ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

BOUNDARY VALUE PROBLEM FOR A B-HYPERBOLIC EQUATION WITH AN INTEGRAL CONDITION OF THE SECOND KIND

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ABSTRACT

In the paper we consider the boundary value problem with an integral condition of the second kind for a hyperbolic partial differential equation of the second order with the Bessel operator. We prove the uniqueness of the solution of the problem. In the paper we use the apparatus of the theory of partial differential equations and ordinary differential equations, methods of functional analysis, the apparatus of special functions. While solving the problem, we obtain some restrictive conditions on the functions that define the initial data of the problem. The solution of the problem is obtained explicitly, as a series, using the variable separation method. The substantiation of the solution is carried out by the method of spectral expansions. The work is theoretical.

Keywords: *Hyperbolic Equation, Nonlocal Integral Condition, Bessel Operator.*

1. INTRODUCTION

The modern problems of natural science have led to the necessity to generalize the classical problems of mathematical physics and to formulate qualitatively new problems, as well as to develop the methods for their studying. One class of qualitatively new problems consists of the problems with nonlocal conditions. The conditions are usually called nonlocal if they relate the values of the solution, to be found in a domain D, on some interior manifold in D to the values on the boundary of D. It has been found in the study that the greatest difficulty is connected with the case when nonlocal conditions do not involve the values of the function at the boundary points. This case includes nonlocal conditions given in the form of integrals. The problems with integral conditions arose in the study of certain physical processes, for which the boundaries of the occurrence regions may not be available for direct measurements, but the average values of the unknown quantities are known. The conditions of this kind may appear in the mathematical simulation of the phenomena related to plasma physics, heat distribution and demography. Nonlocal integral conditions can be regarded as a generalization of discrete nonlocal conditions.

Thus, a number of processes studied in physics, chemistry and biology often lead to the formulation of the so-called nonlocal problems for differential equations. Nonlocal problems are such problems, in which, together with the classical initial and boundary conditions or instead of them, the conditions are set which relate the values of the solution (and, possibly, its derivatives) at the points of interior and boundary manifolds. For example, nonlocal problems with integral conditions appear in the mathematical simulation of some processes of thermal conductivity, moisture transfer in capillary-porous media, the processes occurring in the turbulent plasma, in the study of the problems of mathematical biology, and also in studying some inverse problems of mathematical physics.

The problems with integral conditions for parabolic and elliptic equations have been considered by many authors, and for quite some time. The question of formulation and solvability of the problems for hyperbolic equations is studied much less. Systematic study of the problems with integral conditions for hyperbolic equations began in the 90s of the XX-th century. The study results showed that the choice of the method of proving the solvability of nonlocal problems with integral conditions is conditioned by the form of these conditions themselves. Over the last couple of decades, a large number of publications have appeared in the mathematical literature dedicated to studying nonlocal problems for partial differential equations of hyperbolic type. Two classes of problems are distinguished: the problems in which the integral condition is set along the characteristics, and the mixed problems with the classical initial conditions and the nonlocal

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

boundary conditions. Nonlocal problems of the second type are considered in the works by A. Bouziani [1], [2], Gordeziani and G.A. Avalishvili [3], L.S. Pulkina [4], [5], S.A. Beilin [8], [9], V.B. Dmitriev [6], O.M. Kechina [11], and other authors.

In the works [6], [7] the terms "conditions of the first and second kind" were introduced and the lemmas on the equivalence of the conditions of the first and second kinds was proved. If a nonlocal condition contains only an integral operator, then this condition is called an integral condition of the first kind. If a nonlocal condition, besides an integral operator, contains the values of the sought-for solution or its derivatives on the boundary of the domain under consideration, then such condition is called an integral condition of the second kind. This work is devoted to the study of a mixed problem for a hyperbolic equation with the Bessel operator with nonlocal integral condition of the second kind. In the paper by A.I. Kozhanov and L.S. Pulkina [12], there was proved the unique solvability of the boundary value problems with a nonlocal boundary condition of integral type for multidimensional hyperbolic equations, which was an important step forward in the study of such problems.

Our work is devoted to studying a problem with a nonlocal integral condition for a hyperbolic equation with the Bessel operator. The topicality of the work is connected with the need of studying certain problems of hydrodynamics and gas dynamics. Consider the following physical problem. As is known, the state of a gas is determined by three quantities: velocity, pressure and density. The equations describing small oscillations of the gas are derived from the general equations of hydrodynamics, namely, the continuity equation, Euler's equation and the equation of the relationship between pressure and density in the form of the Poisson adiabat. We will consider the oscillations of the gas in a cylindrical tube. Suppose that the gas is radially inhomogeneous, and there is a power-law dependence of the density on the radial coordinate. In addition, we use Boyle's law at constant temperature. Then in the cylindrical coordinate system, provided that the sought-for functions depend only on the spatial variable r and the time t, the considered hydrodynamic equations after their linearization will assume the form of an equation, in which the Bessel operator is applied to the spatial variable r. Thus, this physical problem can be interpreted as the problem of studying small oscillations of a gas in a cylindrical tube within a single cross-section, while considering the behavior of the gas constant within each radius. Therefore, it became necessary to solve the boundary value problem for the hyperbolic equation with the Bessel operator in a rectangular domain, which is considered in this paper. We prove the uniqueness of solution of this boundary value problem. The solution of the problem is obtained explicitly. The existence of solution is justified by the method of spectral expansions. The research results are new in the sense of formulation of the problem with a nonlocal integral condition and have practical applications in the field of gas dynamics.

