



A NECESSARY AND SUFFICIENT CONDITION FOR OUTPUT FEEDBACK STABILIZABILITY: APPLICATION FOR ATTITUDE CONTROL SYSTEMS OF MICRO SATELLITE.

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ABSTRACT

In this paper, we deal with the attitude control of a satellite equipped with wheels of reaction, using the static output feedback. In the first part, we set the control and its integral in the state-feedback form. Then, by using the algorithm of Kucera, we establish the necessary and sufficient conditions so that the nonlinear system which models the satellite will be stabilizable by the Static Output Feedback. Thereafter, we present the robust stabilization of the attitude control; this robustness is influenced by the parametric uncertainty of the model and by the input noise in the system. The main results of this paper show that the sufficient conditions for the existence of the robust static output feedback control can stabilize the studied system. The controller proposed here is based upon a generalization of Gronwall's lemma.

Keywords: - *Pole placement; satellite; attitude control; attitude stabilization; robustness; Bellman – Gronwall; Output feedback.*

1 INTRODUCTION

In the aeronautic area, the attitude control is essential in the automatic control researches because of the severe criteria imposed by schedules of conditions, in terms of accuracy and robustness, the interest of such a system also resides in structural considerations, these nonlinear systems, actually multivariable and strongly sensitive to disturbances, represent benchmarks for checking the control laws to be synthesized for less constrained industrial systems. In addition, several works were established on the attitude control using various approaches, in particular PID regulators [8],

quadratic regulation (LQR) [6], robust control [3, 4]. In several works, simplified linear models have been considered but they neglect a part of the dynamic system or are based on an adaptive identification which may weigh down the calculations of the control laws.

Our results are based on a nonlinear model which avoids any simplifications, as opposed to many reference works. Generally a model is only one approximate representation of the studied system. Therefore one should take into account two parameters: the origin of uncertainties and the validity of the control laws calculated on the basis of imperfect representation of the system. The



modelling of an earth-pointing satellite is subject to a certain number of idealizations which many result from the uncertainties of the model. Among the various of uncertainties, we can quote in particular:

- The inaccuracy on the parameters of the model,
- Parametric variations, often related to the operating conditions of the satellite,
- External disturbances (solar pressure of radiation for example which acts on surfaces of the vehicle, disturbing the orbit and the attitude...).

Static output – feedback, (SOF), despite its apparent simplicity, is one of the most researched problems in control theory and applications; see for example the survey by V.L.SYRMOS et al (1997), [13] which includes analytical and computational methods in the control of linear time invariant (LTI) systems. The static output feedback problem is to find a static or constant feedback gain to fulfil certain desirable characteristics. Stability is the most important one.

In this paper, we present a necessary and sufficient condition, so that the attitude control system of micro – satellite can be stabilizable by static output feedback, in the second part, we will study the robustness of the static output feedback; control related to uncertainties of the parameters and to input noise of the system. [1,2]. The proof of controlled system stability is established using the generalization of the Bellman-Gronwall lemma [5, 9].

2 STRUCTURE OF MICRO SATELLITE:

The dynamic model of a satellite with earth pointing, using a SCA 3axes maneuvered by the reaction wheels and the magneto couplers (MC), and evolving under the disturbances, results from the equations of the dynamics and the cinematic movement of the satellite[11,8,12].The mathematical model of the system is that given by the following state space representation: [3]:

$$\dot{x}(t) = Ax(t) + Bu(t) + G \int_0^t u(s)ds + \sum_1^3 (\int_0^t u_i(s)ds)(E_i x(t)) \tag{1}$$

where :

$$x(0) = x_0$$

$x(t) \in \mathfrak{R}^n$, the state representing respectively the Eulerian angles and their derivatives.

$u(t) = (u_1(t), u_2(t), u_3(t)) = -h \bullet_w(t)$ is the control with $hw(t)$ is the angular momentum of the wheel cluster;

A, B, C, E_i and G are matrices with respectively $(n, n); (n, p); (p, n); (n, n); (n, p)$ dimensions.

