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LIGHT POLARIZATION IN AN ANISOTROPIC MEDIUM IN TERMS OF ROTATIONS

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ABSTRACT

Stokes vector evolution in a homogeneous birefringent medium is treated in terms of Cayley-Klein parameters inside 8D space of geometric Clifford algebra. Scaled to unity, Stokes parameters are compared with directional cosines of a unit vector in geometric algebra and with a unit magnetization vector in NMR. The unusual interpretation of Cayley-Klein parameters in matrix optics is regarded as a stimulus to revise 4D entities in geometric algebra.

Keywords: *Polarization, Stokes vector, Mathematical methods in physics, Quantum optics.*

1. INTRODUCTION

The idea to treat simplicity as pretty organized complexity has deep roots in optics. It can be easily traced from the simple laws of geometric optics. Actually, it is well known that in specular reflection, the angle of incidence equals the angle of reflection for all wavelengths of light. Knowing that the specular mirror is composed of two complementary gratings the removal of any one of them turns reflection into diffraction. This phenomenon is more complicated than the first one. Laws of wave optics have to be used to describe it. Note, that nothing was added, but something was removed to sophisticate the phenomenon.

Recombination of two complementary gratings into the specular mirror turns the superposition of two diffraction patterns into simple specular reflection pattern again. So simple specular reflection can be treated as a proper superposition of diffraction patterns from two complementary gratings.

This idea, applied to a unit scalar, , inside geometric Clifford algebra [1], reveals an existence of two new 4D entities complementary idempotent paravectors, and, for any arbitrary unit vector with a positive square, . Paravectors are related to the unit scalar, , in the same way as complementary gratings are related to the specular mirror. They are 4D objects of mixed grade, hermitian and invariant to multiplication by themselves (idempotents). Their linear independence, treated as a new kind of `orthogonality', gives way to a fourth dimension

within a 3D Euclidean vector space. In one-sided products idempotent paravectors behave as projectors. They project all algebra elements into four conjugated spinor ideals [2]. This approach provides a new insight onto the meaning of matrices, their parts, raws and columns, single matrix elements and their coefficients [3],[4], and it gives some restrictions on their usage. Here we apply these results to revise matrix representations of light polarization and to compare scaled to unity Stokes parameters with directional cosines of a unit vector in geometric algebra and with a unit magnetization vector in NMR. The comparison is made in terms of Cayley-Klein parameters which are commonly used to describe spatial rotations.

2. STOKES VECTOR AND STOKES PARAMETERS

Stokes vector is a simple way to describe the polarization state of light. For a light beam propagating in $z\hat{e}_3$ direction and decomposed into two components polarized along the orthogonal directions $\hat{e}_1$ and $\hat{e}_2$ the electric field is given by

$$\mathbf{E}(z,t) = (\hat{e}_1\alpha_1 + \hat{e}_2\alpha_2)e^{ikz-i\omega t} + c.c. \qquad (1)$$

the four Stokes parameters characterizing such a beam are defined by [5]

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$$S_0 = |\alpha_1|^2 + |\alpha_2|^2 \qquad S_1 = |\alpha_1|^2 - |\alpha_2|^2$$
$$S_2 = (\alpha_1^* \alpha_2 + \alpha_1 \alpha_2^*) \qquad S_3 = -i(\alpha_1^* \alpha_2 - \alpha_1 \alpha_2^*) \quad (2)$$

which lead to the representation of light polarization by a point on the Poincaré sphere.

All points of equator line in the $\hat{e}_1\hat{e}_2$ plane represent states of linear polarization, whereas north and south poles represent states of circular polarization. All other points on Poincaré sphere represent states of elliptic polarization.

It is supposed, that α_1 and α_2 are instantaneous amplitudes of two transversely-spaced electric field components, E_x and E_y. They oscillate along $\hat{e}_1$ and $\hat{e}_2$ directions, at the same frequency ω with a possible phase shift $\delta=\varphi_1-\varphi_2$ among them. This model describes Stokes vector as a vector in $\hat{e}_1\hat{e}_2$ plane in terms of intensities of these simple oscillations, which are real for linear polarization, imaginary for a circular one, and complex for an elliptical one.