Nonlocal problems for a hyperbolic equation with the Bessel operator with integral conditions of the first and second kind were investigated in the works [13], [14]. Nonlocal problems for a parabolic equation with the Bessel operator with integral conditions of the first and second kind were studied in the works [15]-[17].

The studies of nonlocal problems with integral conditions have showed that the classical methods are not always applicable to the solution of these problems. The presence of non-local conditions causes a number of difficulties, which do not allow using the standard methods for the study of nonlocal problems, and, therefore, the question of development of the research methods still remains topical today. The results of the present work are a continuation of the studies of the mixed problems with nonlocal integral conditions for hyperbolic equations.

2. FOURIER METHOD FOR SOLVING A MIXED PROBLEM WITH AN INTEGRAL CONDITION OF THE SECOND KIND

2.1. Formulation of the problem

Let $D=\left\{(x,t)\middle|\,0<x<1, 0<t<T\right\}$ be a rectangular domain in the coordinate plane Oxt; we denote the boundary of the domain as $\Gamma_0=\left\{(x,t)\middle|\,x=0, 0\le t\le T\right\}$.

In the domain D we consider a hyperbolic equation with the Bessel operator or a B-hyperbolic equation of the form

$$\frac{\partial^2 U}{\partial t^2}-B_x U=0, \qquad (1)$$

Journal of Theoretical and Applied Information Technology
31st August 2015. Vol.78. No.3

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

where $B_x = x^{-k}\dfrac{\partial}{\partial x}\left(x^k \dfrac{\partial}{\partial x}\right) = \dfrac{\partial^2}{\partial x^2} + \dfrac{k}{x}\dfrac{\partial}{\partial x}$ is the Bessel operator, $1 < k < 2$ is a given real number.

It is required to find a function $U(x,t)$ satisfying the following conditions:

$$U(x,t) \in C^2(D) \cap C^1(D \cup \Gamma_0) \cap C(\bar{D}),$$

(2)

$$\frac{\partial^2 U}{\partial t^2} - B_x U = 0, \quad (x,t) \in D,$$

(3)

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0, \qquad 0 < t < T,$$

(4)

$$U|_{t=0} = \text{¨}(x), \quad U_t|_{t=0} = \cdot(x), \quad 0 < x < 1,$$

(5)

$$\frac{\partial U(1,t)}{\partial x} + \int_0^1 U(x,t)\, x^k dx = 0, \qquad 0 \le t \le T,$$

(6)

where ¨ (x) and · (x) are given sufficiently smooth functions.

The nonlocal integral condition (6), in addition to the integral operator, contains the value of the derivative of the sought-for solution with respect to the spatial variable, and, therefore, is an integral condition of the second kind. Or, according to [6], [7], if the values of the sought-for solution or its derivatives are included in the relation, then such relations are called non-local conditions of the second kind.

Let us formulate the purpose of the work: to study the posed boundary value problem for the hyperbolic equation (1) with the Bessel operator with the nonlocal integral condition (6) of the second kind and to prove the unique solvability of the problem.

2.2. Uniqueness of the solution of the mixed problem

Theorem 2.2.1. The mixed problem (2)-(6) with the integral condition (6) cannot have more than one solution.

Proof. We prove the theorem using the method of reductio ad absurdum. Let U_1 and U_2 be two supposed solutions of the problem (2)-(6). Then their difference $\mu = U_1 - U_2$ satisfies the conditions (2)-(4) of the problem (2)-(6), the homogeneous initial conditions

$$\mu|_{t=0} = 0, \qquad \mu_t|_{t=0} = 0$$

(5_0)

and the homogeneous integral condition

$$\frac{\partial \mu(1,t)}{\partial x} + \int_0^1 \mu(x,t)\, x^k dx = 0.$$

(6_0)

It is not difficult to check that there holds the identity

$$x^k V_t\left(\frac{\partial^2 V}{\partial t^2} - B_x V\right) = \frac{1}{2}\frac{\partial}{\partial t}\left\{x^k\left[\left(\frac{\partial V}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]\right\} - \frac{\partial}{\partial x}\left(x^k \frac{\partial V}{\partial t}\frac{\partial V}{\partial x}\right)$$

Putting in this identity $V = \mu$ and taking into account that μ is a solution of equation (1), we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{x^k\left[\left(\frac{\partial \mu}{\partial t}\right)^2 + \left(\frac{\partial \mu}{\partial x}\right)^2\right]\right\} = \frac{\partial}{\partial x}\left(x^k \frac{\partial \mu}{\partial t}\frac{\partial \mu}{\partial x}\right)$$

Integrating the last identity with respect to x over the segment $[0,1]$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}E(t) = \mu_t(1,t)\,\mu_x(1,t),$$

(7)

where

$$E(t) = \int_0^1 \left[\left(\frac{\partial \mu}{\partial t}\right)^2 + \left(\frac{\partial \mu}{\partial x}\right)^2\right] x^k dx.$$

(8)

Journal of Theoretical and Applied Information Technology
31st August 2015. Vol.78. No.3

