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ROBUST STABILITY OF SPACECRAFT TRAFFIC CONTROL SYSTEM USING LYAPUNOV FUNCTIONS

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ABSTRACT

A new approach to the construction of Lyapunov functions as vector functions is developed based on a geometrical interpretation of the second method of Lyapunov. The negative of the gradient is determined from the components of the time derivative of the state vector (i.e., the right-hand side of the state equation). The region of stability of a closed-loop linear, stationary system with uncertain parameters is governed by inequalities in the matrix elements of the closed-loop system.

Keywords: *Control systems, Robust stability, Spacecraft, Traffic control, Lyapunov's direct method, Modelling.*

1. INTRODUCTION

Currently, control problems are characterised by increasingly complex, high-order systems, requirements for high efficiency and stability, numerous uncertainties and incomplete information. Robust stability can be viewed as one of the outstanding issues in control theory, but it is also of a great practical interest. Control system design is one of the main tasks in automation in all branches of industry including manufacturing, energy, electronics, chemicals, medical devices, metals, textiles, transportation, robotics, aviation, space systems, and high-precision military/defence systems. In these systems, uncertainty can occur because of the presence of uncontrolled disturbances acting on the system [1] or because the true values of the parameters of the system are unknown, either initially or as the system changes over time [1, 2, 3, 4, 5].

The main goal in control system design is, in some sense, to provide the best protection against uncertainty in the knowledge of the system. The ability of a control system to maintain stability in the presence of parametric or nonparametric uncertainties is known as system robustness. In general, robust stability analysis consists of determining the ranges of values of uncertain parameters for which the closed-loop system remains stable [2, 17]. A considerable volume of work has been devoted to the development of robust stability theory.

In this study, we investigate a new approach to the construction of vector Lyapunov functions [6, 13]. Vector Lyapunov functions are constructed using a geometrical interpretation of the second method of Lyapunov presented in [8, 10, 14]. The components of the time derivative of the state vector (i.e., the right-hand side of the state equation) are used to form the negative of the gradient. The robust stability of the system is ensured by choosing the controller parameters so that the scalar product of the gradient vector and the time derivative of the state vector are a negative function [11, 12]. Stability conditions can be obtained from the positivity of the Lyapunov function in the form of a system of inequalities involving the uncertain parameters of the system (i.e., plant) and the parameters (i.e., gains) of the controller.

We investigate the robust stability of single-input, single-output (SISO) and multi-input, multi-output (MIMO) linear.

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In the study of stability, the state equation is defined in terms of perturbations Δx about a nominal state; i.e., the state vector $x(t)$ is defined as the difference between the perturbed state $X(t)$ and unperturbed state $X_s(t)(x(t) = \Delta x(t) = X(t) - X_s(t)$

This difference is called a perturbation. Therefore, the origin corresponds to a predetermined condition of the system, the unperturbed state $X_s(t)$. Hence, the right-hand side of the state equation expresses the rates of the perturbations (deviations) of $x(t)$, and we can assume that the vector of perturbation rates for a stable system is directed toward the origin.

Using a geometric interpretation of the second method of Lyapunov, determining stability is reduced [7, 8, 9, 10, 11, 13, 16] to the construction of a family of closed surfaces surrounding the origin with the property that the integral curves corresponding to the solutions of the state equation (with respect to perturbations), i.e., the trajectories of the system, cross these surfaces from the exterior to the interior, where the interior contains the origin. The unperturbed condition is stable if it is possible to construct such families of surfaces.

If the total time derivative of the Lyapunov function is negative and the rate vector is directed toward the origin, then each integral curve emanating from a sufficiently small neighbourhood of the origin will necessarily cross each of the surfaces from the exterior to the interior because the Lyapunov function monotonically decreases. In this case, the integral curves approach the origin, so the unperturbed condition is asymptotically stable.

2. MATHEMATICAL MODEL FORMULATION

2.1. Single-input, single-output systems

We now consider a system with one input and one output [11, 12]

Let the open-loop system be described by the equation

$$\frac{dx}{dt} = Ax + bu, x \in R^n, u \in R^1 \qquad (1)$$

where

$$A = \begin{vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & a_1 \end{vmatrix}, \quad b = \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{vmatrix}, \quad x = \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix}.$$

The state feedback control law is given by the scalar function

$$u(t) = -k^T x(t) \qquad (2)$$

where $k^T = \|k_1 \quad k_2, \quad \cdots, \quad k_n\|$ (dimensions $1 \times n$). Then, system (1) in explicit form can be represented as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -(a_n + k_1)x_1 - (a_{n-1} + k_2)x_2 - \ldots - (a_1 + k_n)x_n \end{cases} \qquad (3)$$

We apply Lyapunov's direct method [2, 16, 17] to determine the stability of the system in (3): for the system to be asymptotically stable, it is necessary and sufficient that there exists a positive Lyapunov function *V(x)* such that the derivative with respect to time along the solution of the state equation (3) is negative; i.e., The time derivative of the Lyapunov function in (4) with regard to the state equation (3) is given by the scalar product of the gradient vector $\frac{\partial V(x)}{\partial x}$ and the state rate vector $\frac{dx}{dt}$. To determine the stability of a system [1, 10], the nominal, or unperturbed, state must be chosen.

The equations of system (1) or (3) are always formed in terms of deviations Δ from a steady state $X_S(x = \Delta x = X - X_S)$.

