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EXPONENTIAL DECAY FUNCTION APPROXIMATION BY THIELE CONTINUED FRACTION AND SPLINE

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ABSTRACT

We provide a technique for the approximation of a function with an exponential decay rate, using Thiele interpolating continued fraction coupled with spline. This new method benefits from computational advantages of rational approximation while avoiding alteration caused by pole appearance. We then provide an application of this technique in the approximation of the packet loss probability function used as a quality of service parameter in communication networks.

Keywords: *Rational Approximation, Thiele Continued Interpolating Fraction, Pole Appearance, Cubic Spline, Packet/Cell Loss Probability, Quality Of Service, Communication Networks.*

1. INTRODUCTION

Exponential decay functions can be found in physics and natural sciences. Examples are the discharge of a capacitor, amplitude of an oscillator subject to linear friction, cooling processes or radioactive decays [1]. The evaluation of those functions is sometime very tough and requires a lot of calculations. It is the case when it comes to evaluate the packet loss probability as a function of the buffer size in queuing systems. Exact solutions of this function are time-consuming for large state systems. The CPU time to evaluate this function for a set of buffer sizes can take several days using a supercomputer [2]. Hence the interest of the use of the approximation techniques.

Due to the asymptotic behavior of a function with exponential decay, the approximation by a rational function of numerator degree $n+1$ and denominator degree n is a natural choice.

Rational approximation techniques are well known for their computational advantages. However, the pole appearance in denominator polynomial of the approximating fraction can ruin the quality of the approximation.

In this paper we propose a new approximation technique to approximate functions with exponential decays and with known decay rate. The technique is based on Thiele interpolating continued fractions and spline. The advantage of this new method is that it takes benefits from computational advantages of rational approximation and in the same time avoids the alteration that might be caused by pole appearance.

We propose an application of our method to approximate the packet loss rate in queuing systems. It is assumed that the packet loss rate is known for a number of small capacities and also that the exponential asymptote is known for the capacity going to infinity [2, 3, 14].

The problem of the estimation of the packet loss rate in queuing systems was treated in literature with different approaches. However, the techniques introduced in the literature suffered from potential introduction of poles [2]. Alternative solutions were not satisfactory either; the improved approach introduced in [4] avoids pole appearance by the artificial choice of denominator polynomial with optimal pole placement. This is an artificial rational interpolation that is susceptible to spoil the performance of the method; in real rational interpolation, the numerator and denominator polynomials are both unknown, they are sought simultaneously [5, 6].

In order to address the issue of pole appearance in Thiele rational approximation, we propose an approach based on the following idea: locate poles of the Thiele denominator polynomial and combine Thiele method with cubic spline interpolation; Thiele will be used to extrapolate beyond the area that contains poles, and the cubic spline interpolation will be used to interpolate within this

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area. The interest is to benefit from the strengths of the two methods.

2. THIELE RATIONAL APPROXIMATION

Let f be an exponential decay function, real-valued of a variable N, known only for a set of values N_i, $i = 0, \dots, 2n+1$.

We consider that the decay rate, that we denote ξ, is known:

$$f(N) \sim \xi N \quad \text{as} \quad N \to \infty. \tag{1}$$

Due to the asymptotic behavior of the function f(N), the approximation by a rational function $r_{2n+1}(N)$ of numerator degree $n+1$ and denominator degree n is a natural choice.

$$r_{2n+1}(N) = \frac{p_n(N)}{q_n(N)} = \frac{\sum_{i=0}^{n+1} a_i N^i}{\sum_{i=0}^{n} b_i N^i}. \tag{2}$$

The rational approximate $r_{2n+1}(N)$ can be obtained as the $(2n+1)^{th}$ convergent of a Thiele type continued fraction [7].

$$r(N) = \varphi[N_0] + \cfrac{N - N_0}{\varphi[N_0, N_1] + \cfrac{N - N_1}{\varphi[N_0, N_1, N_2] + \cfrac{N - N_2}{\ddots}}}. \tag{3}$$