3 STATIC OUTPUT FEEDBACK STABILIZATION:

In this section, we consider the system of attitude control for the satellite described by the nonlinear model above. We will carry out the synthesis, by steps, of the stabilizing control.

LINEAR SYSTEM STABILIZATION

Neglecting the quasi-bilinear term in the system of equation (1), we obtain the following linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) + G \int_0^t u(s)ds \tag{2}$$

Where $x(0) = x_0$

Proposition:

Consider the system described by the simplified model (2), and let the pair (A_1, B_1) be controllable, with:

$$A_1 = \begin{pmatrix} A & G \\ 0 & 0 \end{pmatrix} \text{ is } (n+p, n+p) \text{ and } B_1 = \begin{pmatrix} B \\ I \end{pmatrix} \text{ is } (n+p, p)$$

There exists a state feedback of the form $u(t) = -Kx(t)$ that stabilizes the system.

Proof:

Let:

$$z = \int_0^t u(s)ds$$

Hence:

$$\dot{z} = u(t)$$

(3)

From (2) and (3), the new system becomes:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & G \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ I \end{pmatrix} u$$

(4)



Where I and 0 are respectively identity and null matrix.

Let:

$$X = \begin{pmatrix} x \\ z \end{pmatrix};$$

It follows:

$$\dot{X} = A_1 X + B_1 u$$

(5)

Where: $A_1 = \begin{pmatrix} A & G \\ 0 & 0 \end{pmatrix}$ is $(n+p, n+p)$ and $B_1 = \begin{pmatrix} B \\ I \end{pmatrix}$ is $(n+p, p)$

If the pair (A_1, B_1) is controllable, then there is a linear state feedback allowing the system (5) to be stable. To design this control, we will use pole placement; this pole placement allows having a relation between $u(t)$ and its integral

$$u(t) = -K_1 x(t) - K_2 \int_0^t u(s) ds$$

(6)

Poles placement:

Let us seek a gain matrix K such as $(A_1 - B_1 K)$ is asymptotically stable, and then stabilizes the system (5).

Hence:

$$u = -KX = -(K_1 \quad K_2) \begin{pmatrix} x \\ z \end{pmatrix} = -K_1 x - K_2 z = -K_1 x - K_2 \int_0^t u(s) ds \quad FC + B_2^T P = G$$

$$\begin{cases} \dot{x} = A_2 x(t) + B_2 u(t) \\ y = Cx(t) \end{cases} \quad (9)$$

Let us consider the following static output feedback

$$u = -Fy \quad (10)$$

F is a matrix. We say that the system (9) is output feedback stabilizable if there exists a feedback control law (10) such that the matrix $A_2 - B_2 FC$ has all its eigenvalues with negative real part. Such matrices will be called stable.

We recall some standard terminology.

The pair (A_2, B_2) is said to be stabilizable if there exists a real matrix K such that $A_2 + B_2 K$ is stable.

The pair (A_2, C) is said to be detectable if there exists a real matrix L such that $A_2 + LC$ is stable.

Theorem 3.1. The system (9) is output feedback stabilizable if and only if:

- (i) (A_2, B_2) is stabilizable and (A_2, C) is detectable, and
- (ii) There exist real matrices F and G such that

(11)

Where P is the real symmetric nonnegative-definite solution of:

$$A_2^T P + PA_2 - PB_2 B_2^T + C^T C + G^T G = 0$$

(12)

Proof. To prove necessity, suppose that $A_2 + B_2 FC$ is stable for some F. Then (A_2, B_2) is stabilizable, since $A_2 + B_2 K$ is stable for $K = FC$, and (A_2, C) is detectable, since $A_2 + LC$ is stable for $L = B_2 F$. This proves (i). Since $A_2 + B_2 FC$ is stable, there exists (Wonham, 1985) [14] a unique symmetric nonnegative-definite matrix P such that

$$(A_2 + B_2 FC)^T P + P(A_2 + B_2 FC) + C^T C + C^T F^T FC = 0$$

(13)

Rearranging (13), we obtain

$$A_2^T P + PA_2 - PB_2 B_2^T P + C^T C + (FC + B_2^T P)^T (FC + B_2^T P) = 0$$

Hence setting $G = FC + B_2^T P$ implies that (ii) is verified.

