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THE FREQUENCY RESPONSE OF A DYNAMIC SYSTEM

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ABSTRACT

The purpose of the solutions, proposed in the article, is further enhancement of reliability of mathematical and program support of modern computer systems which require accurate calculations in ill-conditioned problems by step by step control of the priori error. For this purpose, a computational method for calculating the frequency response of the system, based on the reduction of the real Schur form of the initial matrix of coefficients of the system to a triangular form of the general form using orthogonal similarity transformation, has been developed. Next, the obtained matrix by permutations is transformed into a new triangular form, in the lower part of the main diagonal of which the multiple eigenvalues are placed. The value of the matrix frequency response for each predefined frequency value is found as the solution of a system of linear algebraic equations with a triangular coefficient matrix with an extracted diagonalizable submatrix for multiple eigenvalues. A computational algorithm for calculating the matrix frequency response of the system is presented. A comparative assessment of the computational cost in the proposed method and the method, based on the Hessenberg form, is conducted in the problem of calculating the frequency response of the system. The article shows an obvious advantage in the complexity of computations in the absence of multiple eigenvalues, which can be assessed as a linear function of the dimension of the matrix, over the quadratic dependence for an alternative method. In the absence of multiple eigenvalues the proposed method is not inferior in performance to the method based on the Hessenberg form.

Keywords: *Frequency Response, Schur Form, Hessenberg Form, Computational Methods, Linear System*

1. INTRODUCTION

Currently, the number of publications in the development of algorithmic support to compute the frequency response of dynamic systems is very large. However, if you consider the publications from the standpoint of their practical use the results are no longer so impressive. The reason is a visible gap between the purely academic approach to the development of methods of analysis, synthesis and modeling of dynamic systems, as sections of Mathematics, and the need for methods for solving specific applied problems.

Plotting the frequency response is used today to check the adequacy of the model of a dynamic system, where it is compared with the frequency response obtained in the experimental research of the real system [13], [14]. The stability analysis, determining the stability margins, the system bandwidth are conducted using the frequency response of an open-loop and closed-loop control systems [1], [5], [15].

In practice the frequency methods of adjusting – proportional-integral-derivative (PID) controllers, including the synthesis of robust PID controllers [1], [15], [19], the modern algorithms for optimal control [15], [3], [17], [19] and algorithms for predictive control [6], [8] are applied.

To date the method for computing the frequency response of the system, based on the reduction of the system matrix to the Hessenberg form using orthogonal similarity transformation constructed by House holder's method [10], [11], [12], is the most widespread. It is natural to assume that there are other approaches to the computation of the frequency response with less computational cost.

In this paper, we propose such an algorithm for computing the frequency response, based on the reduction of the real Schur form of a matrix system to a triangular form with an extracted diagonalizable block and finding a solution to the system of equations with a triangular matrix and a diagonal submatrix. We present a computational algorithm and conduct a comparative assessment of

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the computational cost in the proposed method and the method, based on the Hessenberg form. .

Usually a dynamic system is described by a system of linear differential equations of state with constant coefficients in the form:

$$\dot{x}(t) = Ax(t) + bu(t);$$
$$y(t) = Cx(t) + du(t), \quad (1)$$

where x is the state vector of dimension n, u is the input variable of the system, y is the output vector of dimension m, the components of which are the output variables of the system, relative to which we need to compute the frequency response, the state matrix $A \in R^{n\times n}$, matrix $C \in R^{m\times n}$, vectors $b \in R^n$ and $d \in R^m$.

Applying the Laplace transform with zero initial conditions to equations (1), we get:

$$sX(s) = AX(s) + bU(s);$$
$$Y(s) = CX(s) + dU(s). \quad (2)$$

Let's define the vector transfer function from the input to the state vector as $W(s) = X(s)/U(s)$, the vector transfer function from the input to the output vector as $W_y(s) = Y(s)/U(s)$ and using equations (2) we'll write:

$$(sI - A)W(s) = b;$$
$$W_y(s) = CW(s) + d. \quad (3)$$

To compute the vector frequency response $W(j\omega)$ and $W_y(j\omega)$ we'll go to the frequency domain at $s = j\omega$:

$$(j\omega I - A)W(j\omega) = b;$$
$$W_y(j\omega) = CW(j\omega) + d. \quad (4)$$

Since the computation $W_y(j\omega)$ is reduced to the computation $W(j\omega)$, the focus is on finding a solution to the first matrix equation of the system (4).

Search for the complex function $W(j\omega)$ is reduced to the computation of values of this function at the defined values of its real argument ω. The problem can be considered solved if we find an algorithm for finding a solution to the system of linear algebraic equations written in the form:

$$(j\omega I - A)w = b \ , \quad (5)$$

where A, b are taken from the system of equations (1), ω is a parameter, w is a complex vector of unknowns, which, as a result of the solution, will give the value of the frequency response of the system described by the model (1), at the defined frequency value ω.

It is obvious that the complexity of solving the system of linear algebraic equations (5) is determined by the coefficient matrix form, and the problem feature is the multiple solution of this system at variation of ω on the investigated interval. On the other hand, the matrix form depends on the chosen basis of the state space on which the model (1) is defined.