The $(2n+1)^{th}$ convergent is written as

$$r_{2n+1}(N) = \varphi[N_0] + \cfrac{N - N_0}{\varphi[N_0, N_1] + \cfrac{N - N_1}{\ddots \; \overline{\varphi[N_0, \dots, N_{2n}] + \cfrac{N - N_{2n}}{\varphi[N_0, \dots, N_{2n+1}]}}}} \tag{4}$$

where the inverse differences $\varphi[N_0, \dots, N_{j+1}]$ are computed recursively from:

$$\varphi[N_j] = f(N_j), \quad j = 0, 1, \cdots \tag{5}$$

$$\varphi[N_0, \dots, N_{j-1}, N_j, N_l] = \frac{N_l - N_j}{\varphi[N_0, \dots, N_{j-1}, N_l] - \varphi[N_0, \dots, N_{j-1}, N_j]}, \quad l = j+1, j+2, \cdots \tag{6}$$

In order that the asymptotic behavior of $r_{2n+1}(N)$ matches that of f(N), we only compute $\varphi[N_0, \dots, N_j]$ with $j = 1, \dots, 2n$ from equation (6). The remaining inverse difference $\varphi[N_0, \dots, N_{2n+1}]$ is computed from the decay rate ξ.

The coefficient of highest degree in the numerator of $r_{2n+1}(N)$, namely, a_{n+1} equals

$$a_{n+1} = \frac{1}{\sum_{j=0}^{n} \varphi[N_0, \dots, N_{2j+1}]}. \tag{7}$$

For $r_{2n+1}(N)$ to behave asymptotically like ξN, we need to require $a_{n+1} = \xi$, in other words,

$$\varphi[N_0, \dots, N_{2n+1}] = \frac{1}{\xi} - \sum_{j=0}^{n-1} \varphi[N_0, \dots, N_{2j+1}]. \tag{8}$$

Thiele rational approximation is very suited to approximate functions with exponential decay; however, in some cases, the generated denominator polynomial could have poles. In such a case, the approximation will be altered.

3. CALCULATION OF THIELE DENOMINATOR POLYNOMIAL COEFFICIENTS

In this section we will calculate the Thiele denominator polynomial coefficients. Those coefficients will be used to calculate the Cauchy bound [8] and hence locate Thiele denominator polynomial poles.

The r_{2n+1} function could be written [9] as

$$\begin{cases} p_n = \varphi[N_0, \cdots, N_{2n+1}] p_{2n} + (N - N_{2n}) p_{2n-1} \\ q_n = \varphi[N_0, \cdots, N_{2n+1}] q_{2n} + (N - N_{2n}) q_{2n-1} \end{cases} \tag{9}$$

with

$p_{-1} = 0$, $p_0 = b_0$,
and $q_{-1} = 0, q_0 = 1$.

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Let us denote

$$\varphi_n = \varphi[N_0, \dots, N_n]. \qquad (10)$$

We have deg $q_{2k} = k$ and deg $q_{2k+1} = k$.

q_{2k} can be written as:

$$\begin{aligned} q_{2k} &= \varphi_{2k} q_{2k-1} + (N - N_{2k-1}) q_{2k-2} \\ &= \varphi_{2k}(b_{k-1}^{(2k-1)} N^{k-1} + \dots + b_0^{(2k-1)}) \\ &\quad +(N - N_{2k-1})(b_{k-1}^{(2k-2)} N^{k-1} \\ &\quad + \dots + b_0^{(2k-2)}) \\ &= b_{k-1}^{(2k-2)} N^k + (\varphi_{2k} b_{k-1}^{(2k-1)} \\ &\quad + b_{k-2}^{(2k-2)} - N_{2k-1} b_{2k-1}^{(2k-2)}) N^{k-1} \\ &\quad + \dots + (\varphi_{2k} b_i^{(2k-1)} + b_{i-1}^{(2k-2)} \\ &\quad - N_{2k-1} b_i^{(2k-2)}) N^i + \dots \\ &\quad + (\varphi_{2k} b_0^{(2k-1)} - N_{2k-1} b_0^{(2k-2)}) \end{aligned}$$