Applying a geometric interpretation of Lyapunov's theorem [11, 12], we define negative gradients of the candidate Lyapunov function as

$$\begin{cases} -\frac{dx_1}{dt} = \frac{\partial V_1(x)}{\partial x_2} = x_2 \\ -\frac{dx_2}{dt} = \frac{\partial V_2(x)}{\partial x_3} = x_3 \\ \ldots \\ -\frac{dx_{n-1}}{dt} = \frac{\partial V_{n-1}(x)}{\partial x_n} = x_n \\ -\frac{dx_n}{dt} = \frac{\partial V_n(x)}{\partial x_1} + \frac{\partial V_n(x)}{\partial x_2} + \frac{\partial V_n(x)}{\partial x_3} + \ldots + \frac{\partial V_n(x)}{\partial x_n} = \\ = -[(a_n + k_1)x_1 + (a_{n-1} + k_2)x_2 + \ldots + (a_1 + k_n)x_n]^2 \end{cases} \qquad (4)$$

Then, we obtain the complete time derivative of the candidate vector Lyapunov function as

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$$\begin{cases} \dfrac{dV_1(x)}{dt} = -x_2^2 \\ \dfrac{dV_2(x)}{dt} = -x_3^2 \\ \dots \\ \dfrac{dV_2(x)}{dt} = -x_n^2 \\ \dfrac{dV_n(x)}{dt} = -[(a_n + k_1)x_1 + \\ +(a_{n-1} + k_2)x_2 + \dots + (a_1 + k_n)x_n]^2 \end{cases} \quad (5)$$

From (5), it follows that the complete time derivative of a candidate vector Lyapunov function will always be a negative function.

The complete time derivative of the Lyapunov function $V(x) = V_1(x) + V_2(x) + \dots + V_n(x)$ can be expressed in scalar form as

$$\frac{dV(x)}{dt} = -x_2^2 - x_3^2 - \dots - [(a_n + k_1)x_1 + \\ +(a_{n-1} + k_2)x_2 + \dots + (a_1 + k_n)x_n]^2 \quad (6)$$

From (4), we can obtain a candidate vector Lyapunov function [11]:

$$V_1(x) = (0, -\frac{1}{2}x_2^2, 0, \dots, 0)$$

$$V_2(x) = (0, 0, -\frac{1}{2}x_3^2, \dots, 0)$$

$$\dots$$

$$V_{n-1}(x) = (0, 0, 0, \dots, -\frac{1}{2}x_n^2)$$

$$V_n(x) = (\frac{1}{2}(a_n + k_1)x_1^2, \frac{1}{2}(a_{n-1} + k_2)x_2^2, \dots, \frac{1}{2}(a_1 + k_n)x_n^2)$$

The entries of the candidate vector Lyapunov function $V_i (i = 1, \dots, n)$ are constructed from the gradient vector. The Lyapunov function can be expressed in scalar form as

$$V(x) = \frac{1}{2}(a_n + k_1)x_1^2 + \frac{1}{2}(a_{n-1} + k_2 - 1)x_2^2 + \\ +\frac{1}{2}(a_{n-2} + k_3 - 1)x_3^2 + \dots + \frac{1}{2}(a_1 + k_n - 1)x_n^2 \quad (7)$$

Given that the function in (7) must be positive and the quadratic forms in (5) are negative, we obtain the following conditions for the stability of the system in (3):

$$\begin{cases} a_n + k_1 > 0 \\ a_{n-1} + k_2 - 1 > 0 \\ a_{n-2} + k_3 - 1 > 0 \\ \dots \\ a_1 + k_n - 1 > 0 \end{cases} \quad (8)$$

in control systems, a precise mathematical formulation is often inaccessible. in reality, systems inevitably contain uncertainty. for a system to satisfy the constraints (8) in the presence of uncertainties in the parameters, we can determine a robust stability radius

$$G = ((g_{ij})),\ g_{ij} = g_{ij}^0 + \Delta_{ij},\ |\Delta_{ij}| < \gamma m_{ij},\ i = 1, \dots, n$$

where the nominal system matrix $G_0 = g_{ij}^0$ is super-stable, $g_{ij} = a_{ij} - b_j k_j$ are the entries of the closed-loop system matrix, $G_0 = ((g_{ij}^0))$ is the nominal system matrix (1), $\Delta = ((\Delta_{ij})), |\Delta| < m_{ij}$ is the matrix of uncertainties, the matrix $m = ((m_{ij}))$ scales changes in the entries g_{ij} of matrix G, and $\gamma > 0$ is the uncertainty range.

We define the system using the negative of the gradient of a candidate function, i.e., $\dot{x} = \Delta_x V$, which was obtained previously in the form of a Lyapunov function:

$$\begin{cases} \dot{x}_1 = -(a_n + k_1)x_1 \\ \dot{x}_2 = -(a_{n-1} + k_2 - 1)x_2 \\ \dot{x}_3 = -(a_{n-2} + k_3 - 1)x_3 \\ \dots \\ \dot{x}_n = -(a_1 + k_n - 1)x_n \end{cases} \quad (9)$$

Super-stability of nominal system (9) is defined using (4)

$$\delta(G_0) = \min(-g_{ij}^0 - \sum_{j \neq 1} g_{ij}^0) = \min((a_n + k_1), \min(a_{n-i} + k_i - 1)) \geq 0$$

$$i = 2, \dots, n \quad (10)$$

Suppose that the condition of super-stability is preserved for all matrices of the family

$$-\left(g_{ii}^0 + \Delta_{ii}\right) - \sum_{j \neq i} \left|g_{ij}^0 + \Delta_{ij}\right| \geq 0, i = 1, \dots, n$$