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

Multiplying equation (1) by x^k and integrating it with respect to x over the segment $[0,1]$, we get

$$\int_0^1 \mu_{tt} x^k dx = \frac{\partial \mu(1,t)}{\partial x}. \quad (9)$$

In the condition (6_0) we substitute $\mu_x(1,t)$ by its value from (9), as a result of which we have

$$\int_0^1 \mu_{tt} x^k dx + \int_0^1 \mu(x,t) x^k dx = 0. \quad (10)$$

Setting here $\int_0^1 \mu(x,t) x^k dx = Z(t)$, we obtain the equation $Z'' + Z = 0$, whose general solution has the form $Z(t) = c_1 \cos t + c_2 \sin t$, thus, $\int_0^1 \mu(x,t) x^k dx = c_1 \cos t + c_2 \sin t$.

By virtue of initial conditions (5_0), $c_1 = 0$ and $c_2 = 0$, and, hence, $\int_0^1 \mu(x,t) x^k dx = 0$. It follows from here and (6_0) that $\frac{\partial \mu(1,t)}{\partial x} = 0$. Now, from this and from (8) and (9) we get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 \left[\left(\frac{\partial \mu}{\partial t}\right)^2 + \left(\frac{\partial \mu}{\partial x}\right)^2\right] x^k dx = 0.$$

Whence we conclude that

$$\int_0^1 \left[\left(\frac{\partial \mu}{\partial t}\right)^2 + \left(\frac{\partial \mu}{\partial x}\right)^2\right] x^k dx = C = const. \quad (11)$$

Putting in (11) $t = 0$ and taking into consideration initial conditions (5_0), we get $C = 0$, and hence

$$\int_0^1 \left[\left(\frac{\partial \mu}{\partial t}\right)^2 + \left(\frac{\partial \mu}{\partial x}\right)^2\right] x^k dx = 0.$$

It follows that $\frac{\partial \mu}{\partial t} = 0$ and $\frac{\partial \mu}{\partial x} = 0$. Therefore, $\mu(x,t) = c$. It follows from this equality and initial conditions (5_0) that $c = 0$. Thus, we have obtained $\mu = 0$ and $U_1 \equiv U_2$. The theorem is proved.

2.3. The construction of particular solutions of equation (1) in a rectangular region by the Fourier method

First we construct a system of particular solutions of equation (1) satisfying the conditions

$$U(x,t) \in C^2(D) \cap C^1(D \cup \Gamma_0) \cap C(\bar{D}), \quad (12)$$

$$\frac{\partial^2 U}{\partial t^2} - B_x U = 0, \quad (x,t) \in D, \quad (13)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0, \quad 0 < t < T, \quad (14)$$

$$\frac{\partial U(1,t)}{\partial x} + \int_0^1 U(x,t) x^k dx = 0, \quad 0 \le t \le T. \quad (15)$$

We look for a particular solution of equation (1) of the form

$$U(x,t) = X(x) T(t), \quad (16)$$

where X and T are yet undetermined functions. We find them from the requirement that the function (16) satisfy conditions (12)-(15). To this end, we substitute it into equation (1) and boundary conditions (14) and (15).

$$XT'' - TB_x X = 0, \quad (17)$$

$$X'(0) T(t) = 0, \quad (18)$$

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$$\left(X'(1)+\int_0^1 X(x)x^k dx \right) T(t)=0 .$$

(19)

Separating variables in equation (17) and dividing equations (18) and (19) by T, we get the ordinary differential equations and conditions for the undetermined functions:

$$T''+ ª^2 T=0 ,$$

(20)

$$B_x X + ª^2 X = 0 ,$$

(21)

$$X'(0)=0 ,$$

(22)

$$X'(1)+\int_0^1 X(x)x^k dx=0 .$$

(23)

With respect to function X we have come to a Sturm—Liouville problem concerning eigenvalues and eigenfunctions. Let us find the general solution of equation (21), i.e. the equation $X''+\frac{k}{x}X'+ª^2 X=0$. Multiply this equation by x^2 and perform the variable change according to the formulas $X=\left(\frac{\P}{ª}\right)^{\frac{1-k}{2}} ³$, $x=\frac{\P}{ª}$. As a result, the equation is reduced to the Bessel equation, which appears in many questions of physics, mechanics and astronomy:

$$\P^2 \frac{d^2 ³}{d\P^2}+\P\frac{d³}{d\P}+\left[\P^2-\left(\frac{k-1}{2}\right)^2\right]³=0 .$$

(24)

Let $\frac{k-1}{2}$ be a non-integer number. Then it is known [20] that among the particular solutions of the Bessel equation there are functions which have the same name with the equation, the Bessel functions of the first kind, defined by the formulas

$$J_{\neg}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n+\neg}}{n!\Gamma(\neg+n+1)} ,$$

$$J_{-\neg}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{-\neg+2n}}{n!\Gamma(-\neg+n+1)} ,$$

where $\Gamma(\neg)$ is the Gamma-function defined for all positive values by the formula [20]: $\Gamma(\neg)=\int_0^{\infty}e^{-x}x^{\neg-1}dx$.