Then the goal is obvious – before we enter the cycle of change of parameter ω we need to choose a basis, in which the matrix A will be of such a form that at each step of the cycle the number of operations, required to solve the system of equations (5), should be minimal.

Let's proceed from the primitive basis to the new basis by the transformable nonsingular matrix $T \in C^{n\times n}$, that is, make the change of variables $q = Tx$ in the system of equations (1). Then in the new basis the model in the state variables will appear as

$$\dot{q}(t) = T^{-1}ATq(t) + T^{-1}bu(t);$$
$$y(t) = CTq(t) + du(t), \quad (6)$$

where the state matrix is similar to the state matrix of the initial system (1). There are several canonical forms which can be obtained for the matrix by similarity transformations: diagonal form, Jordan form, triangular form, Hessenberg form, real Schur form [18], [9], [4], [7], [16].

The diagonal form gives the minimum number of operations, but in the case of multiple eigenvalues it may not exist. With close eigenvalues, the problem

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of obtaining eigenvalues and eigenvectors may be ill-conditioned and the error of their computation may be exacerbated by the inability to solve the problem by unitary similarity transformation. The triangular canonical form always exists and can be obtained by unitary similarity transformation without impairing the conditionality of the problem, but at each step of the cycle the solution of the triangular system of linear equations will require more time than of the decoupled system of equations in the case of a diagonal coefficient matrix.

In this paper, we propose a hybrid version of the solution, based on the state matrix triangulation of the model in the state variables, but using, where possible, the diagonalization of the submatrix of the state matrix.

2. MATERIALS AND METHODS

It is known that the real Schur form A_S for an arbitrary real matrix can be obtained by an orthogonal similarity transformation performed by the practical QR algorithm with preliminary scaling of the initial matrix and reducing it to the Hessenberg form (Voevodin, 2009; Ikramov, 1991; Demmel, 1997; Golub, et al., 1998; Smith, et al., 1976). Therefore, in our paper we consider that the problem of computing the real Schur form A_S of the initial matrix A has been solved, that is, such an orthogonal matrix S has been defined, that the relation holds:

$$A_S = S^{-1} A S \qquad (7)$$

Let's determine a similarity transformation which will perform the reduction of the matrix A_S to a triangular form A_T with an extracted diagonalizable block.

2.1. The reduction of the real Schur form to a triangular form with an extracted diagonalizable block

At the first stage we'll reduce the matrix A_S to a strictly triangular, in the general case, a complex form, which we'll denote $\widetilde{A}_T$. Such transformation is performed by unitary similarity transformation:

$$A_T = N^{-1} \widetilde{A}_T N, \qquad (8)$$

where the matrix of multiple permutations N is formed as:

$$N = \prod_{ij} N_{ij}, \qquad (9)$$

where N_{ij} is the matrix of insert of i-th diagonal element to j-th place of the diagonal with the shift of all the diagonal elements from ($i+1$)-th place to j-th one, in the direction of i-th place. So, the matrix N_{ij} is formed as a product of elementary permutations of two adjacent diagonal elements:

$$N_{ij} = \widetilde{N}_{i,i+1} \cdot \widetilde{N}_{i+1,i+2} \cdot \dots \cdot \widetilde{N}_{j-1,j},$$

where the matrix of elementary permutations $\widetilde{N}_{l,l+1}$ is different from the identity one by the presence of the block of the second order on the diagonal, beginning from its l-th element:

$$n_i = \frac{1}{(a_{l,l+1}^2 + (a_{l+1,l+1} - a_{l,l})^2)^{1/2}} \begin{bmatrix} a_{l,l+1} & a_{l,l} - a_{l+1,l+1} \\ a_{l+1,l+1} - a_{l,l} & a_{l,l+1} \end{bmatrix},$$

constructed from the elements of the transformed matrix.

The choice of indices i, j is based on the task of grouping the multiple eigenvalues in the upper part of the diagonal and comparing the diagonal elements according to the chosen closeness criterion. It is important to note that the matrix of multiple permutations N is orthogonal.

Combining the transformations, set by (8) and (10) ones, we'll obtain a transformation which transforms the real Schur form A_S into a triangular form A_T:

$$A_T = N^{-1} \widetilde{A}_T N = N^{-1} H^{-1} A_S H N = (HN)^{-1} A_S HN.$$

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(101)

Taking into account the properties of the unitary and orthogonal matrices the expression (11) is replaced by the equivalent relation:

$$A_T = N^T H^H A_S HN. \tag{11}$$

Thus, the algorithm on the basis of the expression (12) is preferable to its counterpart – the relation (11) from a computational point of view.