This inequality will be satisfied for all admissible Δ_{ii} if and only if

$$a_n^0 + k_1^0 - \gamma m_{11} > 0$$

$$a_{n-1}^0 + k_2^0 - 1 - \gamma m_{22} > 0$$

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…

$$a_1^0 + k_n^0 - 1 - \gamma m_{nn} > 0$$

i.e., $\gamma < \gamma^* = \min(\frac{a_n^0 + k_1^0}{m_{11}}, \min \frac{a_{n-1}^0 + k_i^0 - 1}{m_{ii}}),$

$i = 1,..,n-1$

Thus, we can explicitly find the radius of robust stability for the family of systems.

2.2. Multi-input, multi-output systems

We will investigate a method for determining the robust stability of linear systems with m inputs and n outputs based on a vector Lyapunov function [15], and we will obtain the conditions for robust stability [11, 12].

Assume a linear system given by

$$\dot{x} = Ax + Bu, x \in R^n, u \in R^m$$

$$y = cx \quad y \in R^{\ell} \qquad (11)$$

and a state feedback controller

$$u = -Kx \qquad (12)$$

where

$$A = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix}, B = \begin{Vmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{Vmatrix},$$

$$C = \begin{Vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{22} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{l1} & c_{l2} & \dots & c_{\ln} \end{Vmatrix}, x = \begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Vmatrix}, y = \begin{Vmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{Vmatrix}$$

$$K = \begin{Vmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \dots & \dots & \dots & \dots \\ k_{m1} & k_{m2} & \dots & k_{mn} \end{Vmatrix}, u = \begin{Vmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{Vmatrix},$$

$$u_i = -k_{i1}x_1 - k_{i2}x_2 - \dots - k_{in}x_n, i = 1,2,\dots,n$$

Equation (11) can be expanded as

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m \\ \dots\dots\dots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m \end{cases} \qquad (13)$$

Let the matrix $G = A\text{-}BK$ represent the closed-loop system. Expressing system (13) in matrix-vector form, we can write $\dot{x} = Gx, x \in R^n$

where

$$G = \begin{Vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \dots & \dots & \dots & \dots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{Vmatrix} \quad g_{ij} = a_{ij} - \sum_{k=1}^{m} b_{ik}k_{kj}$$

Hence, (13) can be written as

$$\begin{cases} \dot{x}_1 = (a_{11} - \sum_{k=1}^m b_{1k}k_{k1})x_1 + (a_{12} - \sum_{k=1}^m b_{1k}k_{k2})x_2 + \dots + (a_{1n} - \sum_{k=1}^m b_{1k}k_{kn})x_n \\ \dot{x}_2 = (a_{21} - \sum_{k=1}^m b_{2k}k_{k1})x_1 + (a_{22} - \sum_{k=1}^m b_{2k}k_{k2})x_2 + \dots + (a_{2n} - \sum_{k=1}^m b_{2k}k_{kn})x_n \\ \dots\dots \\ \dot{x}_n = (a_{n1} - \sum_{k=1}^m b_{nk}k_{k1})x_1 + (a_{n2} - \sum_{k=1}^m b_{nk}k_{k2})x_2 + \dots + (a_{nn} - \sum_{k=1}^m b_{nk}k_{kn})x_n \end{cases} \qquad (14)$$

A Lyapunov function $V(x)$ is defined as a vector $V(V_1(x), V_2(x), \dots, V_n(x))$, and the gradient of the vector Lyapunov function can be written as

$$\begin{cases} \frac{\partial V_1(x)}{\partial x_1} = -\left(a_{11} - \sum_{k=1}^m b_{1k}k_{k1}\right)x_1, \\ \frac{\partial V_1(x)}{\partial x_2} = -\left(a_{12} - \sum_{k=1}^m b_{1k}k_{k2}\right)x_2, \dots, \frac{\partial V_1(x)}{\partial x_n} = -\left(a_{1n} - \sum_{k=1}^m b_{1k}k_{kn}\right)x_n \\ \frac{\partial V_2(x)}{\partial x_1} = -\left(a_{21} - \sum_{k=1}^m b_{2k}k_{k1}\right)x_1, \\ \frac{\partial V_2(x)}{\partial x_2} = -\left(a_{22} - \sum_{k=1}^m b_{2k}k_{k2}\right)x_2, \dots, \frac{\partial V_2(x)}{\partial x_n} = -\left(a_{2n} - \sum_{k=1}^m b_{2k}k_{kn}\right)x_n \\ \dots \\ \frac{\partial V_n(x)}{\partial x_1} = -\left(a_{n1} - \sum_{k=1}^m b_{nk}k_{k1}\right)x_1, \\ \frac{\partial V_n(x)}{\partial x_2} = -\left(a_{n2} - \sum_{k=1}^m b_{nk}k_{k2}\right)x_2, \dots, \frac{\partial V_n(x)}{\partial x_n} = -\left(a_{nn} - \sum_{k=1}^m b_{nk}k_{kn}\right)x_n \end{cases} \qquad (15)$$

The time derivatives of the components of the vector Lyapunov function can be obtained from the state equation (13) or (14) using the scalar product of the components of the gradient of the vector Lyapunov function and the components of the state rate vector $\frac{dx_i}{dt}$, i.e.,

$$\frac{dV_i(x)}{dt} = -\left[\left(a_{i1} - \sum_{k=1}^m b_{ik}k_{k1}\right)x_1 + \left(a_{i2} - \sum_{k=1}^m b_{ik}k_{k2}\right)x_2 + \dots + \left(a_{in} - \sum_{k=1}^m b_{ik}k_{kn}\right)x_n\right]^2$$

$$i = 1,2,\dots,n \qquad (16)$$

The time derivatives of the elements of the vector Lyapunov function $V_i(x)$ are given in (16). From the geometrical interpretation of Lyapunov's

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theorem, these functions will be negative; i.e., the conditions for asymptotic stability of system (14) will always be satisfied.

