Journal of Theoretical and Applied Information Technology
30th June 2017. Vol.95. No 12

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

PROOFS OF IMPLICATIONS INVOLVING QUANTIFIERS DISTRIBUTION OVER LOGICAL OPERATORS

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ABSTRACT

Mathematics is considered the base for computer science. In particular, discrete mathematics is commonly used in many disciplines of computer science. One of the main topics that are discussed in discrete mathematics is quantifiers and their relations with logical operators. Accordingly, this paper proposes a new method to prove the validity of some implications involving quantifiers and logical operators. The proposed method is based on the idea of showing that whenever the premise of the implication is true, the conclusion cannot be false (must be true), so the implication is valid. On the other hand, if the conclusion can be false then the implication is not valid.

Keywords: *Predicate logic, Propositional logic, Quantifiers and Logical operators, Validity of implications.*

1. INTRODUCTION

Predicate logic and propositional logic in discrete mathematics bring a great tool for reasoning properly about mathematics, algorithms, and computers. They are used widely in computer science, and we need to identify their basic concepts in order to study many of the more advanced subjects in computer science. Propositional logic is used widely in information retrieval, digital circuit design and computer architecture, designing programming languages and software engineering [1]. On the other hand, predicate logic is used to form the basis of Prolog (Programming in logic) which is a language that is used widely to stimulate intelligent through programs in the field of artificial intelligence [2]. In addition, Structured Query Language (SQL) which is used widely in designing database systems is formed based on the predicate logic [3]. However, predicate logic is considered as an extension to propositional logic by adding two quantifiers which make it more expressive than the propositional logic and more applicable in complex problems.

One of the main issues in predicate logic is proving the validity of some assertions using different methods of proofs. In this paper, a new method is proposed for proving the quantifiers distribution over logical operators, which is considered an essential problem that is associated with predicate logic. This subject has been mentioned in few books and researches, and the processes of proving these relations are not clear enough. Our goal is to propose a new method that is more comprehensive, clearer, and deals with the problem in details. Additionally, the importance of this subject comes from the necessity of knowing which of these relationships are valid and which ones are invalid; this can assist in applying such topic in artificial intelligence, designing computer circuits, etc.

Firstly, some definitions that are related to our work are introduced. Also, two methods that were used to prove the validity of some logical implications are presented in this section.

Definition 1. A predicate is an assertion that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables from the domain of the predicate or when these variables are bound using quantifiers [4].

Definition 2. The domain of discourse of a predicate variable is the set of all values that may be substituted in place of the variable [5].

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

Definition 3. " The logical implication is a logical connective (or a binary operator) denoted by (→). The implication is used to form statement of the form (p → q) (termed a conditional statement) which is read as : (if p then q) or (p only if q) ". In logical implications, p is called the premise and q is called the conclusion . The converse of (p→q) is (q → p) [6]. The truth table of (p→q) is presented in table 1 [7] [8].

Table 1: The Truth Table of p → q

p	q	p → q
0	0	1
0	1	1
1	**0**	**0**
1	1	1

Definition 4. Quantification is a concept that specifies the quantity of cases in the domain of discourse that satisfy an open formula [9]. The two main quantifiers are: the universal (∀) and the existential (□).

Let P(x) be a predicate with domain D. A universal statement is a statement in the form ∀x P(x) . It is true if and only if (iff) P(x) is true for every x from D. It is false iff P(x) is false for at least one x from D. Let P(x) be a predicate with domain D. An existential statement is a statement in the form □x P(X). It is true iff P(x) is true for at least one x from D. It is false iff P(x) is false for every x from D [10].

It is proven that the existential quantifier (∃) does not distribute over the AND (∧) logical operator. Consequently,
∃x [P(x) ∧ Q(x)] ¬↔[∃x P(x)∧∃x Q(x)] [7] [10], where P(x) and Q(x) are predicates with domain D, but one side implies the other is valid, so the following implication is valid:
∃x [P(x) ∧Q(x)] →[∃x P(x) ∧ ∃x Q(x)], and the converse is not valid [4] [7].

Also, the universal quantifier (∀) is not distributable over the logical operator OR (∨). So,
∀x [P(x)∨Q(x)] ¬↔ [∀x P(x) ∨ ∀x Q(x)] [7] [10], but one side implies the other is valid, so the following implication is valid:
[∀x P(x) ∨ ∀x Q(x)] → ∀x[P(x) ∨ Q(x)] , and the converse is not valid [7] [11].

Furthermore, the (∃) quantifier does not distributes over the implication logical operator. As a consequent,
∃x [P(x) → Q(x)] ¬↔ [∃x P(x) → ∃xQ(x)] [7], but one side implies the other is valid, so the following implication is valid:
[∃x P(x) → ∃xQ(x)] → ∃x [P(x) → Q(x)], and the converse is not valid [7].

