q)}(t,\varepsilon) - \bar{y}^{(q)}(t)\right| \le C\left(\varepsilon + \frac{\varepsilon}{\varepsilon^q}\exp\left(-\frac{\nu t}{\varepsilon}\right) + \frac{1}{\varepsilon^q}\exp\left(-\frac{\nu(1-t)}{\varepsilon}\right)\right). \tag{27}$$

Proof. Let us introduce the function $u(t,\varepsilon) = y(t,\varepsilon) - \bar{y}(t)$, where $y(t,\varepsilon)$ is the solution of problem (1), (2) and $\bar{y}(t)$ is the solution of problem (17), (18). Substituting the function $y(t,\varepsilon)$ in problem (1), (2), we obtain the problem

$$L_\varepsilon u = \varepsilon A(t)\bar{y}'',\ L_1 u = a_1 - L_1\bar{y},\quad L_2 u = a_2 - L_2\bar{y},\quad L_3 u = a_3 - L_3\bar{y}. \tag{28}$$

Applying Theorem 3 to the boundary-value problem (28), we get

$$\left|u^{(q)}(t,\varepsilon)\right| \le C\Big(\alpha_{21}\beta_{30}(a_1 - L_1\bar{y}) - \alpha_{11}\beta_{30}(a_2 - L_2\bar{y}) + (\alpha_{11}\beta_{20} - \alpha_{21}\beta_{10})(a_3 - L_3\bar{y}) +$$
$$+\varepsilon + \frac{\varepsilon}{\varepsilon^q}\exp\left(-\frac{\nu t}{\varepsilon}\right) + \frac{1}{\varepsilon^q}\exp\left(-\frac{\nu(1-t)}{\varepsilon}\right)\Big). \tag{29}$$

Here, it is easy to prove that expression

$$\Omega = \alpha_{21}\beta_{30}(a_1 - L_1\bar{y}) - \alpha_{11}\beta_{30}(a_2 - L_2\bar{y}) + (\alpha_{11}\beta_{20} - \alpha_{21}\beta_{10})(a_2 - L_3\bar{y}) = \widetilde{a} - \widetilde{\alpha}\,\bar{y}(0) \equiv 0.$$

This yields the required estimates (27). Theorem 4 is proved.

Thus, it follows from Theorem 4 that the solution $y(t,\varepsilon)$) of problem (1), (2) tends to the solution $\bar{y}(t)$ of problem (25), (26) as $\varepsilon \to 0$, namely

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \bar{y}(t),\quad 0 \le t < 1,$$
$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \bar{y}'(t),\quad 0 < t < 1.$$

Now, from (21), by virtue of the estimates (8),(9), (17), we obtain

$$\Delta_1^0 = \lim_{\varepsilon\to 0} y(1,\varepsilon) - \bar{y}(1) = \frac{1}{\widetilde{\alpha}}\left(\bar{a} - H_1^0\bar{y}\right),\ y'(1,\varepsilon) = \frac{\mu_3(1)}{\varepsilon\widetilde{\alpha}}\left(\bar{a} - H_0^1\bar{y} + O(\varepsilon)\right), \tag{30}$$

$$\Delta_0^1 = \lim_{\varepsilon\to 0} y'(0,\varepsilon) - \bar{y}'(0) = \frac{1}{\widetilde{\alpha}}\left(\bar{\bar{a}} - H_0^1\bar{y}\right),\ y''(1,\varepsilon) = \frac{\mu_2(0)}{\varepsilon\widetilde{\alpha}}\left(\bar{\bar{a}} - H_0^1\bar{y} + O(\varepsilon)\right),$$

where

$$H_1^0\bar{y} = \widetilde{\alpha}y(1),\ \ \bar{a} = \alpha_{11}\alpha_{30}a_2 - \alpha_{21}\alpha_{30}a_1 + (\alpha_{11}\beta_{20} - \alpha_{21}\alpha_{10})a_3,$$

$$H_0^1\bar{y} = \widetilde{\alpha}\widetilde{y}'(0),\ \bar{\bar{a}} = \beta_{20}\alpha_{30}a_1 - (\alpha_{10}\beta_{30} - \alpha_{30}\beta_{10})a_2 + \alpha_{10}\beta_{20}a_3\,.$$

Proceeding from (30), we conclude that the singularly perturbed boundary value problem (1), (2) has a jump of the first order in the neighborhood of the point t = 0, and in the neighborhood of the point t = 1 it has a jump of zero order, which is one of the features of the problem under study.

The established algorithm for estimating the solution of the boundary value problem makes it possible to investigate the asymptotic behavior of the solution of a general boundary-value problem for higher-order linear equations. However, the proposed algorithm does not allow us to investigate the asymptotic behavior to the solution of essentially nonlinear boundary value problems possessing the phenomena of boundary jumps. The natural direction of further research is the construction of the asymptotic of the solution of linear, nonlinear singularly perturbed boundary value problems with boundary jumps.

Therefore, the investigation of asymptotic behavior and the construction of the asymptotic of solutions of singularly perturbed boundary-value problems with boundary jumps is still topical and represents a certain theoretical interest and is important in applications.

The results of the work open up possibilities for the further development of the theory of boundary value problems for ordinary differential equations with a small parameter at the highest derivatives. The constructed approximations can be useful in considering various applied problems.

5. CONCLUSION

6.

Thus, using the initial and boundary functions, we construct an analytic representation of the solution of a singularly perturbed boundary value problem (1), (2); asymptotic estimates are obtained for the solution of the boundary value problem(1), (2); an unperturbed boundary-value problem is found which solution tends to the solution of the original problem when the small parameter tends to zero; the difference of solutions of degenerate and of original boundary value problems is estimated, and the character of growth of the derivatives with respect to the small parameter were establish; a class of boundary-value problems possessing the phenomenon of boundary jumps were extracted.

The obtained results give the chance for further researches of in the theory is singular perturbed boundary value problems and the established growth of the derivatives enables one to reduce the boundary-value problem (1), (2) to the Cauchy problem with initial jumps, which, in turn, can be regarded as a basis for the construction of the asymptotic expansions of certain singularly perturbed boundary-value problems with boundary jumps.

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