Using the components of the gradient vector, we construct the elements of the vector Lyapunov function:

$$\begin{cases} V_1(x_1,x_2,...,x_n) = -\left(a_{11}-\sum_{k=1}^{m} b_{1k}k_{k1}\right)x_1^2 - \\ -\left(a_{12}-\sum_{k=1}^{m} b_{1k}k_{k2}\right)x_2^2 -,...,-\left(a_{1n}-\sum_{k=1}^{m} b_{1k}k_{kn}\right)x_n^2 \\ V_2(x_1,x_2,...,x_n) = -\left(a_{21}-\sum_{k=1}^{m} b_{2k}k_{k1}\right)x_1^2 - \\ -\left(a_{22}-\sum_{k=1}^{m} b_{2k}k_{k2}\right)x_2^2 -,...,-\left(a_{2n}-\sum_{k=1}^{m} b_{2k}k_{kn}\right)x_n^2 \\ \ldots\ldots \\ V_n(x_1,x_2,...,x_n) = -\left(a_{n1}-\sum_{k=1}^{m} b_{nk}k_{k1}\right)x_1^2 - \\ -\left(a_{n2}-\sum_{k=1}^{m} b_{nk}k_{k2}\right)x_2^2 -,...,-\left(a_{nn}-\sum_{k=1}^{m} b_{nk}k_{kn}\right)x_n^2 \end{cases} \quad (17)$$

The positiveness of the vector Lyapunov function can be expressed as

$$\begin{cases} -\left(a_{11}-\sum_{k=1}^{m} b_{1k}k_{k1}\right)>0, -\left(a_{12}-\sum_{k=1}^{m} b_{1k}k_{k2}\right)>0,...,-\left(a_{1n}-\sum_{k=1}^{m} b_{1k}k_{kn}\right)>0 \\ -\left(a_{21}-\sum_{k=1}^{m} b_{2k}k_{k1}\right)>0, -\left(a_{22}-\sum_{k=1}^{m} b_{2k}k_{k2}\right)>0,...,-\left(a_{2n}-\sum_{k=1}^{m} b_{2k}k_{kn}\right)>0 \\ \ldots\ldots \\ -\left(a_{n1}-\sum_{k=1}^{m} b_{nk}k_{k1}\right)>0, -\left(a_{n2}-\sum_{k=1}^{m} b_{nk}k_{k2}\right)>0,...,-\left(a_{nn}-\sum_{k=1}^{m} b_{nk}k_{kn}\right)>0 \end{cases} \quad (18)$$

We will consider the radius of robust stability of the vector Lyapunov function components. For this purpose, we can address parametric families of coefficients of the vector Lyapunov function components in the form [3, 14]

$d_{ij} = d_{ij}^0 + \Delta_{ij}, |\Delta_{ij}| \le \gamma\, m_{ij}, i,j = 1,2,...,n$ where the coefficients $d_{ij}^0 = -\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{ij}^0\right)$ of nominal matrix D_0 correspond to a strictly positive Lyapunov function, i.e.,

$$\sigma(D_0) = \min_i \min_j -\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0\right) > 0$$

We will require that the coefficients be strictly positive for all functions in the family

$$-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0\right) + \Delta_{ij} > 0, i = 1,2,...,n;\ j = 1,2,...,n$$

This inequality holds for all admissible Δ_{ij} if and only if

$$-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0\right) + \gamma\, m_{ij} > 0, i = 1,2,...,n;\ j = 1,2,...,n$$

i.e.,

$$\gamma < \gamma^* = \min_i \min_j \frac{-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0\right)}{m_{ij}}$$

In particular, if $m_{ij} = 1$ (in which case the scales of all of the coefficients of the components of the Lyapunov function are identical), then $\gamma^* = \sigma(D_0)$

Thus, the stability radius of the family of positive functions is equal to the smallest value among the coefficients of the vector Lyapunov function.

3. CASE STUDY

3.1. Case of linearized spacecraft equation

We investigate the stability of the spacecraft (SC) automatic traffic control systems with proportional control law (for the linearized system):

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = \frac{1}{I_x}(-M_{xu} + M_{xf}) \\ \frac{dx_3}{dt} = x_4 \\ \frac{dx_4}{dt} = \frac{1}{I_y}(-M_{yu} + M_{yf}) \\ \frac{dx_5}{dt} = x_6 \\ \frac{dx_6}{dt} = \frac{1}{I_z}(-M_{zu} + M_{zf}) \end{cases} \quad (19)$$

where I_x, I_y, I_z - main central moments of spacecraft inertia relative to relevant axis; M_{xu}, M_{yu}, M_{zu} & M_{xf}, M_{yf}, M_{zf} - respectively projections of momentum control and disturbing moment relative to relevant axis.