In addition, the (∀) quantifier does not distribute over the implication logical operator. So,
∀x [P(x)→Q(x)] ¬↔ [∀x P(x) → ∀x Q(x)] [7]. However, one side implies the other is valid as follows:
∀x [P(x) → Q(x)] → [∀x P(x) → ∀x Q(x)], but the converse is not valid [7].

Nabulsi (2000) [12] showed that the validity of the previous implications can be proven using the method of truth table.

For example, to prove that
∃x [P(x) ∧ Q(x)] →[∃x P(x)∧ ∃x Q(x)] is valid, the author assumed that the universe is the set {0,1}, then proved the validity of this implication as follows:

- ∃x[P(x)∧Q(x)] ↔ [P(0)∧ Q(0)]∨[P(1) ∧Q(1)] .

If we assumed that [P(0) ∧ Q(0)] is denoted by A and [P(1) ∧ Q(1)] is denoted by B then
∃x [P(x) ∧Q(x)]↔ A ∨ B.

-[∃xP(x)∧∃xQ(x)] ↔ [P(0)∨P(1)] ∧ [Q(0)∨Q(1)] .

If we assumed that [P(0)∨P(1)] is denoted by C and [Q(0) ∨Q(1)]] is denoted by D then
∃x [P(x)∧ ∃x Q(x)] ↔ C ∧ D .

- P(0), P(1), Q(0) and Q(1) are considered as four variables to construct the truth table of 16 cases as shown in table 2. The remaining implications can be proved in the same way.

Rasiowa (1973) [13] discussed that the implication:
[∀x P(x)∨ ∀x Q(x)] → ∀x[P(x) ∨ Q(x)] is proved to be valid by assuming that the conclusion is false; which means that [P(x) ∨ Q(x)] is not satisfied for every x□D, so there is an element (a) □ D, such that [P(x) ∨ Q(x)] is a false proposition. So, p(x) is false and q(x) is false. Consequently,
∀x P(x) is false and ∀x Q(x) is false in the Left-Hand Side (LHS), this will result in a false premise;

Journal of Theoretical and Applied Information Technology
30th June 2017. Vol.95. No 12

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

as a result, the implication is valid. Also, the implication:
∀x [P(x) → Q(x)] → [∀x P(x) → ∀x Q(x)] is proved to be valid, by assuming that the conclusion [∀x P(x) → ∀x Q(x)] is false, and because it is false, then ∀x P(x) must be true and ∀x Q(x) must be false , so every element of D satisfies p(x). Also, there must be an element (a) such that q(x) is false. Therefore, P(x) → Q(x) in the LHS must be false, and this will result in a valid implication.

The method proposed in [12] is long, time-consuming, and provided one example only. On the other hand, the method in [13] is somewhat difficult, not very clear, and provided two examples only. This paper proposes a clearer and easier solution in comparison with the other two methods.

The rest of this paper is organized as follows: Section 2 presents our proposed method. How to solve some examples using the proposed method is presented in Section 3.Finally, the drawn conclusions and the planned future work are discussed in Section 4.

Table 2: The Truth Table of ∃ x [P(x) ∧ Q(x)] →[∃ x P(x)∧ ∃ x Q(x)]

P(0)	P(1)	Q(0)	Q(1)	A	B	A ∨ B	C	D	C ∧ D	(A ∨ B) → (C ∧ D)
0	0	0	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	1	0	1
0	0	1	0	0	0	0	0	1	0	1
0	0	1	1	0	0	0	0	1	0	1
0	1	0	0	0	0	0	1	0	0	1
0	1	0	1	0	1	1	1	1	1	1
0	1	1	0	0	0	0	1	1	1	1
0	1	1	1	0	1	1	1	1	1	1
1	0	0	0	0	0	0	1	0	0	1
1	0	0	1	0	0	0	1	1	1	1
1	0	1	0	1	0	1	1	1	1	1
1	0	1	1	1	0	1	1	1	1	1
1	1	0	0	0	0	0	1	0	0	1
1	1	0	1	0	1	1	1	1	1	1
1	1	1	0	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1

2. METHODOLOGY

This section will propose the new approach to prove some implications involving quantifiers and logical operators.

Our method is based on the following idea: *If we could prove that whenever the premise is true, the conclusion cannot be false which means it must be true, so the implication is valid, otherwise, the implication is not valid*. This is due to the fact that an implication is false only when the premise is true and the conclusion is false according to third line of the truth table shown in table 1.