As in matrix optics [5] and in geometric algebra [6] unit vectors of the Cartesian frame of reference $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are associated with Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3)$$

In matrix optics the choice is

$$\hat{e}_1 = \sigma_3, \quad \hat{e}_2 = \sigma_1, \quad \hat{e}_3 = \sigma_2. \quad (4)$$

It helps to associate basis vectors $\hat{e}_1$ and $\hat{e}_2$ in the transverse plane of the light beam with real matrices σ_3 and σ_1. While it is good for the simple plane model described above, it seems unsatisfactory in more complicated phenomena, such as photon echo [7] or light propagation through a homogeneous birefringent medium, where light is regarded in a 3D Euclidean vector space.

In geometric algebra and its applications to magnetic resonance [2], the choice[1] is

$$e_1=\sigma_1,\ e_2=\sigma_2, e_3=\sigma_3, \quad (5)$$

It is made to associate the direction, e_3, of static magnetic field in 3D Euclidean space with the unique properties of Pauli matrix σ_3.

3. UNIT VECTOR IN GEOMETRIC ALGEBRA

In a given frame of reference, $\{e_1, e_2, e_3\}$, spatial orientation of a unit vector **m** is usually described by its directional cosines ($\boldsymbol{m}_1$, $\boldsymbol{m}_2$, $\boldsymbol{m}_3$,) as

$$\mathbf{m} = m_1 e_1 + m_2 e_2 + m_3 e_3 = \begin{bmatrix} m_3 & m_1 - i m_2 \\ m_1 + i m_2 & -m_3 \end{bmatrix}. \quad (6)$$

Here basis vectors are expressed in terms of Pauli matrices according to Eq. (5).

Matrix in the right side of Eq. (6) can be written as a direct sum of its matrix elements:

$$\mathbf{m} = m_3 P_3 + (m_1 + i m_2)(e_1 P_3) + (m_1 - i m_2)(e_1 N_3) - m_3 N_3 \quad (7)$$

where

$$P_3 = \frac{1}{2}(e_0 + e_3) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad (e_1 P_3) = \frac{1}{2}(e_1 + e_{13}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad (8)$$
$$N_3 = \frac{1}{2}(e_0 - e_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad (e_1 N_3) = \frac{1}{2}(e_1 - e_{13}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(3)
are the unique `paravector names' of unit matrix elements, or their `addresses' inside the 2×2 matrix. Each term in Eq. (7) is a named quantity. As their names are different (linearly independent), they do not mix in the direct sum.

Idempotent paravectors P_3 and N_3 are projectors. They project vector **m** into left (contravariant) spinor ideals [2]:

Positive:

$$\mathbf{m}P_3 = m_3 P_3 + (m_1 + i m_2)(e_1 P_3) = \begin{bmatrix} m_3 & 0 \\ m_1 + i m_2 & 0 \end{bmatrix} \quad (9)$$

Negative:

$$\mathbf{m}N_3 = (m_1 - i m_2)(e_1 N_3) - m_3 N_3 = \begin{bmatrix} 0 & m_1 - i m_2 \\ 0 & -m_3 \end{bmatrix}. \quad (10)$$

Spinor $\mathbf{m}P_3$ describes vector **m** in the right frame of reference, which is associated with the upper half of the Poincaré sphere. Spinor $\mathbf{m}\boldsymbol{N}_3$ describes the same vector **m** in the left frame of reference, which is associated with the lower half of the Poincaré

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sphere. Basis vector e_3 points to the North Pole of the sphere.

Both projections have only two terms, which in matrix representation are composed in columns. They could be treated as column-vectors, but for a little complication: their `paravector names' in Eq. (9) contain zero matrix elements of the adjacent column to avoid ambiguities.

Another way to describe spatial orientation of vector **m** is to treat it as the result of some spatial rotation from a certain fixed position, e. g. from the privileged direction e_3. To avoid ambiguities usually the shortest one is selected. The plane of this rotation is spanned on both vectors and its axis is perpendicular to this plane. The rotation is described as

$$m = Q(m)e_3Q(m) \qquad (11)$$

where $Q(\mathbf{m})$ and $\tilde{Q}(\mathbf{m})$ are two conjugated (mutually reversed) unitary quaternions [2]. In matrix representation they are written as

$$\begin{cases} Q(\mathbf{m}) = \alpha P_3 + \beta(e_1 P_3) - \beta^*(e_1 N_3) + \alpha^* N_3 = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}. \\ Q(\mathbf{m}) = \alpha^* P_3 + \beta^*(P_3 e_1) - \beta(N_3 e_1) + \alpha N_3 = \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}. \end{cases} \qquad (12)$$