q_{2k+1} can be written as

$$\begin{aligned} q_{2k+1} &= \varphi_{2k+1} q_{2k} + (N - N_{2k}) q_{2k-1} \\ &= \varphi_{2k+1}\big(b_k^{(2k)} N^k + \dots + b_0^{(2k)}\big) \\ &\quad +(N - N_{2k})(b_{k-1}^{(2k-1)} N^{k-1} \\ &\quad + \dots + b_0^{(2k-1)}) \\ &= (\varphi_{2k+1} + b_{k-1}^{(2k-1)}) N^k + (\varphi_{2k+1} b_{k-1}^{(2k)} \\ &\quad + b_{k-2}^{(2k-1)} - N_{2k} b_{k-1}^{(2k-1)}) N^{k-1} + \dots \\ &\quad + (\varphi_{2k+1} b_i^{(2k)} + b_{i-1}^{(2k-1)} \\ &\quad - N_{2k} b_i^{(2k-1)}) N^i + \dots \\ &\quad + (\varphi_{2k+1} b_1^{(2k)} + b_0^{(2k-1)} \\ &\quad - N_{2k} b_1^{(2k-1)}) N + (\varphi_{2k+1} b_0^{(2k)} \\ &\quad - N_{2k} b_0^{(2k-1)}) \end{aligned}$$

with, for $i = 1, \dots, k-1$

$$\begin{cases} b_k^{(2k)} = b_{k-1}^{(2k-2)} = \dots = b_1^{(2)} = 1 \\ b_i^{(2k)} = \varphi_{2k} b_i^{(2k-1)} + b_{i-1}^{(2k-2)} - N_{2k-1} b_i^{(2k-2)} \\ b_0^{(2k)} = \varphi_{2k} b_0^{(2k-1)} - N_{2k-1} b_0^{(2k-2)} \end{cases} \qquad (11)$$

By iterative calculation based on the equation system in (11) we can get the coefficients b_i for $i = 0, \dots, n$ of Thiele denominator polynomial q_n.

4. COMBINED THIELE AND SPLINE APPROXIMATION

Let $r_{2n+1}(N)$ be the Thiele continued fraction

$$r_{2n+1}(N) = \frac{p_n(N)}{q_n(N)} = \frac{\sum_{i=0}^{n+1} a_i N^i}{\sum_{i=0}^{n} b_i N^i}, \qquad (12)$$

whose numerator is of degree $n+1$ and denominator is of degree n, which interpolates $f(N)$ on the support supp where $\#supp = 2n+1$.

Henceforth, supp will be indexed by $2n+1$, we will note $supp = supp_{2n+1}$.

We know that all the roots of the denominator polynomial

$$q_n(N) = N^n + b_{n-1} N^{n-1} + \dots + b_0, \qquad (13)$$

are contained in the disk centered at the origin and with radius

$$1 + \max_{0 \le i \le n-1} |b_i|, \qquad (14)$$

called Cauchy bound [8] that we note C_n.

Thiele method will be accurate as far as we move away from C_n.

The idea is to stick the part of the Thiele approximation that is not affected by the occurrence of poles (beyond C_n) with the cubic spline approximation of the function f on the interval containing the poles.

It is clear that for all n, the rational function $r_{2n+1}(N)$ as it is constructed verifies

$$\lim_{N \to \infty} r_{2n+1}(N) = \xi N, \qquad (15)$$

in other terms,

$$r_{2n+1}(N) \sim f(N), \qquad (16)$$

in the neighborhood of $+\infty$.

Let us compare the quantity

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$$\frac{\| r_{2n+1} - r_{2n-1} \|}{\| r_{2n-1} \|}, \qquad (17)$$

with a given tolerance ε for each iteration n, where the uniform norm is taken over the set $[\max(C_n, C_{n-1}), +\infty[$.