Now we start proving the validity of the following four implications:

1. ∃x [P(x) ∧Q(x)] →[∃x P(x) ∧ ∃x Q(x)]

2. [∀x P(x) ∨ ∀x Q (x)] → ∀x[P(x) ∨ Q(x)]

3. [∃x P(x) → ∃x Q(x)] → ∃x [P(x) → Q(x)]

4. ∀x [P(x)→ Q(x)] → [∀x P(x) → ∀x Q(x)]

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

Formula 1. $\exists x[P(x)\wedge Q(x)]\rightarrow[\exists xP(x)\wedge\exists xQ(x)]$ is valid.
In this implication, the LHS (the premise) of the implication means that the same value of x can be chosen to satisfy both predicates P(x) and Q(x). However, the Right- Hand Side (RHS) (the conclusion) of the implication means that different values of x can be chosen in order to satisfy P(x) and Q(x).

Proof. The LHS is true if there is at least one value of x that satisfies both P(x) and Q(x).

→ $\exists x\, P(x)$ is true and $\exists x\, Q(x)$ is true.

→ The right side cannot be false, therefore the implication is valid.

Formula 2. $[\forall xP(x)\vee\forall xQ(x)]\rightarrow\forall x[P(x)\vee Q(x)]$ is valid.
The LHS consist of two parts; $\forall x\, P(x)$ and $\forall x\, Q(x)$, it will be true if one or both of them is true. On the other hand, the RHS will be true if each value of x makes either P(x) or Q(x) true.

Proof. In the LHS there are three cases which make it true as follows:

Case 1: - $\forall x\, P(x)$ is true.

- $\forall x\, Q(x)$ is true.

→ Each value of x makes both P(x) and Q(x) true in the RHS.

→ The RHS is true.

Case 2: - $\forall x\, P(x)$ is true.

- $\forall x\, Q(x)$ is false.

→ Each value of x makes P(x) true in the RHS.

→ The RHS is true.

Case 3: - $\forall x\, P(x)$ is false.

- $\forall x\, Q(x)$ is true.

→ Each value of x makes Q(x) true in the RHS.

→ The RHS is true.

According to the previous three cases the RHS will always be true (cannot be false), therefore, the implication is valid.

Formula 3. $[\exists xP(x)\rightarrow\exists xQ(x)]\rightarrow\exists x[P(x)\rightarrow Q(x)]$ is valid.

In order to prove Formula 3, it needs to be rewritten into an equivalent one for simplicity; in classical logic , $(p\rightarrow q)$ is logically equivalent to $(\neg p \vee q)$ [6]. Therefore,

$$\{[\exists x\ P(x)\rightarrow\exists xQ(x)]\rightarrow\exists x[P(x)\rightarrow Q(x)]\} \leftrightarrow$$

$$\{[\neg\exists x\ P(x)\vee\exists xQ(x)]\rightarrow\exists x[\neg P(x)\vee Q(x)]\} \leftrightarrow$$

$$\{[\neg\exists xP(x)\vee\exists xQ(x)\ ]\rightarrow[\ \exists x\neg\ P(x)\vee\exists xQ(x)]\}$$

It is known that $\neg\exists xP(x)$ is equivalent to $\forall x\neg\ P(x)$ [4]. So, after replacing $\neg\exists xP(x)$ by $\forall x\neg P(x)$,we are going to prove that
$[\forall x\neg P(x)\vee\exists xQ(x)]\rightarrow[\exists x\neg P(x)\vee\exists xQ(x)]$ is valid.

Proof. In the LHS there are three cases which make it true as follows:

Case 1: - $\forall x\neg P(x)$ is true.

- $\exists x\, Q(x)$ is true.

→ $\exists x\,\neg P(x)$ is true because $[\forall x\neg\ P(x)\rightarrow\exists x\,\neg P(x)\ ]$ is valid [7], and $\exists x\, Q(x)$ is also true in the RHS.

→ The RHS is true.

Case 2 : - $\forall x\neg P(x)$ is true.

- $\exists x\, Q(x)$ is false.

→ $\exists x\neg P(x)$ is true in the RHS of the implication.

→ The RHS is true.

Case 3: - $\forall x\neg P(x)$ is false.

- $\exists x\, Q(x)$ is true.

→ $\exists x\, Q(x)$ is true in the RHS.

→ The RHS is true.

According to the previous three cases of Formula 3, the RHS will always be true (cannot be false), therefore, the implication is valid.