To prove sufficiency, suppose that (i) and (ii) hold. It follows from (ii) that (13) is satisfied.

Remark: K2 is chosen to be invertible (p, p) matrix.

So we have:

$$K_2 \int_0^t u(s) ds = -K_1 x - u$$

$$\int_0^t u(s) ds = (K_2^{-1})[-K_1 x - u] = -K_2^{-1} K_1 x - K_2^{-1} u.$$

Let us replace this integral in the linear system (2), we obtains:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + G[-K_2^{-1} K_1 x - K_2^{-1} u] \\ &= (A - GK_2^{-1} K_1)x + (B - GK_2^{-1})u \\ &= A_2 x + B_2 u \end{aligned}$$

(8)

Such as:

$$A_2 = A - G.K_2^{-1}.K_1 \text{ et } B_2 = B - G.K_2^{-1}$$

The system (2) becomes:



From (i), the detectability of (A_2, C) , implies that $A_2 + LC$ is stable for some L. Noting that

$$A_2 + LC = (A_2 + B_2FC) + \begin{bmatrix} L & -B_2 \\ C & FC \end{bmatrix}$$

it follows that $((A_2 + B_2FC), \begin{bmatrix} C \\ FC \end{bmatrix})$ is detectable

as well. Since P is symmetric and nonnegative-definite, we conclude (Wonham, 1985, Lemma 12.2) [14] from (13) that $A_2 + B_2FC$ is stable.

Algorithm:

The problem is how to calculate a stabilizing feedback gain F; many algorithms that provide an output stabilizing gain F use different principles. We use in this section, the following algorithm (proposed by Kucera) [7] with iterates the algebraic Riccati equation (12) up until a weighting matrix G is found that satisfies the constraint (11).

Step 1. set $i=0$ and $G_i=0$.

Step 2. Solve the equation

$$A_2^T P_i + P_i A_2 - P_i B_2 B_2^T + C^T C + G_i^T G_i = 0$$

For P_i symmetric and nonnegative-definite.

Step 3. Set

$$G_{i+1} = B^T P_i C^T (CC^T)^{-1} C + B^T P_i,$$

Increase i by 1 and go to step 2.

If the sequence $P_0, P_1, P_2 \dots$ converges, say to P, both (11) and (12) will be satisfied for

$$F = B^T P C^T (CC^T)^{-1},$$

Thus yielding one particular output stabilizing gain. The convergence remains to be proved, however.

The closed loop system is:

$$\dot{x}(t) = (A_2 - B_2 K_0)x = F_0 x(t)$$

(14)

The eigenvalues of the matrix F_0 , λ_i ($\lambda_i \neq \lambda_j$ pour $i \neq j$) are such as $\text{Re}(\lambda_i) < 0$. Hence the system that models the satellite attitude movement is stable.

The following result enables us to give the stability conditions of the nonlinear system.

APPLICATION TO THE NON LINEAR SYSTEM STABILIZATION

Consider the non linear system (1) that models the satellite:

$$\dot{x}(t) = Ax(t) + Bu(t) + G \int_0^t u(s)ds + \sum_{i=1}^3 \left(\int_0^t u_i(s)ds \right) (E_i x(t))$$

It's easy to see that the non linear system can be written in the following form:

$$\dot{x}(t) = F_0 x(t) + \sum_{i=1}^3 \left(\int_0^t u_i(s)ds \right) E_i x(t).$$

(15)

Where, according to the previous calculation, F_0 is asymptotically stable.