Their matrix elements complex numbers α, β, $-\beta^*$ and α^*, which meet a condition

$$\alpha\alpha^* + \beta\beta^* = 1 \qquad (13)$$

are well known Cayley-Klein parameters. In a given frame of reference they contain information about the axis ($\mathbf{a}=a_1e_1+ a_2e_2+ a_3e_3$), the direction and the angle, ϑ, of rotation:

$$\begin{cases} \alpha = \cos\frac{\vartheta}{2} - i a_3 \sin\frac{\vartheta}{2}; \\ \beta = -i(a_1 + i a_2)\sin\frac{\vartheta}{2}. \end{cases} \qquad (14)$$

Cayley-Klein parameters are complex numbers, complementary in the sense that only pair of them in concert contains full information about the rotation. Half-angle in Eq. (14) is a sequence of the fact that each quaternion in Eq. (12) is a direct sum of two spinors [2], positive and negative, respectively. This kind of spinors also can be expressed in terms of matrix columns and rows as one-sided products of unitary quaternions Eq. (12) with idempotent paravectors P_3 and N_3, defined in Eq. (8):

$$\psi(\mathbf{m}) = Q(\mathbf{m})P_3 = \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix};\ \bar{\psi}(\mathbf{m}) = Q(\mathbf{m})N_3 = \begin{bmatrix} 0 & -\beta^* \\ 0 & \alpha^* \end{bmatrix};$$
$$\tilde{\psi}(\mathbf{m}) = P_3\tilde{Q}(\mathbf{m}) = \begin{bmatrix} \alpha^* & \beta^* \\ 0 & 0 \end{bmatrix};\ \tilde{\bar{\psi}}(\mathbf{m}) = N_3\tilde{Q}(\mathbf{m}) = \begin{bmatrix} 0 & 0 \\ \beta & \alpha \end{bmatrix}. \qquad (15)$$

It is worth noting, that there is no information in Cayley-Klein parameters about what is rotated, and about the initial and final conditions for the rotated entities. It is due to the fact that each simple rotation is a unitary operation. It preserves the lengths of all vectors being rotated and all angles among them. So it describes, the rotation of the entire distribution of vectors as well. For example, if a book is rotated, it may be granted that neither its form nor its contents will be changed.

(11)
The initial state e_3 is transformed by the rotation into the final state:

$$\begin{aligned} m &= P(\mathbf{m}) - N(\mathbf{m}) = \\ &\begin{bmatrix} m_3 & m_1 - i m_2 \\ m_1 + i m_2 & -m_3 \end{bmatrix} = \begin{bmatrix} \alpha\alpha^* - \beta\beta^* & 2\alpha\beta^* \\ 2\alpha^*\beta & \beta\beta^* - \alpha\alpha^* \end{bmatrix} = \\ &= (\alpha^*\beta + \alpha\beta^*)e_1 + i(\alpha^*\beta - \alpha\beta^*)e_2 + (\alpha\alpha^* - \beta\beta^*)e_3. \end{aligned} \qquad (16)$$

where

$$P(\mathbf{m}) = \frac{1}{2}(e_0 + \mathbf{m}) = \psi(\mathbf{m})\tilde{\psi}(\mathbf{m}) = \begin{bmatrix} \alpha\alpha^* & \alpha^*\beta \\ \alpha\beta^* & \beta\beta^* \end{bmatrix}; \qquad (17)$$

and

$$N(\mathbf{m}) = \frac{1}{2}(e_0 - \mathbf{m}) = \bar{\psi}(\mathbf{m})\tilde{\bar{\psi}}(\mathbf{m}) = \begin{bmatrix} \alpha\alpha^* & -\alpha^*\beta \\ -\alpha\beta^* & \beta\beta^* \end{bmatrix} \qquad (18)$$

are 4D paravector `images' [4] of unit vector **m**.
So, taking into account Eq. (13), directional cosines of the unit vector **m** in terms of Cayley-Klein parameters are:

$$\begin{aligned} |m| &= |\alpha|^2 + |\beta|^2 = 1; \qquad m_3 = |\alpha|^2 - |\beta|^2; \\ m_1 &= \alpha^*\beta + \alpha\beta^*; \qquad m_2 = i(\alpha^*\beta - \alpha\beta^*). \end{aligned} \qquad (19)$$

They are identical with scaled to unity Stokes parameters in Eq. (2) with respect to Pauli matrices Eq. (3), but they differ in notations according to Eq. (4) and Eq. (5), and their contents and interpretation is quite different. More than that, complex numbers α_1 and α_2 in definition Eq. (2) of Stokes parameters are identical with Cayley-Klein parameters α and β in Eq. (19).