Since the expansions in the right-hand side of the formulas for the definition of the Bessel functions begin with different powers of x, then the partial solutions $J_{\frac{k-1}{2}}(\P), J_{\frac{1-k}{2}}(\P)$ of Bessel equation (24) will be linear independent, whereas the general solution has the form

$$³(\P)=c_1 J_{\frac{k-1}{2}}(\P)+c_2 J_{\frac{1-k}{2}}(\P) .$$

(25)

Coming back to the old variables in (25), we have

$$X(x)=c_1 x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(ª x)+c_2 x^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(ª x) ,$$

(26)

where $c_1, c_2, ª$ are arbitrary constants. We will find them from the requirements that the general solution (26) satisfy conditions (22) and (23). To this end, we substitute it into these conditions.

Due to the known formula of differentiation of the Bessel functions [20]:

$$\frac{dX}{dx}=-c_1 ª x^{\frac{1-k}{2}} J_{\frac{k+1}{2}}(ª x)+c_2 ª x^{\frac{1-k}{2}} J_{-\frac{k+1}{2}}(ª x) .$$

Also, by virtue of the known asymptotic formula for the Bessel function as $x \to 0$ [18],

Journal of Theoretical and Applied Information Technology
31st August 2015. Vol.78. No.3

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

we have for $c_2 \neq 0$: $\lim\limits_{x\to 0}\dfrac{dX}{dx}=\infty$, whereas for $c_2=0, c_1\neq 0$: $\left.\dfrac{dX}{dx}\right|_{x=0}=0$

Then, according to condition (22), for the general solution (26) to be bounded we must put $c_2=0$; as a result, we get $X=c_1x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)$. Here we also put $c_1=1$, since the Eigen functions are determined up to a constant factor. Thus, the solution of equation (21), satisfying condition (22), has the form

$$X=x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)$$

(27)

Substitute (27) into condition (23): $\dfrac{dX}{dx}=-\alpha x^{\frac{1-k}{2}}J_{\frac{k+1}{2}}(\alpha x)$, whence $\dfrac{dX(1)}{dx}=-\alpha J_{\frac{k+1}{2}}(\alpha)$.

As a result of substitution we have

$$-\alpha J_{\frac{k+1}{2}}(\alpha)+\int_0^1 J_{\frac{k-1}{2}}(\alpha x)x^{\frac{k+1}{2}}dx=0$$

(28)

It is known [18] that

$$\int J_p(x)x^{p+1}dx=x^{p+1}J_{p+1}(x)+c$$

(29)

Calculating the integral in (28) with the help of formulas (29) and the Newton—Leibniz formula, we obtain $\int_0^1 J_{\frac{k-1}{2}}(\alpha x)x^{\frac{k+1}{2}}dx=\dfrac{1}{\alpha}J_{\frac{k+1}{2}}(\alpha)$ From this and (28) we have: $-\alpha J_{\frac{k+1}{2}}(\alpha)+\dfrac{1}{\alpha}J_{\frac{k+1}{2}}(\alpha)=0$ or

$$J_{\frac{k+1}{2}}(\alpha)=0.$$

(30)

It is known [19] that the transcendental equation (30) has an infinite set of real roots. Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be positive roots of equation (30) placed in the order of increasing. Then the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ determine the eigenvalues of the spectral problem. Setting in (27) $\alpha=\alpha_n$, we obtain the corresponding Eigen functions of the Sturm—Liouville problem

$$X_n=x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha_n x), n=1,2,3,\ldots$$

(31)

Let us prove that the system of functions (31) is orthogonal on the interval [0,1] with the weight x^k. The function (27) is a solution of equation (21), i.e.

$$x^{-k}\frac{d}{dx}\left(x^k\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)\right)\right)+\alpha^2\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)\right)=0$$

(32)

Multiplying equation (32) by x^k, we get

$$\frac{d}{dx}\left(x^k\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)\right)\right)+\alpha^2x^k\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)\right)=0$$

or

$$\alpha^2x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}(\alpha x)=-\frac{d}{dx}\left(x^k\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha x)\right)\right)$$

(33)

Setting in this equality $\alpha=\alpha_1$ and $\alpha=\alpha_2$, we have

$$\alpha_1^2x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}(\alpha_1 x)=-\frac{d}{dx}\left(x^k\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha_1 x)\right)\right)$$

$$\alpha_2^2x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}(\alpha_2 x)=-\frac{d}{dx}\left(x^k\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha_2 x)\right)\right)$$

We multiply the first of these equalities by $x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha_2 x)$ and the second one, by $x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\alpha_1 x)$, then we subtract the first equality from the second. As a result, we get after some easy transformations

Journal of Theoretical and Applied Information Technology
31st August 2015. Vol.78. No.3

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$$\left(\alpha_2^2-\alpha_1^2\right)xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha_2 x\right)=$$

$$=\frac{d}{dx}\left[x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_1 x\right)\right)-x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}\left(\alpha_1 x\right)\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\right)\right]+$$

$$+x^k\left[\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_1 x\right)\right)\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\right)-\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\right)\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_1 x\right)\right)\right]$$

Since the second bracket turns to zero, the last equality assumes the form

$$\left(\alpha_2^2-\alpha_1^2\right)xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha_2 x\right)=$$

(34)

$$=\frac{d}{dx}\left[x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_1 x\right)\right)-x^{\frac{k+1}{2}}J_{\frac{k-1}{2}}\left(\alpha_1 x\right)\frac{d}{dx}\left(x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\right)\right]$$