Finally, considering the similarity transformation, performing the real triangularization of the initial matrix, to be known and taking into account the orthogonality of this transformation, we can write a general similarity transformation connecting an arbitrary matrix with a triangular form:

$$A_T = (N^T H^H S^T)A(SHN) \tag{12}$$

or

$$A = (SHN)A_T(N^T H^H S^T). \tag{13}$$

The result is a triangular matrix A_T, which has the form by construction:

$$\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix},$$

where the matrices A_1, A_2 are complex, in the general case, they are triangular, and the matrix A_2 has different eigenvalues, therefore, is diagonalizable, that is, there is a well-conditioned matrix of eigenvectors S_2 such that $A_2 = S_2 \Lambda S_2^{-1}$, where Λ is a diagonal matrix, on the diagonal of which the diagonal elements of the matrix A_2 (its eigenvalues) are placed.

Note that the computation of the matrix of eigenvectors S_2 for the triangular matrix A_2 can be performed by the known Gaussian back substitution, without bringing the additional method error in the computation.

2.2. The algorithm for solving a triangular system of linear equations with a diagonalizable submatrix

Previously, we have considered a similarity transformation which reduces the coefficient matrix of the system to a complex triangular form with an extracted block having different eigenvalues. Below is an algorithm for solving a system of linear algebraic equations when the coefficient matrix is triangular with an extracted lower triangular block having different eigenvalues:

$$\left(j\omega I_n - \begin{bmatrix} A_1 & A_{12} \\ 0_{n2\times n1} & A_2 \end{bmatrix} \right) w = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{14}$$

,

from where, presenting $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, we get:

$$\begin{bmatrix} j\omega I_{n1} - A_1 & -A_{12} \\ 0_{n2\times n1} & j\omega I_{n2} - A_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

.

(15)

Let's consider a subsystem, consisting of the last n2 equations of the system (16), in which the coefficient matrix is represented in the diagonal form:

$$(j\omega I_{n2} - S_2 \Lambda S_2^{-1}) w_2 = b_2,$$

from where, opening the brackets and multiplying both parts of this equation on the left-hand side by S_2^{-1}, we get:

$$j\omega(S_2^{-1} w_2) - \Lambda(S_2^{-1} w_2) = S_2^{-1} b_2.$$

Denoting $w_2^d = S_2^{-1} w_2$, $A_{12}^d = A_{12} S_2$ and $b^d = S_2^{-1} b_2$ in the second equation of the system (16) and performing the change of variables $w_2 = S_2 w_2^d$ in the first equation (16), we obtain an equivalent vector-matrix equation relative to the vector of unknowns $\begin{bmatrix} w_1 \\ w_2^d \end{bmatrix}$:

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$$\begin{bmatrix} j\omega I_{n1} - A_1 & -A_{12}^d \\ 0_{n2\times n1} & j\omega I_{n2} - \Lambda \end{bmatrix} \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix} = \begin{bmatrix} b_1 \\ b^d \end{bmatrix} \quad (16)$$

.

The elements of the vector of unknowns w_2^d are determined obviously:

$$w_{2\,i}^d = \frac{b_i^d}{j\omega - \lambda_{ii}}, \text{ at } i = 1,\dots,n_2 \quad (17)$$

The elements of the vector of unknowns w_1 are determined using the vector which has already been computed w_2^d by solving the system of equations:

$$(j\omega I_{n1} - A_1)w_1 = b_1 + A_{12}^d w_2^d.$$

It is obvious that the system is triangular, and the solution $\mathbf{w}_1$ is found by the Gaussian back substitution.
It is not too difficult to see that the solution of the initial matrix equation (15) is connected with the solution of the matrix equation (17) as follows:

$$w = \begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & S_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix} \quad (18)$$

2.3. The frequency response compensation

Let the model of a dynamic system be defined by the model in the state variables in the form (1). The frequency response of this system is determined by the relation (4).

Let the similarity transformation (14), transforming the matrix A into a triangular form A_T, be determined. Let's introduce the notation $Q = SHN$, $W_T(j\omega) = Q^{-1}W(j\omega)$ and $b_T = Q^{-1}b$. Then, using the relation $A = QA_TQ^{-1}$, from the equations (4) we get:

$$(j\omega I - A_T)W_T(j\omega) = b_T. \quad (19)$$

Comparing the relation (20) with the initial equations of frequency response (4), we find that $W_T(j\omega)$ is the frequency response in the system with a triangular state matrix. The frequency response of the system is compensated by formula:

$$W(j\omega) = QW_T(j\omega). \quad (20)$$

$W_T(j\omega)$ is the frequency response in the system with a triangular state matrix and is determined as the solution w of the matrix equation (15), on the right-hand side of which is $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_T$.
Above we have described the method for obtaining the solution of the equation (17) with parameter ω.
Then using the relations (19) from the formula (21), we obtain the formula to compute the value of the vector frequency response at the defined frequency value ω:

$$W(j\omega)\big|_{\omega} = Q\begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & S_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix}_{\omega} \quad (21)$$

.

The value of the vector frequency response of the system relative to the output variables is determined as:

$$W_y(j\omega)\big|_{\omega} = CQ\begin{bmatrix} I_{n_1} & 0_{n_1\times n_1} \\ 0_{n_2\times n_2} & S_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix} \quad (22)$$

2.4. The algorithm for computing the frequency response (a case of scalar input)

Let the state equations of a linear stationary model of the dynamic system be defined in the vector-matrix form (1) with matrices (A, b, C, d), and it is necessary to construct the system frequency response on the frequency interval from ω_0 to ω_f with a step Δ_ω.