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Cayley-Klein parameters in Eq. (14) are complex numbers. So they can be represented in the `module-phase' form. They even can oscillate, if the rotation is in progress. But they never can be compared by their phase, because their phases are defined in different (noncoplanar) hyperplanes of 4D spinorial space. So any coherence between them should be treated not in the sense of their phase synchronism, but in the sense of their complementarity. Only a pair of them in concert contains full information about the axis and the angle of rotation. They should not be treated separately.

Now let us return to the Stokes vector and look at its evolution along with the light beam propagation through a birefringent medium.

4. STOKES VECTOR EVOLUTION IN A HOMOGENEOUS BIREFRINGENT MEDIUM

In a homogeneous non-dissipative birefringent medium, Stokes vector evolution equation takes a simple form of the classical torque equation [8]. In geometric algebra [9] it takes the form:

$$\frac{\partial \mathbf{S}(z)}{\partial z} = \mathbf{\Omega}\times\mathbf{S}(z) = -i\frac{1}{2}[\mathbf{\Omega}\mathbf{S}(z) - \mathbf{S}(z)\mathbf{\Omega}]. \qquad (20)$$

where **z** is the distance along the propagation direction, and $\mathbf{S}(z)=[S_1(z),S_2(z), S_3(z)]$ is Stokes polarization vector.

According to Eq. (20), as the wave propagates along the ze_3 axis, Stokes vector **S** precesses with a constant spatial frequency **Ω** about the fixed direction **w**=**Ω**/Ω. In terms of the medium properties, direction **w** and absolute value Ω characterize the type and the strength of the medium birefringence, respectively.

Scaling the length S_0 of the Stokes vector $\mathbf{S}(z)=S_0\mathbf{s}$ to unity, $S_0=e_0$, brings Eq. (20) to the form:

$$\frac{\partial \mathbf{s}(z)}{\partial z} = -i\frac{1}{2}\Omega[\mathbf{w}\mathbf{s}(z) - \mathbf{s}(z)\mathbf{w}]. \qquad (21)$$

To solve it in 4D space, we split the unit Stokes vector **S**(z) into a direct sum of two complementary idempotent paravectors:

$$\mathbf{s}(z) = P(\mathbf{s}) - N(\mathbf{s}) = \frac{1}{2}(e_0 + \mathbf{s}(z)) - \frac{1}{2}(e_0 - \mathbf{s}(z)) \; (22) \qquad (21)$$

and then split each of them into an ordered product of contravariant and covariant spinors:

$$P(\mathbf{s}) = \psi(\mathbf{s})\tilde{\psi}(\mathbf{s}); \qquad N(\mathbf{s}) = \bar{\psi}(\mathbf{s})\tilde{\bar{\psi}}(\mathbf{s}). \quad (23) \qquad (22)$$

The first step turns Eq. (20) into a pair of Liouville-von-Neumann-like equations

$$\begin{cases} \dfrac{\partial P(\mathbf{s})}{\partial z} = -i\dfrac{1}{2}\Omega[\mathbf{w}P(\mathbf{s}) - P(\mathbf{s})\mathbf{w}]; \\ \dfrac{\partial N(\mathbf{s})}{\partial z} = i\dfrac{1}{2}\Omega[\mathbf{w}N(\mathbf{s}) - N(\mathbf{s})\mathbf{w}]. \end{cases} \quad (24) \qquad (23)$$

Substitation of Eq. (23) into Eq. (24) gives a system of four Schrödinger-like linear equations for conjugated spinors

$$\frac{\partial \psi(\mathbf{s})}{\partial z} = -i\frac{1}{2}\Omega\mathbf{w}\,\psi(\mathbf{s}); \qquad \frac{\partial \tilde{\psi}(\mathbf{s})}{\partial z} = i\frac{1}{2}\Omega\,\tilde{\psi}(\mathbf{s})\mathbf{w}; \quad (25)$$
$$\frac{\partial \bar{\psi}(\mathbf{s})}{\partial z} = -i\frac{1}{2}\Omega\mathbf{w}\,\bar{\psi}(\mathbf{s}); \qquad \frac{\partial \tilde{\bar{\psi}}(\mathbf{s})}{\partial z} = i\frac{1}{2}\Omega\,\tilde{\bar{\psi}}(\mathbf{s})\mathbf{w}. \qquad (24)$$