Calculating the inner derivatives in equality (34), we get

$$\left(\alpha_2^2-\alpha_1^2\right)xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha_2 x\right)=\frac{d}{dx}\left[\alpha_2 xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k+1}{2}}\left(\alpha_2 x\right)-\alpha_1 xJ_{\frac{k+1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\right]$$

Integrating this equality with respect to x over the segment [0,1], we get

$$\left(\alpha_2^2-\alpha_1^2\right)\int_0^1 xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha_2 x\right)dx=$$

$$=\int_0^1\frac{d}{dx}\left[\alpha_2 xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k+1}{2}}\left(\alpha_2 x\right)-\alpha_1 xJ_{\frac{k+1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha_2 x\right)\right]dx=$$

$$=\alpha_2 J_{\frac{k-1}{2}}\left(\alpha_1\right)J_{\frac{k+1}{2}}\left(\alpha_2\right)-\alpha_1 J_{\frac{k+1}{2}}\left(\alpha_1\right)J_{\frac{k-1}{2}}\left(\alpha_2\right)=0$$

(35)

Thus, it is proved that the system of Bessel functions $\left\{J_{\frac{k-1}{2}}\left(\alpha_n x\right)\right\}$ is orthogonal with the weight x on the interval [0,1]. It follows from here that the system of functions (31) is orthogonal with the weight x^k on the interval [0,1].

In (35) we replace α_2 with α, $\alpha_1<\alpha<\alpha_2$

$$\left(\alpha^2-\alpha_1^2\right)\int_0^1 xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha x\right)dx=\alpha J_{\frac{k-1}{2}}\left(\alpha_1\right)J_{\frac{k+1}{2}}\left(\alpha\right)-\alpha_1 J_{\frac{k+1}{2}}\left(\alpha_1\right)J_{\frac{k-1}{2}}\left(\alpha\right)$$

Since $J_{\frac{k+1}{2}}\left(\alpha_1\right)=0$, the last equality assumes the form

$$\int_0^1 xJ_{\frac{k-1}{2}}\left(\alpha_1 x\right)J_{\frac{k-1}{2}}\left(\alpha x\right)dx=\frac{\alpha J_{\frac{k-1}{2}}\left(\alpha_1\right)J_{\frac{k+1}{2}}\left(\alpha\right)}{\alpha^2-\alpha_1^2}$$

(36)

Now let us take the limit as $\alpha\to\alpha_1$ in the equality (36). Since both the numerator and denominator converge to zero as $\alpha\to 0$, we evaluate the indeterminate form in the right-hand side of this equality by L'Hopital's rule: $\int_0^1 xJ_{\frac{k-1}{2}}^2\left(\alpha_1 x\right)dx=\frac{J_{\frac{k-1}{2}}^2\left(\alpha_1\right)}{2}$. Thus we have

$$\int_0^1 X_i(x)X_j(x)x^k dx=\int_0^1 x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_i x\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_j x\right)x^k dx=\int_0^1 J_{\frac{k-1}{2}}\left(\alpha_i x\right)J_{\frac{k-1}{2}}\left(\alpha_j x\right)xdx=$$

(37)

$$=\begin{cases}0,\ j\neq i\\ \frac{1}{2}J_{\frac{k-1}{2}}^2\left(\alpha_i\right),\ j=i\end{cases}$$

where α_i and α_j are positive roots of the equation $J_{\frac{k+1}{2}}\left(\alpha\right)=0$.

Let an arbitrary function $f\left(x\right)$ be representable in the form of the series

$$f\left(x\right)=\sum_{i=1}^{\infty}a_i x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_i x\right).$$

(38)

To determine the coefficients a_i, we multiply both sides of the expansion (38) by $x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(\alpha_j x\right)$ and integrate with the weight x^k over the segment [0,1], considering term-by-term integration possible. Then, taking into account (37), we get

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$$a_i = \frac{2}{J^2_{\frac{k-1}{2}}(\lambda_i)} \int_0^1 f(x) x^{\frac{k+1}{2}} J_{\frac{k-1}{2}}(\lambda_i x)\, dx = f_i$$

(39)

The expansion (38), whose coefficients are determined by formula (39), is the expansion of function f(x) into the Fourier—Bessel series.

Setting in equation (20) $\lambda^2 = \lambda_n^2$, we get $T_n'' + \lambda_n^2 T_n = 0.$ The general solution of this ordinary differential equation has the form

$$T_n = a_n \cos \lambda_n t + b_n \sin \lambda_n t, n = 1, 2, ...$$

Thus, the system of partial solutions of equation (1), satisfying conditions (12)-(15), is determined by the formula

$$U_n(x,t) = (a_n \cos \lambda_n t + b_n \sin \lambda_n t) x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x), n = 1, 2, ... \qquad (40)$$

3. RESULTS OF SOLVING THE MIXED PROBLEM WITH AN INTEGRAL CONDITION

To solve the problem (2)-(6), we apply the method of separation of variables, which was described above. Let us justify the existence of solution of the problem (2)-(6) using the method of spectral expansions.