The first stage is the similarity transformation which transforms the coefficient matrix of the system A into a triangular form A_T:

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$$A_T = (N^T H^H S^T) A(SHN) = Q^{-1} AQ = Q^H AQ \quad (23)$$

and the lower block A_2 of dimension n_2, for which the matrix of eigenvectors S_2 is computed, and the other two blocks A_1 and A_{12} are determined. Using the transformable matrices Q and S_2 the matrices are computed:

$$A_{12}^d = A_{12} S_2;$$

$$b_T \equiv \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = Q^H b \; ; b^d = S_2^{-1} b_2; \quad (24)$$

$$C_T \equiv [C_1 \quad C_2] = CQ \; ; C^d = C_2 S_2,$$

which will be used at each step of the cycle of frequency change.

Then we'll go into the cycle of frequency change and at each new frequency value we'll compute vector w_2^d, based on it we'll determine w_1. The vector frequency response relative to the output variables of the system is determined as:

$$W_y(j\omega)\Big|_{\text{at a defined step of the cycle}} = \begin{bmatrix} C_1 & C^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix} + d \quad (25)$$

This value is transferred to the results table for further processing and is (or) plotted.

The algorithm for computing the frequency response is presented below:

Algorithm 1.

The_computation_of_the_frequency_response (

$A, b, C, d,$ /* – matrices and vectors of the model coefficients in the state variables */

$\omega_0, \omega_f, \Delta_\omega$ /* – start, end frequency value and step of its changes */

):

$A_s = A$

Obtaining_the_real_Shur_form (A_s, S)

Obtaining_ a_ triangular-diagonal_form $(A_s, H, N, A_1, A_{12}, A_2, \lambda, S_2, S_2^{-1})$

$A_{12}^d = A_{12} S_2$

$b_T = N^T H^H S^T b \equiv \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ /*– in accordance with dimensions A_1, A_2 */

$b^d = S_2^{-1} b_2$

/*– block b_2 of dimension n_2 corresponds to the diagonalizable submatrix $A_2 = S_2 \Lambda S_2^{-1}$ */

$C_T = CSHN \equiv [C_1 \quad C_2]$ /* – in accordance with dimensions A_1, A_2 */

$C^d = C_2 S_2$

$\omega = \omega_0$

While $\omega \le \omega_f$ /* perform in the cycle */

Solve $(j\omega I - diag(\lambda)) w_2^d = b^d$ relative to w_2^d

/* – solve a decoupled system of linear algebraic equations *

Solve $(j\omega I_{n1} - A_1) w_1 = b_1 + A_{12}^d w_2^d$ relative to w_2^d

/* – solve a triangular system of linear algebraic equations *

$$W_y = \begin{bmatrix} C_1 & C_2^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix} + d$$

/* – the value of the vector frequency response at a defined frequency value *

The_procedure_for_using_the value_of_frequency_response (ω, W_y)

$\omega = \omega + \Delta_\omega$

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End While

End The_computation_of_the_frequency_response

2.5. The algorithm for computing the matrix frequency response

Previously we have considered the algorithm for computing the frequency response of a linear stationary system with scalar input and output vector. Let's generalize the obtained results for the case of a dynamic system with input and output vectors:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t);\\ y(t) &= Cx(t) + Du(t),\end{aligned} \qquad (26)$$

where x is the state vector of dimension n, u is the input vector of dimension k, y is the output vector of dimension m and matrices A, B, C, D have consistent dimensions: $A \in R^{n \times n}$, $B \in R^{n \times k}$, $C \in R^{m \times n}$, $D \in R^{m \times k}$.

The matrix transfer function of a linear stationary system (27) is determined as the matrix $W_y(s)$ of dimension $m \times k$, (i,j)-th element of which $W_{y_{i,j}}(s) = Y_i(s)/U_j(s)$ is the transfer function from the j-th input to the i-th output. Similarly, we'll determine the matrix transfer function $W(s)$, (i,j)-th element of which is $W_{i,j}(s) = X_i(s)/U_j(s)$.

The computation of the matrix frequency response of the system is reduced to the computation of the vector frequency response for each input variable of the system (27) taking into account the fact that for all k components of the input variable the same triangular form of the coefficient matrix of the system and computation of it are used, as well as it is not necessary to determine eigenvectors of its diagonalizable submatrix k times.