If the following condition is satisfied

$$\sup_{N\in[max(C_n,C_{n-1}),+\infty[} |r_{2n+1}(N) - r_{2n-1}(N)| \leq \varepsilon \sup_{N\in[max(C_n,C_{n-1}),+\infty[} |r_{2n-1}(N)|, \qquad (18)$$

then we calculate the following cubic splines which interpolate $r_{2n+1}(N)$ on

$$supp_{2n+1} \cup \{b, b-h, \dots, b-(m-1)h\},$$

where h is a step chosen arbitrarily on the interval $[C_n, b]$ and b is the upper limit of the evaluation interval of the function f.

We note those cubic splines $S_{2n+1,m}(N), m = 1,2, \dots$

For a given tolerance ε, the stop test on the calculation of $S_{2n+1,m}, m = 1,2, \dots$ will be

$$\frac{\| S_{2n+1,m+1} - S_{2n+1,m} \|}{\| S_{2n+1,m} \|} \leq \varepsilon. \qquad (19)$$

Our approach consists on approximating the function f by the resulting cubic spline $S_{2n+1,m+1}$.

The combined Thiele and spline approximation technique introduced in this paper is comparable to Thiele rational approximation introduced in [2] in case the generated Thiele interpolating fraction does not have poles, as both rely on Thiele interpolating fraction.

In case the generated Thiele interpolating fraction do have poles, the Thiele rational approximation method is affected, and the combined Thiele and spline technique is far more adapted to approximate the function as it avoids poles alteration.

We can consider the combined Thiele and spline approximation technique as an alternative to the optimally placed pole technique introduced in [4]. The optimally placed pole technique is about constructing an approximating fraction. This construction is done by fixing a denominator polynomial with optimally placed poles; i.e. poles outside the region of the approximation. The numerator polynomial is then sought so that the constructed fraction approximates the function. This method avoids pole appearance by the artificial choice of denominator polynomial with optimal pole placement. This is an artificial rational interpolation that is susceptible to spoil the performance of the method; in real rational interpolation, the numerator and denominator polynomials are both unknown, they are sought simultaneously [5, 6]. The merit of our method, combined Thiele and spline approximation technique, is that it preserves the Thiele interpolating fraction, where the numerator and denominator polynomials are sought simultaneously. The numerical examples in the following section show that the absolute error of our method is smaller than that of the optimally placed poles method.

5. NUMERICAL EXAMPLES

In this section, we consider a packet loss probability in queuing systems. We assume that the traffic can be described by a discrete time batch Markovian Arrival Process (D-BMAP) model of M/G/1-type [2, 10, 11, 12, 13]. The packet loss probability function, that we denote $P_L(N)$, depends on the parameters in Table 1.

Table 1: Parameters of the packet loss probability function.

Parameter	Signification
N	Buffer size
c	Number of packets serviced per time unit
M	Number of independent and non identical information sources
p	Probability that a source is changing from OFF to ON state
q	Probability that a source is changing from ON to OFF state

For large values of buffer size N, the computation of the exact values of the packet loss probability function is very tough. The Table 2 gives an idea about the CPU time to calculate this function [2]. The calculus was done using a supercomputer with a processing power of 90 teraFLOPS.

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We remark that when the buffer size N is 2900, the exact value of the packet loss probability is computed in about 3 hours 48 minutes of CPU time. The computing of the exact values of $\log P_L(N)$ for the buffer size N going from 2100 to 2900 necessitate a very long time and is practically impossible using an ordinary computer.

Table 2: Total CPU time in seconds with a supercomputer of 90 teraflops processing power.

N	$\log P_L(N)$	CPU time (seconds)
2100	−2.2263	7092.38
2200	−2.2441	7826.95
2300	−2.2617	8508.55
2400	−2.2792	9046.97
2500	−2.2966	9933.09
2600	−2.3138	10657.52
2700	−2.3309	11433.24
2800	−2.3497	12512.29

In the examples below, we consider a system with the following parameters:

$$M = 15, \quad c = 1, \quad p = 2.1900e-05, \quad q = 7.0000e-06, \quad \xi = -1.1332e-03. \tag{20}$$

5.1 Example: Pole Free Thiele Approximation

In this first example, we suppose that the function $\log P_L(N)$ is known in the following support points: $supp_7 = \{4, 14, 25, 34, 62, 76, 684\}$
Thiele continued fraction that approximates $\log P_L(N)$ at support $supp_7$ is given in equation (21).

$$r_7(N) = -0.0011332(-1.28266 + N)(2770.21 + N)(1236.1 - 37.4744N + N^2)\frac{1}{(1.80586 + N)(1296.5 - 40.2723N + N^2)} \tag{21}$$

The denominator of this fraction does not have poles on the real axis. We can see in Figure 1 that, from a support of only 7 points, the constructed Thiele approximate sticks perfectly on the exact packet loss probability curve.