Control law is given as

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$$\begin{cases} -M_{xu} + M_{xf} = k_1 x_1 + k_2 x_2 \\ -M_{yu} + M_{yf} = k_3 x_3 + k_4 x_4 \\ -M_{zu} + M_{zf} = k_5 x_5 + k_6 x_6 \end{cases} \quad (20)$$

System (19) can be expressed as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = ak_1x_1 + ak_2x_2 \\ \frac{dx_3}{dt} = x_4 \\ \frac{dx_4}{dt} = bk_3x_3 + bk_4x_4 \\ \frac{dx_5}{dt} = x_6 \\ \frac{dx_6}{dt} = ck_5x_5 + ck_6x_6 \end{cases} \quad (21)$$

where $a = \frac{1}{I_x}$, $b = \frac{1}{I_y}$, $c = \frac{1}{I_z}$

We investigate the stability of the system (21) developed by the Lyapunov function method [11, 12]. The gradient of the vector Lyapunov function can be written as:

$$\frac{\partial V_1(x_1,...,x_6)}{\partial x_1} = 0, \frac{\partial V_1(x_1,...,x_6)}{\partial x_2} = -x_2,$$
$$\frac{\partial V_1(x_1,...,x_6)}{\partial x_3} = 0,..., \frac{\partial V_1(x_1,...,x_6)}{\partial x_6} = 0$$

$$\frac{\partial V_2(x_1,...,x_6)}{\partial x_1} = -ak_1x_1, \frac{\partial V_2(x_1,...,x_6)}{\partial x_2} = -ak_2x_2,$$
$$\frac{\partial V_2(x_1,...,x_6)}{\partial x_3} = 0,..., \frac{\partial V_2(x_1,...,x_6)}{\partial x_6} = 0$$

$$\frac{\partial V_3(x_1,...,x_6)}{\partial x_1} = 0, \frac{\partial V_3(x_1,...,x_6)}{\partial x_2} = 0,$$
$$\frac{\partial V_3(x_1,...,x_6)}{\partial x_3} = 0, \frac{\partial V_3(x_1,...,x_6)}{\partial x_4} = -x_4,$$
$$\frac{\partial V_3(x_1,...,x_6)}{\partial x_5} = 0, \frac{\partial V_3(x_1,...,x_6)}{\partial x_6} = 0$$

$$\frac{\partial V_4(x_1,...,x_6)}{\partial x_1} = 0, \frac{\partial V_4(x_1,...,x_6)}{\partial x_2} = 0,$$
$$\frac{\partial V_4(x_1,...,x_6)}{\partial x_3} = -bk_3x_3, \frac{\partial V_4(x_1,...,x_6)}{\partial x_4} = -bk_4x_4,$$
$$\frac{\partial V_4(x_1,...,x_6)}{\partial x_5} = 0, \frac{\partial V_4(x_1,...,x_6)}{\partial x_6} = 0$$

$$\frac{\partial V_5(x_1,...,x_6)}{\partial x_1} = 0,..., \frac{\partial V_5(x_1,...,x_6)}{\partial x_5} = 0,$$
$$\frac{\partial V_5(x_1,...,x_6)}{\partial x_6} = -x_6$$

$$\frac{\partial V_6(x_1,...,x_6)}{\partial x_1} = 0,..., \frac{\partial V_6(x_1,...,x_6)}{\partial x_5} = -ck_5x_5,$$
$$\frac{\partial V_6(x_1,...,x_6)}{\partial x_6} = -ck_6x_6$$

The full derivatives from Lyapunov function V (x) in a scalar form is defined as:

$$\frac{dV_1(x_1,...,x_6)}{dt} = -x_2^2,$$
$$\frac{dV_2(x_1,...,x_6)}{dt} = -a^2k_1^2x_1^2 - a^2k_2^2x_2^2,$$

$$\frac{dV_3(x_1,...,x_6)}{dt} = -x_4^2,$$
$$\frac{dV_4(x_1,...,x_6)}{dt} = -b^2k_3^2x_3^2 - b^2k_4^2x_4^2,$$

$$\frac{dV_5(x_1,...,x_6)}{dt} = -x_6^2,$$
$$\frac{dV_6(x_1,...,x_6)}{dt} = -c^2k_5^2x_5^2 - c^2k_6^2x_6^2$$

The full derivatives from Lyapunov function V (x) is defined as:

$$\frac{dV(x_1,...,x_6)}{dt} = -(a^2k_1^2x_1^2 + a^2k_2^2x_2^2 + x^2 + b^2k_3^2x_3^2 + \quad (22)$$
$$+ b^2k_4^2x_4^2 + x_4^2 + c^2k_5^2x_5^2 + c^2k_6^2x_6^2 + x_6^2)$$

Full derivative from Lyapunov vector function in this construction is definitely negative function. We can find components of the gradient vector of Lyapunov function:

$$V_1(x_1,...,x_6) = -\frac{1}{2}x_2^2,$$
$$V_2(x_1,...,x_6) = -\frac{1}{2}ak_1x_1^2 - \frac{1}{2}ak_2x_2^2,$$

$$V_3(x_1,...,x_6) = -\frac{1}{2}x_4^2,$$
$$V_4(x_1,...,x_6) = -\frac{1}{2}bk_3x_3^2 - \frac{1}{2}bk_4x_4^2,$$

$$V_5(x_1,...,x_6) = -\frac{1}{2}x_6^2,$$
$$V_6(x_1,...,x_6) = -\frac{1}{2}ck_5x_5^2 - \frac{1}{2}ck_6x_6^2$$