Knowing the appropriate similarity transformation, from the system (27) by the change of variables we can go to the model in the state variables which is defined by four matrices A_T, B_T, C_T, D, where the matrices

$$A_T \equiv \begin{bmatrix} A_1 & A_{12} \\ 0_{n2 \times n1} & A_2 \end{bmatrix}, \quad B_T \equiv \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad (27)$$

$$C_T \equiv \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

are divided into blocks in accordance with the extraction of diagonalizable submatrix A_2. The value of the frequency response of such a system is determined as the solution of a matrix equation with parameter ω and unknown block matrix $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \in R^{n \times k}$:

$$\begin{bmatrix} j\omega I_{n1} - A_1 & -A_{12} \\ 0_{n2 \times n1} & j\omega I_{n2} - A_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad (28)$$

Using the spectral decomposition $A_2 = S_2 \Lambda S_2^{-1}$ by the change of variables $W_2 = S_2 W_2^d$ the matrix equation (29) is transformed into the form:

$$\begin{bmatrix} j\omega I_{n1} - A_1 & -A_{12}^d \\ 0_{n2 \times n1} & j\omega I_{n2} - \Lambda \end{bmatrix} \begin{bmatrix} W_1 \\ W_2^d \end{bmatrix} = \begin{bmatrix} B_1 \\ B^d \end{bmatrix} \qquad (29)$$

where $A_{12}^d = A_{12} S_2$, $B^d = S_2^{-1} B_2$ and $C^d = C_2 S_2$. The solution of the matrix equation (30) can be found by solving a system of linear algebraic equations of the form (17) with the matrix column $\begin{bmatrix} B_1 \\ B^d \end{bmatrix}$ on the right side k times.

The matrix frequency response relative to the output variables of the system at a defined frequency value is determined as:

$$W_y(j\omega)\Big|_{\omega} = \begin{bmatrix} C_1 & C^d \end{bmatrix} \begin{bmatrix} W_1 \\ W_2^d \end{bmatrix}_{\omega} + D \qquad (30)$$

The algorithm for computing the matrix frequency response of the system is presented below:

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Algorithm 2.

The_computation_of_the_matrix_frequency_response_of_the_system (

$A, B, C, D,$ /* – the coefficient matrices of the model in the state variables */

$\omega_0, \omega_f, \Delta_\omega$ /* – start, end frequency value and step of its changes */

):

$A_s = A$

Obtaining_the_real_Shur_form (A_s, S)

Obtaining_ a_ triangular-diagonal_form $(A_s, H, N, A_1, A_{12}, A_2, S_2, S_2^{-1}, \lambda)$

$A_{12}^d = A_{12} S_2$

$B_T = N^T H^H S^T B \equiv \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

/*– block B_2 of dimension $n_2 \times k$ corresponds to the diagonalizable submatrix A_2 */

$B^d = S_2^{-1} B_2$

$C_T = CSHN \equiv \begin{bmatrix} C_1 & C_2 \end{bmatrix}$

$C^d = C_2 S_2$

$\omega = \omega_0$

While $\omega \leq \omega_f$ /* perform in the cycle */

For $i = 1$ to k

$b \equiv \begin{bmatrix} b_1 \\ b^d \end{bmatrix} = \begin{bmatrix} B_1 \\ B^d \end{bmatrix}^{(i)}$ /* – i -th column */

$d \equiv D^{(i)}$ /* – i -th column */

Solve $(j\omega I - diag(\lambda)) w_2^d = b^d$ relative to w_2^d

/*– solve a decoupled system of linear algebraic equations */

Solve $(j\omega I_{n1} - A_1) w_1 = b_1 + A_{12}^d w_2^d$ relative to w_1

/*– solve a triangular system of linear algebraic equations */

$W_y^{(i)} = \begin{bmatrix} C_1 & C^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2^d \end{bmatrix} + d$

/*– i -th column of the matrix frequency response of the system */

End For

/* the value of the matrix frequency response of the system W_y has been computed

At defined frequency value ω */

The_procedure_for_using_the value_of_frequency_response (ω, W_y)

$\omega = \omega + \Delta_\omega$

End While

End The_computation_of_the_frequency_response_of_the_system

It is interesting to compare the method, proposed in this article, with the method for computing the frequency response described in the works [10], [11], the principle of which is that the frequency response is computed in the system defined by the model in the state variables with the state matrix in the Hessenberg form.

3. RESULTS

Accurate to the notation, as well as in this work, in the work (Laub, 1981) the problem of computing the matrix frequency response of a linear stationary dynamic system, defined by the model in the state variables, is considered as the problem of computing the complex expression of real matrices

$$P(j\omega I - A)^{-1} Q, \tag{31}$$

for a large number of values of the real parameter ω.

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In this case it is assumed that $Q \in R^{n \times k}$, $P \in R^{m \times n}$, $A \in R^{n \times n}$, at that $n \geq m$ and $n \geq r$, and the number N, defining how many times the expression (32) is to be computed at different values of the parameter ω, is large, so that $N >> n$.

To solve this problem the work [10] proposes a method, based on the preliminary transformation of the basis of the state space in such a basis, in which the state matrix A of the system, being investigated, takes the real Hessenberg form (almost an upper-triangular matrix, different from it by the elements $a_{i+1,i}$, which are non-zero in the general case). This uses the orthogonal similarity transformation, constructed according to Householder's method.

Since the Hessenberg form is form-invariant to the operations of addition with a diagonal matrix, to compute the expression (32), at each value ω we need to solve the system of linear algebraic equations, in which the coefficient matrix has the upper Hessenberg form.