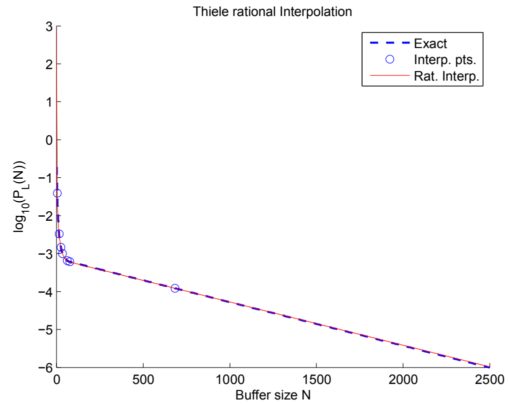

Figure 1: Exact vs Thiele approximation without poles

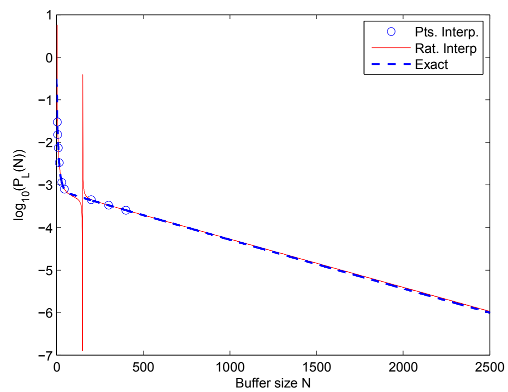

Figure 2: Exact vs Thiele approximation with poles

5.2 Example: Thiele Approximation with Poles

Let us consider the same packet loss probability function $\log P_L(N)$ as in the first example, but known in a different set of support points:

$supp_9 = \{5, 7, 10, 15, 30, 45, 200, 300, 400\}$
Thiele continued fraction that approximates $\log P_L(N)$ in support $supp_9$ is given in equation (22).

$$r_9(N) = -0.001133(-151.322 + N)(-2.95715 + N)(2765.25 + N)(73.9681 + 16.99N + N^2)\frac{1}{(-150.844 + N)(-2.75588 + N)}\frac{1}{(246.532 + 14.9041N + N^2)} \tag{22}$$

The denominator of this fraction has a pole nearby $N = 150$. This pole appears clearly in Figure

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2. Moreover, we can see clearly that once we move away beyond the pole region, the Thiele approximation curve stick back to the exact function curve.

The Figure 3 illustrates our approach, combined Thiele and cubic spline approximation, in comparison with the exact function curve. We observe that the curve of our method sticks perfectly to the curve of the exact function.

Finally, the Figure 4 gives the absolute error graph of our method compared to the method of optimally placed poles cited in [4]. Obviously, the absolute error of our method is smaller than that of the optimally placed poles method.