The matrix F_0 is asymptotically stable, then from the Hille-Yoshida theorem, there exists $M > 0$ and $\omega < 0$ such as:

$$\forall t \geq 0 : \|e^{F_0 t}\| \leq M e^{\omega t}$$

(16)

Theorem 3.2:

Let:

$$R = \frac{-\omega}{M^2 \sum_{i=1}^p \|\delta_i\| \|E_i\|}$$

$$\text{and } M_0 = \frac{M}{1 + \frac{M^2}{\omega} \sum_{i=1}^p \|\delta_i\| \|E_i\| \|x_0\|}$$

For $\|x_0\| \leq R$, the system (15) controlled by the linear state feed back: $u(t) = -Fy$

And then is asymptotically stable.

Where: δ_i is a matrix that depends on the control choice.

Proof:

From (1), (9) and (15), we can write:

$$\dot{x}(t) = F_0 x(t) + \sum_{i=1}^3 \left(\int_0^t u_i(s)ds \right) E_i x(t).$$

$$\text{And: } \int_0^t u(s)ds = -K_2^{-1} K_1 x - K_2^{-1} u.$$

It follows:

$$\int_0^t u_i(s)ds = (-K_2^{-1} K_1 x)_i - (K_2^{-1} u)_i$$

And from the static output feedback: $u(t) = -Fy$

$$\int_0^t u_i(s)ds = (-K_2^{-1} K_1 x)_i + (K_2^{-1} K_0 x)_i = ((-K_2^{-1} K_1 + K_2^{-1} K_0) x)_i$$

Hence:

$$\int_0^t u_i(s)ds = (\delta x(t))_i$$

(17)

With:

$$(\delta)_i = (-K_2^{-1} K_1 + K_2^{-1} K_0)_i$$

$$\int_0^t u_i(s)ds = (-K_2^{-1} K_1 x)_i + (K_2^{-1} FCx)_i = ((-K_2^{-1} K_1 + K_2^{-1} FC)x)_i$$

Hence:

$$\int_0^t u_i(s)ds = (\delta x(t))_i$$

(18)



With: $(\delta)_i = (-K_2^{-1}K_1 + K_2^{-1}FC)_i$

The studied system is:

$$\begin{cases} \dot{x} = F_0 x + \sum_{i=1}^p (\delta x(t))_i E_i x(t) \\ y = Cx(t) \end{cases}$$

(19)

The solution of the system (19) is :

$$x(t) = e^{F_0 t} x_0 + \int_0^t e^{F_0(t-s)} \left(\sum_{i=1}^p (\delta x(t))_i E_i x(t) \right) ds$$

Application of Hille-Yoshida theorem leads to:
The matrix F_0 is asymptotically stable, if there exists $M > 0$ and $\omega < 0$ such as:

$$\forall t \geq 0 : \|e^{F_0 t}\| \leq M e^{\omega t}$$

Hence:

$$\|x(t)\| \leq M e^{\omega t} \|x_0\| + \int_0^t (M e^{\omega(t-s)}) \sum_{i=1}^p \|\delta_i\| \|E_i\| \|x(s)\| ds$$

$$\|x(t) e^{-\omega t}\| \leq M \|x_0\| + \int_0^t (M e^{\omega s}) \sum_{i=1}^p \|\delta_i\| \|E_i\| \|e^{-\omega s} x(s)\| ds$$

Let:

$$M \|x_0\| = k ; g(s) = M \sum_{i=1}^p \|\delta_i\| \|E_i\| e^{\omega s} ; m(t) = x(t) e^{-\omega t} ; n = 2$$

Hence:

$$m(t) \leq k + \int_0^t g(s) [m(s)]^2 ds = k + \int_0^t [g(s)m(s)] m(s) ds$$

From the Bellman – Gronwall lemma [9]:

$$\begin{aligned} m(t) &\leq k \exp \int_0^t g(s) ds \Rightarrow -m(t) \exp \int_0^t -g(s) m(s) ds \geq -k \sum_{i=1}^p (\delta x(t))_i [E_i + \Delta E_i(t)] x(t) \\ &\Rightarrow \frac{d}{dt} \exp \int_0^t -g(s) m(s) ds \geq -k g(s) \end{aligned}$$