We will look for a solution of the problem (2)-(6) in the form of the following series

$$U(x,t) = \sum_{n=1}^{\infty} (a_n \cos \lambda_n t + b_n \sin \lambda_n t) x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x)$$

(41)

where a_n and b_n are yet undetermined constants. We will find them from the requirements that the function $U(x,t)$, determined by the series (41), satisfy the initial conditions (5). To this end, we substitute this function into these initial conditions:

$$\sum_{n=1}^{\infty} a_n x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x) = \varphi(x)$$

(42)

$$\sum_{n=1}^{\infty} \lambda_n b_n x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x) = \psi(x)$$

(43)

The series (42) and (43) are the expansions of the functions $\varphi(x)$ and $\psi(x)$ into the Fourier—Bessel series. Due to (39), the coefficients of these expansions can be represented in the form

$$a_n = \frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)} \int_0^1 \varphi(x) x^{\frac{k+1}{2}} J_{\frac{k-1}{2}}(\lambda_n x)\, dx = \varphi_n,$$

(44)

$$b_n = \frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n} \int_0^1 \psi(x) x^{\frac{k+1}{2}} J_{\frac{k-1}{2}}(\lambda_n x)\, dx = \frac{\psi_n}{\lambda_n}.$$

(45)

Based on the differentiation formulas for cylinder functions, the integrals for φ_n and ψ_n can be represented in the form

$$\varphi_n = \frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n} \int_0^1 \varphi(x) \frac{d}{dx}\left(x^{\frac{k+1}{2}} J_{\frac{k+1}{2}}(\lambda_n x) \right) dx,$$

$$\psi_n = \frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n} \int_0^1 \psi(x) \frac{d}{dx}\left(x^{\frac{k+1}{2}} J_{\frac{k+1}{2}}(\lambda_n x) \right) dx$$

Let $\varphi, \psi \in C^1[0,1]$. Then, integrating these integrals by parts, we get

$$\varphi_n = -\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n} \int_0^1 \varphi'(x) x^{\frac{k+1}{2}} J_{\frac{k+1}{2}}(\lambda_n x)\, dx = -\frac{\varphi_n'}{\lambda_n}$$

(46)

$$\psi_n = -\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n} \int_0^1 \psi'(x) x^{\frac{k+1}{2}} J_{\frac{k+1}{2}}(\lambda_n x)\, dx = -\frac{\psi_n'}{\lambda_n}$$

(47)

Also, we represent the integrals for φ_n' and ψ_n' in the form

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$$\varphi'_n=\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\varphi'(x)}{x}\frac{d}{dx}\left(x^{\frac{k+3}{2}}J_{\frac{k+3}{2}}(\lambda_n x)\right)dx,$$

$$\psi'_n=\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\psi'(x)}{x}\frac{d}{dx}\left(x^{\frac{k+3}{2}}J_{\frac{k+3}{2}}(\lambda_n x)\right)dx.$$

Now, let $\varphi(x),\psi(x)\in C^2[0,1]$ and $\varphi'(1)=0,\psi'(1)=0$. Then, integrating by parts the last integrals, we get

$$\varphi'_n=-\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\varphi''(x)x-\varphi'(x)}{x^2}x^{\frac{k+3}{2}}J_{\frac{k+3}{2}}(\lambda_n x)dx=-\frac{\varphi''_n}{\lambda_n}$$

(48)

$$\psi'_n=-\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\psi''(x)x-\psi'(x)}{x^2}x^{\frac{k+3}{2}}J_{\frac{k+3}{2}}(\lambda_n x)dx=-\frac{\psi''_n}{\lambda_n}$$

(49)

Then we consider integrals for φ''_n and ψ''_n. These integrals will be also represented in the form

$$\varphi''_n=\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\varphi''(x)x-\varphi'(x)}{x^3}\frac{d}{dx}\left(x^{\frac{k+5}{2}}J_{\frac{k+5}{2}}(\lambda_n x)\right)dx,$$

$$\psi''_n=\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\psi''(x)x-\psi'(x)}{x^3}\frac{d}{dx}\left(x^{\frac{k+5}{2}}J_{\frac{k+5}{2}}(\lambda_n x)\right)dx.$$

Let $\varphi(x),\psi(x)\in C^3[0,1]$ and $\varphi(1)=0,\psi(1)=0,\varphi'(1)=0,\psi'(1)=0,\varphi''(1)=0,\psi''(1)=0$. Integrating the last integrals as above, we get

$$\varphi''_n=\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\varphi'''(x)x^2-3\varphi''(x)x+3\varphi'(x)}{x^4}x^{\frac{k+5}{2}}J_{\frac{k+5}{2}}(\lambda_n x)dx=\frac{\varphi'''_n}{\lambda_n}$$

(50)

$$\psi''_n=\frac{2}{J^2_{\frac{k-1}{2}}(\lambda_n)\lambda_n}\int_0^1\frac{\psi'''(x)x^2-3\psi''(x)x+3\psi'(x)}{x^4}x^{\frac{k+5}{2}}J_{\frac{k+5}{2}}(\lambda_n x)dx=\frac{\psi'''_n}{\lambda_n}$$

(51)

Substituting in (48) and (49) φ''_n and ψ''_n by their values from (50) and (51), we will have

$$\varphi'_n=-\frac{\varphi'''_n}{\lambda_n^2},\qquad \psi'_n=-\frac{\psi'''_n}{\lambda_n^2}.$$

(52)