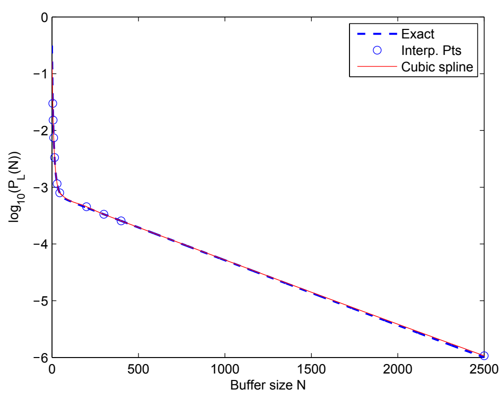

Figure 3: Exact vs Combined Thiele and cubic spline approximation

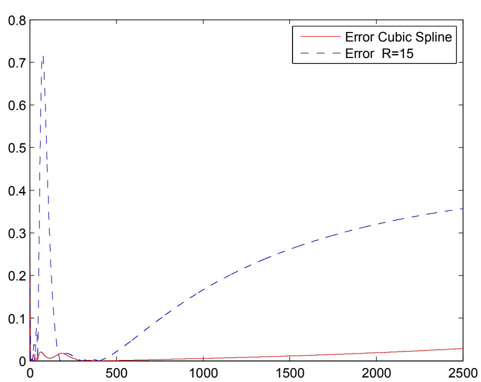

Figure 4: Absolute error comparison: Combined Thiele and cubic spline approximation vs Optimally placed poles (R=15)

6. CONCLUSION

We have contributed through this work to introduce a new technique to approximate functions with exponential decay based on Thiele interpolating fraction and spline. The merit of this new technique is that it benefits from the computational advantages of rational and cubic spline approximations and at the same time avoids the alteration that might be caused by pole appearance. This latter is a major obstacle that restricts the use of rational approximation.

We have applied our technique to approximate the packet loss probability as a function of the buffer size. Exact solutions of this function are time-consuming for large state systems, and sometimes are practically impossible. Though, the evaluation of the function for small values as well as the evaluation of the decay rate is possible. This information is sufficient to approximate the whole function all thanks to the new technique. The success of this latter is confirmed by numerical examples.

REFERENCES

[1] A. Leike, Demonstration of the exponential decay law using beer froth, European Journal of Physics 23 (1). doi:10.1088/0143-0807/23/1/304

[2] A. Cuyt, R. Lenin, G. Willems, C. Blondia, P. Rousseeuw, Computing packet loss probabilities in multiplexer models using rational approximation, IEEE Transactions on Computers 52 (5) (2003) 633-644. doi:10.1109/tc.2003.1197129 .

[3] E. Falkenberg, "On the Asymptotic Behavior of the Stationary Distribution of Markov Chains of M/G/1-Type," Comm. Statistical Stochastic Models, vol. 10, pp. 75-97, 1994.

[4] R. Lenin, A. Cuyt, K. Yoshigoe, S. Ramaswamy, Computing packet loss probabilities of d-bmap/ph/1/n queues with group services, Performance Evaluation 67 (2010) 160-173. doi:10.1016/j.peva.2009.10.001

[5] L. Wuytack, On the osculatory rational interpolation problem, Mathematics of Computation 29 (131) (1975) 837-847.

[6] G. Claessens, The rational hermite interpolation problem and some related recurrence formulas, Computers and Mathematics with Applications 2 (2) (1976) 117-123. doi:10.1016/0898-1221(76)90023-7

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

[7] D. Warner, Hermite interpolation with rational functions, PhD thesis, Univ. of California.

[8] A. L. Cauchy, Exercises de mathematique 2 (9) (1829) 122.

[9] J. Tan, The limiting case of thiele's interpolating continued fraction expansion, Journal of Computational Mathematics 19 (4) (2001) 433-444.

[10] A. Kamal, Efficient solution of multiple server queues with application to the modeling of atm concentrators, Proc. IEEE Infocom '96, San Francisco, CA (1996) 248-254
doi:10.1109/infcom.1996.497900

[11] C. Blondia and O. Casals, "Performance Analysis of Statistical Multiplexing of VBR Sources: A Matrix-Analytical Approach", Performance Evaluation, vol. 16, pp. 5-20, 1992.

[12 M.F. Neuts, Structured Stochastic Matrices of M/G/1 Type and their Applications. Marcel Dekker, 1989.

[13] K. Wuyts and R.K. Boel, "A Matrix Geometric Algorithm for Finite Buffer Systems with B-ISDN Applications", Proc. ITC Specialists Seminar Control in Comm., pp. 265-276, 1996.

[14] Z. Liu, P. Nain, and D. Towsley, "Exponentially Bounds with an Application to Call Admission", Technical Report TR94-63, Computer Science Dept., Univ. of Massachusetts, Amherst, Oct. 1994.