Composition of spatially inverted spinors into direct sums yields unitary quaternions:

$$Q(\mathbf{s}) = \psi(\mathbf{s}) + \bar{\psi}(\mathbf{s}); \qquad \tilde{Q}(\mathbf{s}) = \tilde{\psi}(\mathbf{s}) + \overline{\tilde{\psi}}(\mathbf{s}) \quad (26)$$

This step turns the system of four equations Eq. (25) into a system of two linear equations for conjugated quaternions:

$$\begin{cases} \dfrac{\partial Q(\mathbf{s})}{\partial z} = -i\dfrac{1}{2}\Omega\mathbf{w}Q(\mathbf{s}); \\ \dfrac{\partial \tilde{Q}(\mathbf{s})}{\partial z} = i\dfrac{1}{2}\Omega\tilde{Q}(\mathbf{s})\mathbf{w}. \end{cases} \quad (27)$$

The system Eq. (27) can be easily solved for a simple rotation with spatial frequency Ω=const(z) about the fixed axis **w**=const(z). One easily obtains

$$\begin{cases} Q(\mathbf{s}) = exp(-i\dfrac{\Omega z}{2}\mathbf{w}) = e_0 cos\dfrac{\Omega z}{2} - i\mathbf{w}sin\dfrac{\Omega z}{2}; \\ \tilde{Q}(\mathbf{s}) = exp(i\dfrac{\Omega z}{2}\mathbf{w}) = e_0 cos\dfrac{\Omega z}{2} + i\mathbf{w}sin\dfrac{\Omega z}{2}. \end{cases} \quad (28)$$

This is the point at which Cayley--Klein parameters are expressed in terms of the angle and the axis of the particular rotation. Comparison of Eq. (28) with Eq. (12) yields

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$$\begin{cases} \alpha & = cos\frac{\Omega z}{2} - iw_3 sin\frac{\Omega z}{2}; \\ \beta & = -i(w_1 + iw_2) sin\frac{\Omega z}{2}. \end{cases} \quad (29)$$

For initial state **s**(0)=e_3 the output Stokes vector **s**(z) will have direction cosines as in Eq. (19) where Cayley-Klein parameters α and β are defined by Eq. (29). Any other initial state have to be treated as the result of some previous rotation Eq. (11), which is described by unitary quaternions Eq. (12) with Cayley-Klein parameters α_1 and β_1, followed by the rotation in the homogeneous birefringent medium, which is described by the unitary quaternions Eq. (26) with Cayley-Klein parameters Eq. 29, which we denote now α_2 and β_2. Then Cayley--Klein parameters for the resulting rotation are given by a recurrence relation [2]:

$$\begin{cases} \alpha_{21} & = \alpha_2\alpha_1 - \beta_2^*\beta_1; \\ \beta_{21} & = \beta_2\alpha_1 + \alpha_2^*\beta_1. \end{cases} \quad (30)$$

This is the most elegant way to find the result of two sequential rotations about non-collinear axes [9]. Substitution of Eq. (30) into Eq. (19) yields spatial orientation of the unit Stokes vector in the Cartesian frame of reference.

5. POLARIZATION

As it was mentioned before, every unit Stokes vector corresponds to a point on the Poincaré sphere and vice versa. The right circular polarization is represented by the North Pole. In our model it is in the direction of light beam propagation. The left circular polarization is represented by by the South Pole, which is in the direction of the source of light. Linear polarizations are represented by points in the equatorial plane, and the elliptical polarization by the points between the poles and the equatorial plane.

In all these cases Stokes vector is a vector of 3D Euclidean space. More than that, it is the axis of rotation. The plane of rotation is perpendicular to the axis. Electric field components rotate in the plane of rotation with light frequency, i. e. they are circularly polarised in this plane. But polarization is measured in e_1e_2 plane, i. e. in the transverse section of the light beam. So the result of measurement depends on the mutual orientation of these to planes in 3D Euclidean space.

For example, if **s**=e_3, i. e. Stokes vector points in the direction of light propagation, the resulting Cayley-Klein parameters in Eq. (29) are α=1, β=0. Both of them are real numbers. There are no phase relations between them because phase for β is undefined. The plane of rotation is collinear (coincides) with the plane of measurement. So rotation of some measurable light components in the anticlockwise direction is associated with right circular polarization.