It is obvious that we'll require less computational resources for solving a system of linear algebraic equations in the case of the matrix in the Hessenberg form than in the case of an arbitrary matrix. This conclusion about the effectiveness of this algorithm is confirmed by the quantitative assessments of the computational cost for these cases.

Next, under the assessment of computational cost we'll understand the number of long complex arithmetic operations (multiplications and divisions) required for conducting the assessed computations.

Let the model of a linear dynamic system be defined in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{32}$$

where $A \in R^{n \times n}$, $B \in R^{n \times k}$, $C \in R^{m \times n}$, $D \in R^{m \times m}$.

The whole way of computing the frequency response can be assessed by formula:

$$N_0 = N_p + N(N_b + N_u). \tag{33}$$

Here N_p is the assessment of cost of preliminary computations required to obtain the system model in the chosen basis. These computations are performed once. N_b is the assessment of cost of basic computations required for obtaining the matrix frequency response in the assumption that the output vector is the state vector of the system in the chosen basis. These computations are performed for each predefined frequency value (N times, and $N >> n$) and are significantly different for various bases. N_u is the assessment of cost of computations required to compensate the desired matrix frequency response in the initial basis. These computations are also performed for each predefined frequency value, but in contrast to N_b, the assessment N_u can be considered independent of the chosen basis and equal to about mnk. Thus, the comparative analysis of the presented methods for computing the frequency response of the system is reduced to obtaining the value N_b exactly, for each of the methods. .

The assessment N_b determines the complexity of computing the matrix expression:

$$(j\omega I - A_b)^{-1} B_b, \tag{34}$$

where $A_b = T^{-1}AT$ is the matrix obtained by the similarity transformation with the transformable matrix T from the initial state matrix A, $B_b = T^{-1}B$. It is obvious that the value of the matrix expression satisfies the linear matrix equation with the unknown variable $Z \in C^{n \times k}$:

$$(j\omega I - A_b)Z = B_b, \tag{35}$$

and can be determined by solving k systems of linear algebraic equations of n-th order.

A case of state matrix of the general form

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Let's consider computing the matrix expression (35) for a case of arbitrary dense matrix of the general form $A_b = A$.

Computing the expression (35) is performed in several stages.

First LU -decomposition of the matrix $(j\omega I - A_b)$ is determined:

$$(j\omega I - A_b) = LU, \tag{36}$$

where $L \in C^{n\times n}$ is a lower triangular matrix with a unit diagonal, $U \in C^{n\times n}$ is un upper triangular matrix. It is known that LU -decomposition of an arbitrary matrix with complex elements will require approximately $\frac{1}{3}n^3$ long complex arithmetic operations.

We'll search for the value of the matrix expression (35) as the solution of the matrix equation (36) which, taking into account (38), can be rewritten as:

$$U^{-1}L^{-1}B_b = Z. \tag{37}$$

Let's introduce an unknown matrix $Z_1 \in C^{n\times k}$ such that

$$Z_1 = L^{-1}B_b, \tag{38}$$

and determine it as the solution of the matrix equation

$$LZ_1 = B_b. \tag{39}$$

To determine $Z_1 \in C^{n\times k}$ it is necessary to solve a system of linear algebraic equations with lower-triangular complex matrix $L \in C^{n\times n}$ with a unit diagonal k times. This will require $\frac{1}{2}kn(n-1)$ long complex arithmetic operations.

Assuming that the matrix Z_1 is known, the equation (38) can be rewritten as the matrix equation relative to the unknown $Z \in C^{n\times k}$:

$$UZ = Z_1. \tag{40}$$

In order to find $Z \in C^{n\times k}$, satisfying this equation, it is necessary to solve a system of linear algebraic equations with upper-triangular complex matrix $U \in C^{n\times n}$ with a unit diagonal k times. This will require $\frac{1}{2}kn(n+1)$ long complex arithmetic operations.

In the end, the computational cost N_b, required to compute the expression (35) in the case of arbitrary dense matrix $A_b = A$, denoted as N_a in this case, can be assessed as:

$$N_a = \frac{1}{3}n^3 + kn^2. \tag{41}$$

A case of the matrix in the Hessenberg form

Let's consider computing the matrix expression (35) for the case of the matrix in the Hessenberg form: $A_b = A_h$. It is obvious that the matrix $(j\omega I - A_h)$ will also be in the Hessenberg form.

It is known that as a result of LU -decomposition of the Hessenberg matrix the two-diagonal lower triangular matrix L with a unit diagonal is obtained. Then the solution of the vector-matrix equation (40) will require kn operations.

The solution of the vector-matrix equation (41) in the present case, as in the general case, will require $\frac{1}{2}kn(n+1)$ operations. In the end, the computational cost N_h, required to compute the matrix expression $(j\omega I - A_h)^{-1}B_h$ (see the analogous expression (35) for the general case) in the case of the matrix in the Hessenberg form A_h, can be assessed as:

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$$N_h = N_{LU(H)} + kn + \frac{1}{2}kn(n+1) \quad (42)$$

We can assess the number of operations, required to compute the matrix expression (35), if we apply the Gaussian elimination for solving the system of linear algebraic equations relative to the matrix equation:

$$HZ = B_h, \quad (43)$$

where H is the Hessenberg matrix (almost triangular).