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Lyapunov function V (x) in a scalar form is defined as

$$V(x_1,x_2,...,x_6)=\frac{1}{2}[-ak_1x_1^2-(ak_2+1)x_2^2- \quad (23)$$
$$bk_3x_3^2-(bk_4+1)x_4^2-ck_5x_5^2-(ck_6+1)x_6^2]$$

Conditions of the system (21) robust stability, we obtain, taking into account the negative definition of the function (22) and the existence of a positive definite quadratic form (23), i.e.

$$-ak_1>0,\ -(ak_2+1)>0,\ -bk_3>0,$$
$$-(bk_4+1)>0,\ \ -ck_5>0,\ \ -(ck_6+1)>0$$

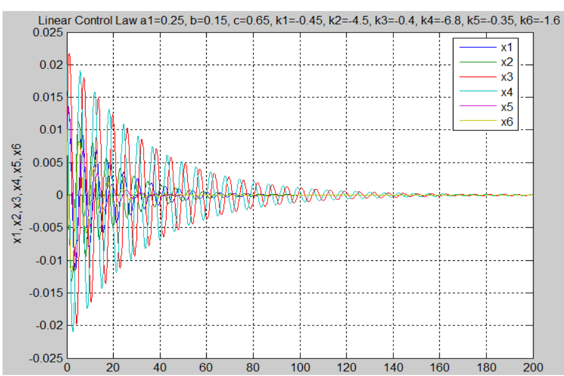
or

$$a>0,\ b>0,\ c>0$$

$$k_1<0,\ k_2<-\frac{1}{a},\ k_3<0,\ k_4<-\frac{1}{b},\ k_5<0,$$
$$k_6<-\frac{1}{c}; \quad (24)$$

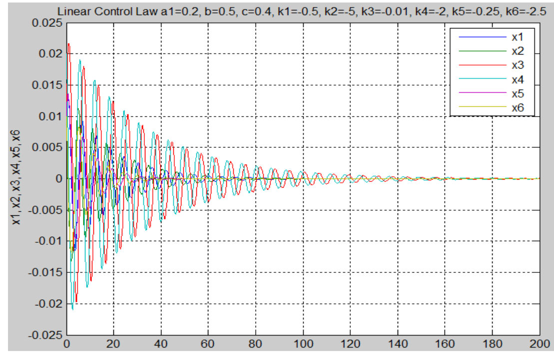
System (21) with the proportional control law (20) will be stable when the system parameters changing in area (24), and loses stability beyond the borders of the area (24). The system can behave chaotically and amplitude of chaotic oscillations can reach infinity.

Figures 1-2 show the results of numerical simulations of the linear spacecraft control system with proportional control law.



Figures - 1. The linear spacecraft control system with proportional control law.

$$k_1=-0.45; k_2=-4.5; k_3=-0.4; k_4=-6.8;$$
$$k_5=-0.35; k_6=-1.6$$



Figures - 2. The linear spacecraft control system with proportional control law.

$$k_1=-0.5; k_2=-5; k_3=-0.01;$$
$$k_4=-2; k_5=-0.25; k_6=-2.5$$

3.2. Case of nonlinear spacecraft equation

We investigate the stability of the spacecraft (SC) automatic traffic control systems with proportional control law (for the nonlinear system) [18, 19, 20, 21, 22]:

$$\begin{cases}\frac{dx_1}{dt}=x_2\\ \frac{dx_2}{dt}=\frac{1}{I_x}(I_y-I_z)x_4x_6+\frac{1}{I_x}(-M_{xu}+M_{xf})\\ \frac{dx_3}{dt}=x_4\\ \frac{dx_4}{dt}=\frac{1}{I_y}(I_z-I_x)x_2x_6+\frac{1}{I_y}(-M_{yu}+M_{yf})\\ \frac{dx_5}{dt}=x_6\\ \frac{dx_6}{dt}=\frac{1}{I_z}(I_x-I_y)x_2x_4+\frac{1}{I_z}(-M_{zu}+M_{zf})\end{cases} \quad (25)$$

Control law is given as

$$-M_{xu}+M_{xf}=k_1x_1+k_2x_2 \quad (26)$$
$$-M_{yu}+M_{yf}=k_3x_3+k_4x_4$$
$$-M_{zu}+M_{zf}=k_5x_5+k_6x_6$$

System (25) can be expressed as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = ak_1x_1 + ak_2x_2 + a\left(\frac{1}{b} - \frac{1}{c}\right)x_4x_6 \\ \frac{dx_3}{dt} = x_4 \\ \frac{dx_4}{dt} = b\left(\frac{1}{c} - \frac{1}{a}\right)x_2x_6 + bk_3x_3 + bk_4x_4 \\ \frac{dx_5}{dt} = x_6 \\ \frac{dx_6}{dt} = c\left(\frac{1}{a} - \frac{1}{b}\right)x_2x_4 + ck_5x_5 + ck_6x_6 \end{cases} \tag{27}$$

where $a = 1/I_x$, $b = 1/I_y$, $c = 1/I_z$

Stationary states of the system can be found by equation system:

$$\begin{cases} x_{2s} = 0 \\ ak_1x_{1s} + ak_2x_{2s} + a\left(\frac{1}{b} - \frac{1}{c}\right)x_{4s}x_{4s} = 0 \\ x_{4s} = 0 \\ bk_3x_{3s} + bk_4x_{4s} + b\left(\frac{1}{c} - \frac{1}{a}\right)x_{2s}x_{6s} = 0 \\ x_6 = 0 \\ ck_5x_5 + ck_6x_6 + c\left(\frac{1}{a} - \frac{1}{b}\right)x_2x_4 = 0 \end{cases} \tag{28}$$

