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APPLICATION OF LINEAR PURE BIRTH-DEATH PROCESSES FOR NETWORK-CENTRIC INFORMATION SYSTEMS MODELING

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ABSTRACT

The solutions of Kolmogorov's equations for the probabilities of state for a discrete-time Markov process with a linear function of birth-death intensities were obtained. We also found expressions for the distribution laws of *k*th point arrival in birth-death processes and their basic numerical characteristics.

Keywords: *Markov Stochastic Processes; Network-Centric Information System; Pure Birth Process; Pure Death Process*

1. INTRODUCTION

We consider a discrete Markov stochastic process in continuous time. If the intensities of all death flows are equal to zero, then it is called a pure birth process (PBP). It permits only positive flux jumps at any given time [1-5]. If the intensities of all birth flows process are equal to zero, it is called a pure death process (PDP). It permits only negative flux jumps at any given time [1-5]. Currently, Markov PBP and PDP in various forms are widely used in the information systems modeling, using the queuing systems theory [1, 4 - 8]. The solution of Kolmogorov's equations for the probabilities of states for PBP and PDP are known only for some special cases, such as when the intensity of birth or death is a constant [1-5]. This significantly limits the scope of application use. However, to obtain solutions of Kolmogorov's equations for probabilities of states for Markov continuous time processes with a finite number of states, allows to approach the solution of a number of modeling network-centric information systems problems.

2. STATEMENT OF THE PROBLEM

The purpose of the paper is to obtain solutions of Kolmogorov's equations for probabilities of states for Markov continuous time processes with a finite number of states and a linear function of birth-death intensities, and find expressions for the distribution laws of the *k*th point arrival in birth-death processes and their basic numerical characteristics. We consider the expectation $m(t)$, dispersion $D(t)$ and the third central moment $M_3(t)$ or factor $k(t) = M_3(t)/D(t)$. as the main numerical characteristics of distribution laws.

2.1 Solution of the problem

2.1.1 Pure birth processes

Pure birth processes $\xi(t)$ satisfy the two conditions [3,5]:

1) the process $\xi(t)$ has a finite number of states $x = 0, 1, 2, \dots, N$;

2) states x change on $+1$ $(x \to x+1)$ with intensity $\lambda(x)$.

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The system of Kolmogorov's differential equations for unconditional probabilities of states has the following form [3]

$$\frac{dp(0,t)}{dt} = -\lambda(0)p(0,t), \quad x = 0; \tag{1}$$

$$\frac{dp(x,t)}{dt} = -\lambda(x)p(x,t) + \lambda(x-1)p(x-1,t); \quad 1 \le x \le N-1; \tag{2}$$

$$\frac{dp(N,t)}{dt} = \lambda(N-1)p(N-1,t); \quad x = N. \tag{3}$$

It should be noted that for any t (including $t = 0$) normalization condition should be met

$$\sum_{x=0}^{N} p(x,t) = 1, \tag{4}$$

and for $t = 0$ initial conditions $p(0,0) = 1$.

Solution of equation (1) with the initial condition is [4]

$$p(0,t) = \exp(-\lambda(0)t), \quad x = 0. \tag{5}$$

The solution of equations (2) can be found by variation of arbitrary constants in the form of the recurrence formula [9]

$$p(x,t) = \exp(-\lambda(x)t) \times \times \int_0^t \lambda(x-1)p(x-1,\tau)\exp(\lambda(x)\tau)d\tau; \quad 1 \le x \le N-1. \tag{6}$$

Taking into account the normalization condition (4) the solution of equation (3) is

$$p(N,t) = 1 - \sum_{x=0}^{N-1} p(x,t), \quad x = N. \tag{7}$$

The solution of (6) with (5) for PBP can also be represented explicitly. Thus three cases are generic:

2.1.2 PBP with intensity (Poisson PBP)

Thus if $\lambda(x) = \lambda$, then we obtain a Poisson distribution

$$p(x,t) = \frac{(\lambda t)^x}{x!}\exp(-\lambda t), \quad 0 \le x \le N-1. \tag{8}$$

The main numerical characteristics of the Poisson distribution (8) are defined by the expression

$$m(t) = D(t) = \lambda t, \quad k(t) = 1.$$

In paper [2] it is shown that the distribution of the arrival of kth point of birth is subordinate to the Erlang distribution

$$p_k(t) = \frac{\lambda}{(k-1)!}(\lambda t)^{k-1}\exp(-\lambda t), \quad 0 \le t < \infty. \tag{9}$$

with numerical characteristics

$$m = \frac{k}{\lambda}; \quad D = \frac{k}{\lambda^2}; \quad M_3 = \frac{2k}{\lambda^3}, \tag{10}$$

where $1 \le k \le N$.