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

Formula 4. $\forall x[P(x)\rightarrow Q(x)]\rightarrow[\forall xP(x)\rightarrow\forall xQ(x)]$ is valid.
In order to prove Formula 4, it needs to be rewritten into an equivalent form for simplicity;

$\{\forall x[P(x)\rightarrow Q(x)]\rightarrow[\forall xP(x)\rightarrow\forall xQ(x)]\} \leftrightarrow$

$\{\forall x[\neg P(x)\vee Q(x)]\rightarrow[\neg\forall xP(x)\vee\forall xQ(x)]\}$

It is known that $\neg\forall xP(x)$ is equivalent to $\exists x\neg P(x)$ [4]. So, after replacing $\neg\forall xP(x)$ by $\exists x\neg P(x)$, we are going to prove that
$\forall x [\neg P(x)\vee Q(x)]\rightarrow[\exists x\neg P(x)\vee\forall xQ(x)]$ is valid.

The LHS becomes true if each value of x makes either $\neg P(x)$ or $Q(x)$ true. In the RHS, there is two parts; $\exists x\neg P(x)$ and $\forall xQ(x)$ and it is sufficient that one of them is true, so the RHS becomes true.

Proof. In the LHS there are three cases that make it true:

Case 1: In the LHS some values of x make $\neg P(x)$ true and the remaining values of x make $Q(x)$ true.

→ $\exists x\neg P(x)$ is true in the RHS.

→ The RHS is true.

Case 2: In the LHS, all values of x will make $\neg P(x)$ true.

→ $\forall x\neg P(x)$ is true.

→ $\exists x\neg P(x)$ is true in the RHS.

→ The RHS will be true.

Case 3: In the LHS, all values of x will make $Q(x)$ true.

→ $\forall x\, Q(x)$ is true in the RHS.

→ The RHS is true.

All the cases of Formula 4 resulted in a true value for the RHS. Therefore, the implication is valid.

3. EXAMPLES OF HOW TO SOLVE PROBLEMS USING THE PROPOSED METHOD

In this section the validity of two examples using the proposed method of proof will be verified. The criteria to evaluate the validity of the implications is the following: whenever the premise of the implication is true, the conclusion must be true, so the implication is valid. If the conclusion can be false, then the implication is not valid.

Example 1. Prove or disprove the following implication:

$[\forall xP(x) \vee \forall xQ(x)]\rightarrow[\exists xP(x) \vee \forall x Q(x)]$

Solution. The left side has three cases in order to be true:

Case 1: $\forall xP(x)$ and $\forall x Q(x)$ are both true.

Case 2: $\forall xP(x)$ is false and $\forall x Q(x)$ is true.

In case 1 and case 2:

- $\forall xQ(x)$ in the LHS is true.

→ $\forall xQ(x)$ is true in the RHS.

→ The RHS is true.

Case 3: - $\forall x P(x)$ is true.

- $\forall x Q(x)$ is false.

→ □x P(x) is true in the RHS because $[\forall x P(x) \rightarrow \exists x P(x)]$ is valid.

→ The RHS is true.

All the three cases resulted in having the RHS to be true. For that reason, the implication is valid.

Example 2. Prove or disprove the following implication:

[□x P(x) ∨ □x Q(x)]→[∀x P(x) ∨ □x Q(x)]

Solution. Let Assume that the LHS is true, so in order to be true, there will be three cases:

Case 1: □x P(x) and □x Q(x) are both true.

Case 2: □x P(x) is false and □x Q(x) is true.

In case 1 and case 2:

- □x Q(x) is true in the LHS.

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

→ □x Q(x) is true in the RHS.

→ The RHS is true.

Case 3: - □x P(x) is true in the LHS.

- □x Q(x) is false in the LHS.

→ □x Q(x) is false in the RHS.

→ ∀x P(x) is true or false in the RHS

→ the RHS can be true or false.

In case 3, the RHS is either true or false, so the implication is not valid.

4. CONCLUSIONS AND FUTURE WORK

This paper proposed a new method to verify the validity of implications involving quantifiers distribution over logical operators. The idea is based on the fact that whenever the premise of the implication is true, the conclusion of that implication cannot be false (must be true), which results in a valid implication. Moreover, if the conclusion can be false the implication is not valid. Accordingly, we used the proposed method to prove the validity of the following four implications:

1. ∃x [P(x) ∧ Q(x)] → [□x P(x) ∧ ∃x Q(x)]
2. [∀x P(x) ∨ ∀x Q(x)] → ∀x [P(x) ∨ Q(x)]
3. [∃x P(x) → ∃x Q(x)] → ∃x [P(x) → Q(x)]
4. ∀x [P(x) → Q(x)] → [∀x P(x) → ∀x Q(x)]

Additionally, the following two problems:

[∀x P(x) ∨ ∀x Q(x)] → [∃x P(x) ∨ ∀x Q(x)] and [∃xP(x) ∨ ∃x Q(x)] → [∀x P(x) ∨ ∃x Q(x)] were verified by using the proposed method. We showed that the first one is valid and the second is not valid.

In future, we will investigate other methods to prove relations between quantifiers and logical operators, which are more simple and efficient.

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