If **s**=e_3, i. e. Stokes vector points in the direction of the source of light, the resulting Cayley--Klein parameters are α=-1, β=0. Again the plane of rotation coincides with e_1e_2 plane. But now measurable light components rotate in the clockwise direction. This corresponds to the left circular polarization of light.

If Stokes vector **s** is in the $\hat{e}_1\hat{e}_2$ plane, rotations occur in a plane that contains e_3 vector which is orthogonal to e_1e_2 plane. So only linear projection of rotating measurable light components (linear polarization) can be measured in e_1e_2 plane. Linear light polarization is perpendicular to Stokes vector direction. For example, if **s**=e_1, $(\alpha = \frac{\sqrt{2}}{2}$, $\beta = -i\frac{\sqrt{2}}{2})$, linear polarization is long e_2 direction. If **s**=e_2 $(\alpha = \frac{\sqrt{2}}{2}, \beta = \frac{\sqrt{2}}{2})$, linear polarization is along e_1 direction.

In all other points of the Poincaré sphere which are not poles or equator points, Stokes vector is tilted to $\hat{e}_3$ vector. Hence the plane of rotation is slanted to $\hat{e}_1\hat{e}_2$ plane, and its Cartesian projection on $\hat{e}_1\hat{e}_2$ plane is an ellipse. This provides an elliptic polarization.

6. DISCUSSION

In this paper, we concentrate on the idea that light might be more complicated substance, than it is supposed to be. Geometric algebra allows it to be even an 8D entity which has some measurable and some immeasurable components [4].

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Matrix representations for such objects have several limitations. First of all, there is only one direction in 3D vector space, associated with Pauli matrix σ_3, for which matrix elements have their unique paravector names (addresses), as in Eq. (8). In this case matrix elements can be treated as some components of a vector, as in Eq.(6), or as some components of a unitary quaternion (Cayley-Klein parameters), as in Eq. (12), describing its orientation, as in Eq. (11). Both kinds of 2×2 matrices can be truncated up to `column-vectors' in contravariant paravector projections, as in Eq.(9) or in Eq. (15). But this can lead to some ambiguities. Spinor $\mathbf{m}P_3$ in Eq.(9) is treated in 3D Euclidean space and can be treated as representation of vector **m** in the right-handed (spherical) frame of reference, e. g. as a point on the surface of the unit Poincaré sphere. Spinor $\Psi(\mathbf{m})$ represents the same vector in 4D Hilbert space. Its components also can be associated with directional cosines of some vector and with a point on the surface a sphere, but it is a sphere in 4D space. An example of correspondence between points on both spheres is described in [3].

In matrix optics $\hat{e}_1 = \sigma_3$. So the unique space direction is in the transverse plane of the light beam. It brings spinorial properties (halved engles) into the plane of measurements. Association of spinorial components (Cayley-Klein parameters α_1 and β_1) with instantaneous amplitudes of two transversely-spaced electric field components, E_x and E_y, mixes vectors defined in 3D and in 4D spaces, which is unfair in geometric algebra. As a sequence vectors σ_3 and $\frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$ with angle 45^0 between them are treated as orthogonal ones.

In our model σ_3 is associated with a light beam propagation direction. In this case, Stokes vector spatial rotations in a homogeneous birefringent medium are similar to temporal ones for unit magnetization vector in pulsed NMR [2]. In both cases, they are the axes of rotations. The main difference is in magnetic and electric nature of these measurable quantities. Magnetization vector is an invariant of steady-state rotations, it coincides with the axis of rotation. Electric field components rotate in the plane of rotation with the frequency of light. Cartesian projection of the plane of rotation onto the plane of measurement defines the polarization of light.

Unusual interpretation of Cayley-Klein parameters in matrix optics is a stimulus to revise those things we thought we knew as about light in optics, and about 4D entities in geometric algebra $\Gamma_{3,0}$.

7. CONCLUSIONS

Stokes vector evolution in a homogeneous birefringent medium is treated in terms of Cayley-Klein parameters inside geometric Clifford algebra $\Gamma_{3,0}$. It is shown that its spatial rotation is completely defined by the properties of the medium but its position on Poincaré sphere depends on its initial orientation. If the north pole of the sphere is associated with the direction of light propagation, Stokes vector has to be treated only as an axis of rotation. The plane of rotation for electric components of light is perpendicular to this axis, and its Cartesian projection on the transverse plane of the light beam defines its polarization. Some sequences of different interpretation of Cayley-Klein parameters in matrix optics and in geomnetric algebra are considered.

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