2.1.3 PBP with intensity $\lambda(x) = \lambda(N-x)$ (binomial PBP)

If $\lambda(x) = \lambda(N-x)$, it follows from the (6) binomial distribution

$$p(x,t) = \frac{N!}{(N-x)!x!} \times \times (1-\exp(-\lambda t))^x \exp(-\lambda t)^{N-x}, \quad 0 \le x \le N. \tag{11}$$

For it, the numerical characteristics are

$$m(t) = N(1-\exp(-\lambda t));$$

$$D(t) = m(t)\exp(-\lambda t),$$

$$k(t) = 2\exp(-\lambda t) - 1,$$

$$0 < k(t) < 1.$$

The distribution laws for the arrival of kth point of birth can be determined by [1] Erlang distribution in a similar way

$$p_k(t) = \frac{N!\lambda}{(N-k)!(k-1)!} \times \times (1-\exp(-\lambda t))^{k-1}\exp(-\lambda t)^{N-k+1}, \quad 0 \le t < \infty. \tag{12}$$

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Its main numerical characteristics, taking into account [6] are

$$m = \sum_{i=1}^{k} \frac{\lambda^{-1}}{N+1-i};$$
$$D = \sum_{i=1}^{k} \frac{\lambda^{-2}}{(N+1-i)^2}; \quad (13)$$
$$M_3 = \sum_{i=1}^{k} \frac{\lambda^{-3}}{(N+1-i)^3}.$$

2.1.4 PBP intensity (negative binomial PBP)

If $\lambda(x) = \lambda(\alpha + x)$, then from (6) follows that the negative binomial distribution

$$p(x,t) = \frac{\Gamma(\alpha + x)}{\Gamma(\alpha)x!} \times$$
$$\times (1 - \exp(-\lambda\ t))^x \exp(-\alpha\ \lambda\ t), \quad (14)$$
$$0 \le x \le N-1,$$

where $\Gamma(z)$ is Gamma function.

For the probability distribution (14) the numerical characteristics are

$$m(t) = \alpha\,(\exp(\lambda t) - 1),$$
$$D(t) = m(t)\exp(\lambda t),$$
$$k(t) = 2\exp(\lambda t) - 1,$$
$$k(t) > 1.$$

The distribution laws for the arrival of *k*th point of birth is determined by the Erlang distribution (9)

$$p_k(t) = \frac{\Gamma(\alpha + N + 1)\lambda}{\Gamma(\alpha + N + 1 - k)\,(k-1)!} \times$$
$$\times (1 - \exp(-\lambda\ t))^{k-1} \exp(-\lambda\ t)^{N+\alpha-k+1}, \quad (15)$$
$$0 \le t < \infty.$$

Its main numerical characteristics, taking into account [10] are

$$m = \sum_{i=1}^{k} \frac{\lambda^{-1}}{N+1-i+\alpha};$$
$$D = \sum_{i=1}^{k} \frac{\lambda^{-2}}{[N+1-i+\alpha]^2}; \quad (16)$$
$$M_3 = \sum_{i=1}^{k} \frac{\lambda^{-3}}{[N+1-i+\alpha]^3}.$$

The analysis of the expressions for numerical characteristics of distributions (8), (11) and (14) shows that for the pure birth process $k(t) > 0$.

The density of the probability distribution of time intervals between consecutive events for considered PBP is exponential with parameter scale $\lambda(x)$.

2.2 Pure death process

Pure death processes $\xi(t)$ satisfy the two conditions [3,5]:

1) the process $\xi(t)$ has a finite number of states $x = 0, 1, 2, \dots, N;$;
2) states x change on $-1\ (x \to x-1)$ with intensity $\mu(x)$.

The system of Kolmogorov's differential equations for unconditional probabilities of states has the following form [3];

$$\frac{dp(0,t)}{dt} = \mu(1)\,p(1,t), \quad (17)$$
$$x = 0;$$

$$\frac{dp(x,t)}{dt} = -\mu(x)\,p(x,t) + \mu(x+1)\,p(x+1,t), \quad (18)$$
$$1 \le x \le N-1;$$

$$\frac{dp(N,t)}{dt} = -\mu(N)\,p(N,t), \quad (19)$$
$$x = N;$$

with the initial condition $p(N,0) = 1$.