By equation (28) we identify stationary state of the system (27) which is:

$$x_{1s} = 0,\ x_{2s} = 0,\ x_{3s} = 0,$$
$$x_{4s} = 0,\ x_{5s} = 0,\ x_{6s} = 0 \tag{29}$$

Lets research robust stability of the system (27) using Lyapunov's method. Gradient vector components shall be marked as:

$$\frac{\partial V_1(x)}{\partial x_1} = 0,\ \frac{\partial V_1(x)}{\partial x_2} = -x_2, \ldots,\ \frac{\partial V_1(x)}{\partial x_6} = 0$$

$$\frac{\partial V_2(x)}{\partial x_1} = -ak_1x_1,\quad \frac{\partial V_2(x)}{\partial x_2} = -ak_2x_2,\ \frac{\partial V_2(x)}{\partial x_3} = 0,$$
$$\frac{\partial V_2(x)}{\partial x_4} = -\frac{ac - ab}{bc}x_4x_6,$$

$$\frac{\partial V_2(x)}{\partial x_5} = 0,\ \frac{\partial V_2(x)}{\partial x_6} = 0$$

$$\frac{\partial V_3(x)}{\partial x_1} = 0, \ldots,\ \frac{\partial V_3(x)}{\partial x_4} = -x_4, \ldots,\ \frac{\partial V_3(x)}{\partial x_6} = 0$$

$$\frac{\partial V_4(x)}{\partial x_1} = 0,\quad \frac{\partial V_4(x)}{\partial x_2} = 0,\quad \frac{\partial V_4(x)}{\partial x_3} = -bk_3x_3,$$
$$\frac{\partial V_4(x)}{\partial x_4} = -bk_4x_4$$

$$\frac{\partial V_4(x)}{\partial x_5} = 0,\ \frac{\partial V_4(x)}{\partial x_6} = -\frac{ab - bc}{ac}x_2x_6,$$

$$\frac{\partial V_5(x)}{\partial x_1} = 0, \ldots,\ \frac{\partial V_5(x)}{\partial x_5} = 0,\ \frac{\partial V_5(x)}{\partial x_6} = -x_6$$

$$\frac{\partial V_6(x)}{\partial x_1} = 0,\ \frac{\partial V_6(x)}{\partial x_2} = -\frac{bc - ac}{ab}x_2x_4,\ \frac{\partial V_6(x)}{\partial x_3} = 0,$$
$$\frac{\partial V_6(x)}{\partial x_4} = 0$$

$$\frac{\partial V_6(x)}{\partial x_5} = -ck_5x_5,\ \frac{\partial V_6(x)}{\partial x_6} = -ck_6x_6$$

Projection of system (27) speed vector on coordinate axis is presented as follows:

$$\left(\frac{dx_1}{dt}\right)_{x_1} = 0,$$

$$\left(\frac{dx_1}{dt}\right)_{x_2} = x_2, \ldots,\ \left(\frac{dx_1}{dt}\right)_{x_3} = 0, \ldots,\ \left(\frac{dx_1}{dt}\right)_{x_1} = 0$$

$$\left(\frac{dx_2}{dt}\right)_{x_1} = ak_1x_1,\ \left(\frac{dx_2}{dt}\right)_{x_2} = ak_2x_2,\ \left(\frac{dx_2}{dt}\right)_{x_3} = 0,$$
$$\left(\frac{dx_2}{dt}\right)_{x_4} = \frac{ac - ab}{bc}x_4x_6,$$

$$\left(\frac{dx_2}{dt}\right)_{x_5} = 0,\ \left(\frac{dx_2}{dt}\right)_{x_6} = 0,$$

$$\left(\frac{dx_3}{dt}\right)_{x_1} = 0,\ \left(\frac{dx_3}{dt}\right)_{x_2} = 0,\ \left(\frac{dx_3}{dt}\right)_{x_3} = 0,$$
$$\left(\frac{dx_3}{dt}\right)_{x_4} = x_4, \ldots,\ \left(\frac{dx_6}{dt}\right)_{x_6} = 0$$

$$\left(\frac{dx_4}{dt}\right)_{x_1} = 0,\ \left(\frac{dx_4}{dt}\right)_{x_2} = 0,\ \left(\frac{dx_4}{dt}\right)_{x_3} = bk_3x_3,$$
$$\left(\frac{dx_4}{dt}\right)_{x_4} = bk_4x_4$$

$$\left(\frac{dx_4}{dt}\right)_{x_5} = 0,\ \left(\frac{dx_4}{dt}\right)_{x_6} = \frac{ab - bc}{ac}x_2x_6,$$