To perform a forward path of the method of Gauss on the Hessenberg matrix with simultaneous carrying out appropriate computations on the right side of the matrix equation $\frac{1}{2}(n-1)(n+2)+k(n-1)$ long complex arithmetic operations will be required.

To perform the Gaussian back substitution with arbitrary diagonal elements $\frac{1}{2}kn(n+1)$ long complex arithmetic operations will be required.

In the end, the computational cost N_h, required to compute the matrix expression $(j\omega I - A_h)^{-1}B_h$ using the Gaussian elimination and taking into account the Hessenberg form of the matrix A_h, can be assessed as:

$$N_h = \frac{1}{2}(n^2+n-2)(k+1) \quad (44)$$

The right side of this expression can be reduced to the form $\frac{1}{2}n(n+1)(k+1)-(k+1)$ and discarding the subtrahend $(k+1)$ we can obtain the upper bound:

$$N_h < \frac{1}{2}n(n+1)(k+1) \quad (45)$$

From (45) we can obtain the lower bound:

$$N_h > \frac{1}{2}n^2(k+1) \quad (46)$$

This lower bound is presented in the work (Ikramov, 1991).

Now we can give an interval assessment of cost in the form of:

$$\frac{1}{2}n^2(k+1) < N_h < \frac{1}{2}n(n+1)(k+1) \quad (47)$$

A case of upper-triangular complex matrix

Let's consider computing the matrix expression (35) for the case of upper-triangular complex matrix (matrix in the Schur form): $A_b = A_t$. It is obvious that the matrix $(j\omega I - A_t)$ will be also upper-triangular. Then the solution of the matrix equation (36) at the defined parameter ω and $A_b = A_t$ and $B_b = B_t$ relative to the unknown matrix $Z \in C^{n\times k}$ will require $\frac{1}{2}kn(n+1)$ long complex arithmetic operations. Thus, for the case of the choice of a basis of the state space such that the state matrix is represented by an upper-triangular complex matrix, the assessment of computational cost of type N_b, required to compute the matrix expression (35), may be taken in the form:

$$N_t = \frac{1}{2}n(n+1)k \quad (48)$$

A case of diagonal complex matrix

Let a diagonal matrix, which is similar to the initial state matrix of the system, defined by the model in the state variables, exist. Setting a problem of computing the frequency response of such a system we'll consider computation of the matrix expression (35) for the case of diagonal complex matrix: $A_b = A_d$. It is obvious that the matrix $(j\omega I - A_d)$ will be also diagonal. Then the solution of the matrix equation (36) at the defined parameter ω and $A_b = A_d$ and $B_b = B_d$ relative to

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the unknown matrix $Z \in C^{n\times k}$ will require kn long complex arithmetic operations. Thus, for the case when the choice of a basis of the state space, such that the state matrix is represented by a diagonal complex matrix, is possible, the assessment of computational cost of type N_b, required to compute the matrix expression (35), may be taken in the form:

$$N_d = nk \quad (49)$$

A case of complex triangular matrix with a diagonalizable block

And, finally, we'll consider the proposed hybrid case. The solution of the matrix equation

$$(j\omega I - A_b)Z = B_b, \quad (50)$$

for this method is replaced by the solution of the matrix equation:

$$\begin{bmatrix} A_t & A_{12} \\ 0_{n2\times n1} & A_d \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \quad (51)$$

where $n = n_1 + n_2$, $A_t \in C^{n_1\times n_1}$ is an upper-triangular matrix, $A_d \in C^{n_2\times n_2}$ is a diagonal matrix, $A_{12} \in C^{n_1\times n_2}$, $B_1 \in C^{n_1\times k}$, $B_2 \in C^{n_2\times k}$. To solve the matrix equation (52) we need to solve a system of linear algebraic equations, the coefficient matrix of which has a triangular form with a diagonal lower block, k times.

$$\begin{bmatrix} A_t & A_{12} \\ 0_{n2\times n1} & A_d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \quad (52)$$

In order to determine n_2 unknown elements of the vector z_2 we'll need to perform n_2 long complex arithmetic operations. In order to determine n_1 unknown elements of the vector z_1, first we'll need to perform $n_1 n_2$ multiplication complex arithmetic operations to obtain a triangular system n_1 of linear algebraic equations, and then the solution of it will require $\frac{1}{2}n_1(n_1+1)$ long complex arithmetic operations. In total, the computational cost of solving the system of equations (53) can be assessed as $\frac{1}{2}(n_1+1)(n+n_2)$.

Thus, for the case when the basis of the state space has been chosen such that the complex state matrix is represented by a triangular matrix with ($n_2 \times n_2$) diagonalizable block, the assessment of the computational cost of type N_b, required to compute the matrix expression (35), may be taken in the form:

$$N_{t/d} = \frac{1}{2}kn(n+1) - \frac{1}{2}kn_2(n_2-1). \quad (53)$$

This shows that this method offers advantage in the number of operations over the use of a triangular matrix and it is assessed as $N_t - N_{t/d} = \frac{1}{2}kn_2(n_2-1)$ long complex arithmetic operations. Also it can be seen that in the absence of multiple eigenvalues the equation (54) gives nk long complex arithmetic operations.