Then, substituting in (46) and (47) φ'_n and ψ'_n by their values from (52), we will obtain

$$\varphi_n=\frac{\varphi'''_n}{\lambda_n^3},\qquad \psi_n=\frac{\psi'''_n}{\lambda_n^3}.$$

(53)

It follows from (44) and (45) that the coefficients of the series (41) are represented in the form

$$a_n=\frac{\varphi'''_n}{\lambda_n^3},\qquad b_n=\frac{\psi'''_n}{\lambda_n^4}.$$

(54)

Substituting the coefficients of the series (41) by their values from (54), we get

$$U(x,t)=\sum_{n=1}^{\infty}\left(\frac{\varphi'''_n}{\lambda_n^3}\cos\lambda_n t+\frac{\psi'''_n}{\lambda_n^4}\sin\lambda_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\lambda_n x)$$

(55)

Let us prove the uniform convergence of the series (55) in the domain $\bar{D}$. It is known (Watson, 1949) that for the Bessel function of the first kind there holds an asymptotic formula as $\mu\to\infty$

$$J_\nu(\mu)=O\left(\frac{1}{\mu^{1/2}}\right).$$

(56)

It is also known (Watson, 1949) that for $f(x)\in C[0,1]$ and $\mu\to\infty$

$$\int_0^1 f(x)xJ_\nu(\mu x)dx=O\left(\frac{1}{\mu^{3/2}}\right).$$

(57)

According to formulas (56) and (57), we will have for $n\to\infty$

$$\varphi'''_n=O\left(\frac{1}{\lambda_n^{1/2}}\right),\qquad \psi'''_n=O\left(\frac{1}{\lambda_n^{1/2}}\right).$$

(58)

It follows from the asymptotic formulas (56)-(58) that in the domain $\bar{D}$ for $n\to\infty$

$$\left(\frac{\varphi'''_n}{\lambda_n^3}\cos\lambda_n t+\frac{\psi'''_n}{\lambda_n^4}\sin\lambda_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(\lambda_n x)=O\left(\frac{1}{\lambda_n^{7/2}}\right).$$

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ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

This yields that for the terms of series (55) there holds in the domain $\bar{D}$ the following estimate

$$\left|\left(\frac{\ddot{\,}'''_n}{a_n^3}\cos a_n t+\frac{\cdot'''_n}{a_n^4}\sin a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(a_n x\right)\right|<\frac{c_1}{a_n^{7/2}}$$

By virtue of the Weierstrass criterion, the series (55) and (41) converge uniformly in $\bar{D}$ and, hence, $U\left(x,t\right)\in C\left(\bar{D}\right)$.

Let us prove now that $U\left(x,t\right)\in C^1\left(D\right)$. To this end, we differentiate series (55) with respect to t and x.

$$\frac{\partial U}{\partial t}=\sum_{n=1}^{\infty}\left(-\frac{\ddot{\,}'''_n}{a_n^2}\sin a_n t+\frac{\cdot'''_n}{a_n^3}\cos a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(a_n x\right)$$

(59)

$$\frac{\partial U}{\partial x}=-\sum_{n=1}^{\infty}\left(\frac{\ddot{\,}'''_n}{a_n^2}\cos a_n t+\frac{\cdot'''_n}{a_n^3}\sin a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k+1}{2}}\left(a_n x\right)$$

(60)

With the help of asymptotic formulas (56), (58) and analogously to what is proved above, it can be showed that for the terms of the series (59) and (60) there hold in the domain D the following estimates

$$\left|\left(-\frac{\ddot{\,}'''_n}{a_n^2}\sin a_n t+\frac{\cdot'''_n}{a_n^3}\cos a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(a_n x\right)\right|<\frac{c_2}{a_n^{5/2}}$$

,

$$\left|\left(\frac{\ddot{\,}'''_n}{a_n^2}\cos a_n t+\frac{\cdot'''_n}{a_n^3}\sin a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k+1}{2}}\left(a_n x\right)\right|<\frac{c_3}{a_n^{5/2}}$$

.

It follows from this that the series (59) and (60) converge uniformly in D and, hence, $U\left(x,t\right)\in C^1\left(D\right)$.

Let us now prove that $U\left(x,t\right)\in C^2\left(D\right)$. Differentiating twice the series (55) with respect to t and applying to it the Bessel operator $B_x=\frac{\partial^2}{\partial x^2}+\frac{k}{x}\frac{\partial}{\partial x}$, we get

$$\frac{\partial^2 U}{\partial t^2}=-\sum_{n=1}^{\infty}\left(\frac{\ddot{\,}'''_n}{a_n}\cos a_n t+\frac{\cdot'''_n}{a_n^2}\sin a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(a_n x\right)$$

(61)

$$B_x U=\sum_{n=1}^{\infty}\left(\frac{\ddot{\,}'''_n}{a_n}\cos a_n t+\frac{\cdot'''_n}{a_n^2}\sin a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(a_n x\right)$$

(62)

Also, with the help of asymptotic formulas (56)-(58), it can be proved that for the terms of series (55) in the domain D there hold the following estimate

$$\left|\left(\frac{\ddot{\,}'''_n}{a_n}\cos a_n t+\frac{\cdot'''_n}{a_n^2}\sin a_n t\right)x^{\frac{1-k}{2}}J_{\frac{k-1}{2}}\left(a_n x\right)\right|<\frac{c_4}{a_n^{3/2}}$$

It follows from this estimate that series (61) and (62) converge uniformly in the domain D and, hence, $U\left(x,t\right)\in C^2\left(D\right)$.