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$$\left(\frac{dx_5}{dt}\right)_{x_1} = 0,\ldots, \left(\frac{dx_5}{dt}\right)_{x_5} = 0, \left(\frac{dx_5}{dt}\right)_{x_6} = -x_6$$

$$\left(\frac{dx_6}{dt}\right)_{x_1} = 0, \left(\frac{dx_6}{dt}\right)_{x_2} = \frac{bc-ac}{ab}x_2x_4, \left(\frac{dx_6}{dt}\right)_{x_3} = 0$$

$$\left(\frac{dx_6}{dt}\right)_{x_4} = 0, \left(\frac{dx_6}{dt}\right)_{x_5} = ck_5x_5, \left(\frac{dx_6}{dt}\right)_{x_6} = ck_6x_6$$

Full derivative of time from Lyapunov's vector function may be presented as follows:

$$\begin{aligned}\frac{dV(x)}{dt} &= \frac{\partial V(x)}{\partial x}\frac{dx}{dt} = -a^2k_1^2x_1^2 - a^2k_2^2x_2^2 \\ &- x_2^2 - \left(\frac{ac-ab}{bc}\right)^2 x_4^2x_6^2 - \\ &- b^2k_3^2x_3^2 - b^2k_4^2x_4^2 - x_4^2 - \\ &- \left(\frac{ab-bc}{ac}\right)^2 x_2^2x_6^2 - x_6^2 - c^2k_5^2x_5^2 - \\ &- \frac{bc-ac}{ab}x_2^2x_4^2 - c^2k_6^2x_6^2,\end{aligned} \tag{30}$$

Full derivative of time from vector function of Lyapunov in this construction is definitely negative function

By component of gradient vector, Lyapunov's function shall be displayed as

$$V_1(x) = -\frac{1}{2}x_2^2$$

$$V_2(x) = -\frac{1}{2}ak_1x_1^2 - \frac{1}{2}ak_2x_2^2 - \frac{ac-ab}{2bc}x_6x_4^2$$

$$V_3(x) = -\frac{1}{2}x_4^2$$

$$V_4(x) = -\frac{1}{2}bk_3x_3^2 - \frac{1}{2}bk_4x_4^2 - \frac{ab-bc}{2ac}x_2x_6^2$$

$$V_5(x) = -\frac{1}{2}x_6^2$$

$$V_6(x) = -\frac{bc-ac}{2ab}x_2^2x_4 - \frac{1}{2}ck_5x_5^2 - \frac{1}{2}ck_6x_6^2$$

in scalar form is:

$$\begin{aligned}V(x) &= -\frac{1}{2}ak_1x_1^2 - \frac{1}{2}(ak_2+1)x_2^2 - \\ &- \frac{ac-ab}{2bc}x_6x_4^2 - \frac{1}{2}bk_3x_3^2 - \\ &- \frac{1}{2}(bk_4+1)x_4^2 - \frac{bc-ac}{2ab}x_2x_6^2 - \\ &- \frac{ab-bc}{2ac}x_2^2x_4 - \frac{1}{2}ck_5x_5^2 - (ck_6+1)x_6^2\end{aligned} \tag{31}$$

Function (31) satisfied all conditions of Morse theorem from catastrophe theory, therefore function (31) may be presented by quadratic form [13].

$$\begin{aligned}V(x) &\approx -\frac{1}{2}ak_1x_1^2 - \frac{1}{2}(ak_2+1)x_2^2 - \\ &- \frac{1}{2}bk_3x_3^2 - \frac{1}{2}(bk_4+1)x_4^2 - \\ &- \frac{1}{2}ck_5x_5^2 - (ck_6+1)x_6^2\end{aligned} \tag{32}$$

Conditions of robust stability of stationary state of the system (27) will be gained with consideration of negative certainty of full derivative (31) from the above quadratic form as follows:

$$-ak_1 > 0,\ \ -ak_2 - 1 > 0,\ \ -bk_3 > 0,$$
$$-bk_4 - 1 > 0,\ -ck_5 > 0,\ \ -ck_6 - 1 > 0$$

or

$$\begin{aligned}&k_1 < 0,\ \ k_2 < -\frac{1}{a}, \\ &k_3 < 0,\ k_4 < -\frac{1}{b},\ k_5 < 0,\ \ k_6 < -\frac{1}{c}\end{aligned} \tag{33}$$

System (27) with the proportional control law (26) will be stable when the system parameters changing in area (33).

4. CONCLUSION

Research in recent years has shown that the method of Lyapunov functions can be successfully used to analyse the robust stability of linear and nonlinear control systems. Widespread application of this method is constrained by the lack of a general method for selecting or constructing Lyapunov functions and difficulties with their algorithmic representation. An inappropriate choice of a Lyapunov function or the inability to construct one does not indicate instability of the system, only that a proper Lyapunov function has not been found.

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An analysis of the robust stability of systems is provided by the new approach, which is derived from a geometric interpretation of the asymptotic stability theorem of Lyapunov. A Lyapunov function is constructed in the form of a vector, and the negative of the gradient is found using the components of the time derivative of the state vector (the right-hand side of the state equation). In this case, the time derivative of the Lyapunov function, which is given by the scalar product of the gradient vector and the time derivative of the state vector, is always a negative function. The region of robust stability of the closed-loop system is defined by the conditions for which the constructed Lyapunov function is positive.

The proposed approach to the construction of Lyapunov functions allows for an evaluation of the region of robust stability in the form of simple inequalities in the uncertain parameters of the controlled system. This study developed a method for analysing the robust stability of SISO and MIMO dynamical systems in canonical forms. The method ensures the stability of the system; i.e., the real parts of the eigenvalues of the closed-loop system are all negative. The efficiency and applicability of the proposed approach are evident.

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