The proposed method, in the absence of multiple eigenvalues, runs as fast as the method based on the diagonal form. The error, in the case of multiple eigenvalues, does not exceed the error of methods based on the use of triangular canonical form, because the matrix S_2 is computed from the triangular matrix A_2 using the Gaussian back substitution, which does not have methodical errors, and is well-conditioned by construction of the matrix A_2.

The obtained assessments of the computational cost, being compared, are presented in Table 1.

Table 1.The Assessment Of The Computational Cost Of The Methods For Calculating The Value Of The Matrix Frequency Response $W(j\omega) \in C^{n\times k}$

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Form of the state matrix	Designation of assessment (type N_b)	The assessment of the number of long complex arithmetic operations
Arbitrary dense matrix $A \in R^{n \times n}$	N_a	$\frac{1}{3}n^3 + kn^2$
Hessenberg form $A_h \in R^{n \times n}$	N_h	$\frac{1}{2}(n^2+n-2)(k+1)$ or $\frac{1}{2}n^2(k+1) < N_h < \frac{1}{2}n(n+1)(k+1)$
Upper-triangular form $A_t \in C^{n \times n}$	N_t	$\frac{1}{2}kn(n+1)$
Diagonal form $A_d \in C^{n \times n}$	N_d	nk
Triangular matrix $A_{t/d} \in C^{n \times n}$ with a diagonal block $A_d \in C^{n_2 \times n_2}$	$N_{t/d}$	$\frac{1}{2}kn(n+1) - \frac{1}{2}kn_2(n_2-1)$

4. DISCUSSION

Using the Hessenberg form solves the problem of computing the frequency response [10], [11], [16] even in the case when the condition numbers of eigenvalues are large. The Hessenberg form can be obtained from any initial matrix using orthogonal similarity transformation, implemented, for example, by direct Householder's_method [9], [4], [7]. In the method discussed, the eigenvalue problem is not set. For solving a system of linear equations the coefficient matrix in the Hessenberg form is preferable to an arbitrary matrix, and orthogonal similarity transformations guarantee high accuracy of both obtaining form and compensating the frequency response in the initial basis.

In the method, proposed in the article, the eigenvalue problem is solved in the general case for the matrix block. Possible ill-conditioning of the eigenvalue problem is taken into account and is avoided. Firstly, this is achieved by using the Schur form, for which the assertion is proved that it exists for any matrix [9], [4], [7], and it can be obtained by an orthogonal similarity transformation that does not degrade the conditioning of the initial problem. Secondly, from the Schur form the upper triangular form by unitary similarity transformation using the QR algorithm with double shift is computed. Thirdly, by permutations a triangular matrix is reduced to the form, in which various eigenvalues are concentrated at the bottom of the matrix. Finally, by unitary and orthogonal similarity transformations a triangular matrix with a diagonalizable block in the lower part of the diagonal is obtained. For this and only for this triangular diagonalizable, well-conditioned block the eigenvalue problem is solved.

The article shows that if an eigenvalue, being well-conditioned, is not assigned to a diagonalizable matrix the method will work with accuracy and speed not worse than ones in the method based on the Hessenberg form.

In the case of absence of multiple eigenvalues, as well as in the method, based on the Hessenberg form [10], the assessed complexity of computations is a quadratic function of the dimension of the system matrix. In the absence of multiple eigenvalues, the complexity of computations in this method is significantly decreased, becoming a linear function of the dimension of the system matrix, whereas the method with the Hessenberg form also gives a quadratic dependence in this case.

So, the method for calculating the frequency response, proposed in this article, can compete with the well-known method which uses the Hessenberg form [10], [11], giving an obvious advantage in time.

5. CONCLUSION

The proposed solutions are important, because they help to enhance reliability of mathematical and program support of modern computer systems which require accurate calculations in ill-conditioned problems by step by step control of the priori error.

The algorithm for computing the matrix frequency response, presented in the article, is based on the similarity transformation of the system matrix and reduction to a triangular form with an extracted diagonalizable block for multiple eigenvalues. Such an approach for each predefined frequency value allows us to determine the value of the matrix frequency response as the solution of a system of linear algebraic equations with a triangular coefficient matrix with a diagonal submatrix.

The method characteristics, obtained in the paper, show that in the absence of multiple eigenvalues, the complexity of computations by the proposed method is dramatically decreased, as it is close to linear dependence on the dimension of the system matrix.

Using the algorithm in the absence of multiple eigenvalues gives assessment of performance, which is a quadratic function of the dimension of the system matrix. In the intermediate case a quadratic component of the dependence is weakened by a coefficient, which is less than unity per value of the squared ratio of dimension of the diagonalizable matrix to the dimension of the system.

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The proposed method is a reliable method, based on matrix factorizations, using the orthogonal, unitary similarity transformations and direct, not having the methodical error computations.

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