The solution of equation (19) with the initial condition is

$$p(N,t) = \exp(-\mu(N)t), \quad (20)$$
$$x = N.$$

Equation (18) can be found by any variations in the form of a constant recurrence formula [9]

$$p(x,t) = \exp(-\mu(x)t) \times$$
$$\times \int_0^t \mu(x+1)\,p(x+1,\tau)\exp(\mu(x)\tau)\,d\tau, \quad (21)$$
$$1 \le x \le N-1.$$

Taking into account the normalization conditions similar to (4) for $t > 0$, the solution of equation (17) with $N < \infty$ is

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$$p(0,t) = 1 - \sum_{x=1}^{N} p(x,t), \quad (22)$$
$$x = 0.$$

The solution (21) with (20) for PDP can also be represented explicitly. Thus three cases are characteristic:

2.2.1 PDP with intensity (Poisson PDP)

So if $\mu(x) = \mu$, we get the Poisson distribution

$$p(x,t) = \frac{(\mu\, t)^{N-x}}{(N-x)!} \exp(-\mu\, t), \quad (23)$$
$$1 \le x \le N.$$

Its main numerical characteristics are

$$m(t) = N - \mu\, t;$$
$$D(t) = \mu\, t;$$
$$k(t) = -1.$$

In accordance with paper [2] it can be shown that the time of arrival of the kth point of death is subordinate to the Erlang distribution

$$p_k(t) = \frac{\mu}{(N-k)!} (\mu\, t)^{N-k} \exp(-\mu\, t), \quad (24)$$
$$0 \le t < \infty,$$

with numerical characteristic

$$m = \frac{N-k+1}{\mu},$$
$$D = \frac{N-k+1}{\mu^2}, \quad (25)$$
$$M_3 = \frac{N-k+1}{0{,}5\, \mu^3}.$$

2.2.2 PDP with intensity (binomial PDP)

If $\mu(x) = \mu x$, then from (21) follows a binomial distribution

$$p(x,t) = \frac{N!}{(N-x)!\, x!} \times$$
$$\times (1 - \exp(-\mu\, t))^{N-x} \exp(-x\mu t), \quad (26)$$
$$0 \le x \le N.$$

In this case, the numerical characteristics are defined by the relations

$$m(t) = N \exp(-\mu t);$$
$$D(t) = m(t)(1 - \exp(-\mu\, t)),$$
$$k(t) = 1 - 2\exp(-\mu\, t),$$
$$-1 < k(t) < 0.$$

Distribution laws for the arrival of kth point of death is defined similarly by the Erlang distribution (9)

$$p_k(t) = \frac{N!\,\mu}{(N-k)!(k-1)!} \times$$
$$\times (1 - \exp(-\mu\, t))^{N-k} \exp(-\mu\, t)^k, \quad (27)$$
$$0 \le t < \infty.$$

Its main numerical characteristics, taking into account [6] are

$$m = \sum_{i=0}^{N-k} \frac{\mu^{-1}}{k+i},$$
$$D = \sum_{i=0}^{N-k} \frac{\mu^{-2}}{(k+i)^2}, \quad (28)$$
$$M_3 = \sum_{i=0}^{N-k} \frac{2\mu^{-3}}{(k+i)^3}.$$

2.2.3 PDP with intensity (negative binomial PDP)

If the intensity of $\mu(x) = \mu(\alpha + N - x)$, then from (21) follows a negative binomial distribution

$$p(x,t) = \frac{\Gamma(\alpha + N - x)}{\Gamma(\alpha)\,(N-x)!} \times$$
$$\times (1 - \exp(-\mu\, t))^{N-x} \exp(-\alpha\, \mu t), \quad (29)$$
$$1 \le x \le N.$$

Its main numerical characteristics are

$$m(t) = N - \alpha(\exp(\mu t) - 1);$$
$$D(t) = \alpha \exp(\mu\, t)(\exp(\mu\, t) - 1),$$
$$k(t) = 1 - 2\exp(\mu t);\ k(t) < -1.$$

Distribution laws for the arrival of kth point of death is defined similarly by the Erlang distribution (9)

$$p_k(t) = \frac{\Gamma(\alpha + N - k + 1)\mu}{\Gamma(\alpha)\,(N-k)!} \times$$
$$\times (1 - \exp(-\mu\, t))^{N-k} \exp(-\alpha\, \mu\, t), \quad (30)$$
$$0 \le t < \infty.$$

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Its main numerical characteristics, taking into account [10] are

$$m = \sum_{i=0}^{N-k} \frac{\mu^{-1}}{i+\alpha},$$
$$D = \sum_{i=0}^{N-k} \frac{\mu^{-2}}{(i+\alpha)^2}, \qquad (31)$$
$$M_3 = \sum_{i=0}^{N-k} \frac{2\mu^{-3}}{(i+\alpha)^3}.$$

The analysis of the expressions for numerical characteristics of distributions (23), (26) and (29) shows that for pure death process $k(t) < 0$.

The density of the probability distribution of time intervals between consecutive events for considered PDP is exponential with parameter scale $\mu(N - x + 1)$.

3. CONCLUSIONS

Thus, solutions of Kolmogorov's equations for unconditional probabilities of states of discrete Markov processes in pure birth-death processes with final number of states were obtained. In this case, birth and death intensities depend on linear functions. It was shown that the distributions for the Markov pure birth-death processes can be distinguished by the value of the coefficient. We also obtained expressions for the distribution laws of the arrival of k th point of birth or death and their basic numerical characteristics.

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