Thus, we have proved the following theorem, which is one of the main results of our study.

Theorem 3.1. If the functions $\ddot{\,},\cdot\in C^3\left[0,1\right]$, $\ddot{\,}\left(1\right)=0,\ddot{\,}'\left(1\right)=0,\ddot{\,}''\left(1\right)=0$ and $\cdot\left(1\right)=0,\cdot'\left(1\right)=0,\cdot''\left(1\right)=0$, then the problem (2)-(6) is uniquely solvable, and the solution is defined by the series (41).

4. CONCLUSION

As it was already mentioned in the introduction, the topicality of the subject of this work is justified by the need of theoretical generalization of the classical problems of mathematical physics. Also it is justified by the applied character of the work, which we will now discuss. The problem, considered in the paper, has the following physical meaning. Suppose that there is an "infinite" (extending in both directions to infinity) immovable cylindrical tube with the cross-sectional radius equal to R. Fill this tube with gas. Let the gas undergo small harmonic oscillations about its equilibrium position. In studying these small oscillations we restrict ourselves, for simplicity, to the case of radial oscillations. Then the function U, the velocity potential, will depend on the distance of the oscillating gas particle to the Oz axis and on time. After transition to cylindrical coordinates, the wave equation describing the motion of the gas in the tube will assume the form:

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ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{a^2}\frac{\partial^2 U}{\partial t^2}. \quad (63)$$

Let us formulate a problem about small oscillations of gas in the cylindrical tube: find a solution of equation (63), satisfying the initial conditions

$$U\big|_{t=0} = \ddot{\ }(r), \qquad U_t\big|_{t=0} = \cdot\ (r) \quad (64)$$

and the boundary condition

$$\left.\frac{\partial U}{\partial r}\right|_{r=R} = 0. \quad (65)$$

We will solve the posed problem by the Fourier method, according to which we will look for particular solutions of the form

$$U(r,t) = T(t)W(r). \quad (66)$$

Substitute (66) into equation (63) and conditions (64), (65). After separation of variables and the calculations, which were demonstrated above, we will obtain a solution of the problem (63)-(65) in the form of the series

$$U(r,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{\text{a}_n at}{R} + b_n \sin\frac{\text{a}_n at}{R}\right) J_0\left(\frac{\text{a}_n r}{R}\right),$$

whose coefficients are determined by the formulas:

$$a_n = \frac{2}{R^2 J_0^2(\text{a}_n)} \int_0^R r \ddot{\ }(r) J_0\left(\frac{\text{a}_n r}{R}\right) dr,$$

$$b_n = \frac{2}{\text{a}_n R a J_0^2(\text{a}_n)} \int_0^R r \cdot (r) J_0\left(\frac{\text{a}_n r}{R}\right) dr,$$

whereas $\text{a}_1, \text{a}_2, \text{a}_3, \ldots$ are positive roots of the transcendental equation $J_1(\text{a}) = 0$, placed in the increasing order.

Obviously, equation (1) is more general compared with equation (63). Equation (63) is obtained from (1) for the value $k=1$. The problem (63)-(65) is usually regarded as a mathematical model of oscillations of a real gas in a cylindrical tube. However, equation (63) is obtained after some mathematical simplifications. Thus, it makes sense to assume that equation (1) can describe more accurately the behavior of a real gas for some minor deviations from the value $k=1$. The problem (2)-(6) can be interpreted as a problem about oscillations of gas in an infinite tube, whose cross-section is a rectangular domain.

The solution of the problem has been obtained explicitly, in the form of a series. The results of further studies of the obtained solution will be displayed as a graphic dependence of the velocity potentials on time and the space variable. It will be possible to evaluate the behavior of gas for some deviations of the value of the parameter k from the value $k=1$. We hope that the planned results will be useful in the study of physical experiments of gas dynamics. In the future, we plan to consider the boundary value problem for the equation of small oscillations of a gas in the case when the considered functions depend not on one spatial variable, as was already mentioned in the introduction, but also on the second spatial variable in a cylindrical coordinate system. We will also consider other problems for the equation of small oscillations of a gas.

In solving the problem, we used the well-known method of separation of variables, which enabled us to investigate the solvability of the problem. The scientific novelty of this work is the following: setting of the problem in a rectangular region for a hyperbolic equation with the Bessel operator and an integral condition of the second kind; the proof of the unique solvability of the problem in a rectangle; in the course of solving the problem there have been identified some restrictive conditions on the functions included in the classical initial data of the problem. The results of this study, as was mentioned in the introduction, are continuation of the studies of the mixed problems with nonlocal integral conditions for hyperbolic partial differential equations.

5. ACKNOWLEDGMENTS

I express my sincere gratitude to my research advisor Nikolai B. Pleshchinsky, Doctor of Physical-Mathematical Sciences, Professor of Kazan Federal University, for his help, valuable criticism and support in carrying out this work.

I express my gratitude for the support in the scientific research to Yuri G. Ignatiev, Doctor of Physical-Mathematical Sciences, Professor of Kazan Federal University.

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

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