(21)

Hence

$$k(m(t))^{-1} \geq 1 - k \int_0^t g(s) ds$$

Under the following hypotheses:

$$1 - k \int_0^t g(s) ds > 0$$

We obtain:

$$m(t) \leq \frac{k}{1 - k \int_0^t g(s) ds}$$

4 ROBUSTNESS RELATIVE TO THE UNCERTAINTIES OF THE PARAMETERS:

Consider the non linear system (1):

$$\dot{x}(t) = Ax(t) + B u(t) + G \int_0^t u(s) ds + \sum_{i=1}^3 \left(\int_0^t u_i(s) ds \right) (E_i x(t))$$

We showed in first section that:

$$Ax(t) + B u(t) + G \int_0^t u(s) ds = A_2 x(t) + B_2 u(t)$$

Thus:

$$\dot{x}(t) = A_2 x(t) + B_2 u(t) + \sum_{i=1}^3 \left(\int_0^t u_i(s) ds \right) (E_i x(t))$$

(20) Where:

$$A_2 = A - GK_2^{-1} K_1$$

$$B_2 = B - GK_2^{-1}$$

And K_1 and K_2 are the gains matrixes chosen at the pole placement;

The system (20) can be written under the form:

$$\dot{x}(t) = (A - GK_2^{-1} K_1) x(t) + (B - GK_2^{-1}) u(t) + \sum_{i=1}^3 \left(\int_0^t u_i(s) ds \right) (E_i x(t))$$

We show that the control given by

$$u(t) = -Fy(t),$$

Where F is such as:

$(A - GK_2^{-1} K_1) - (B - GK_2^{-1}) FC$ is stable, i.e.: there exist $M > 0$ and $\omega < 0$ such as:

$$\forall t \geq 0 : \left\| e^{((A - GK_2^{-1} K_1) - (B - GK_2^{-1}) FC)t} \right\| \leq M e^{\omega t}$$

, stabilizes the following uncertain non-homogeneous non linear system:

$$\dot{x}(t) = [A + \Delta A(t) - [G + \Delta G(t)] K_2^{-1} K_1] x(t) - [B + \Delta B(t) - [G + \Delta G(t)] K_2^{-1}] F [C + \Delta C(t)] x(t) +$$

$$\sum_{i=1}^p (\delta x(t))_i [E_i + \Delta E_i(t)] x(t)$$

Where:

$$(\delta)_i = (-K_2^{-1} K_1 + K_2^{-1} FC)_i$$

We suppose that the uncertainties are bounded in module, i.e.: there exist, some constants a, b, g, c and e such as:

$$\|\Delta A(t)\| \leq a ; \|\Delta B(t)\| \leq b ; \|\Delta C(t)\| \leq c \text{ et } \|\Delta G(t)\| \leq g.$$

$$\|\Delta E(t)\| \leq e$$

One has the following result that shows that the system (21) controlled by

$$u(t) = -Fy(t)$$

can tolerate variations of the parameters of the model.



Theorem 4:

Let:

$$\sigma = M \left[a + g \|K_2^{-1}\| \|K_1\| + b \|K_0\| + g \|K_2^{-1}\| \|K_0\| \right]$$

And let's suppose that:

H1) $w + \sigma < 0$

H2)

$$\|x_0\| < \frac{-(w + \sigma)}{M^2 \sum_{i=1}^p \|E_i\| \|C_i + c_i\|}$$

Let's put:

$$M_0 = 1 + \frac{M^2 \|x_0\| \sum_{i=1}^p \|E_i\| \|C_i + c_i\|}{w + \sigma}$$

Then:

$$\forall t \geq 0 :$$

$$\|x(t)\| \leq M_0 \|x_0\| e^{(w+\sigma)t}$$

$$\|u(t)\| \leq \|FC\| \|M_0\| \|x_0\| e^{(w+\sigma)t}$$

(22)

Proof:

The system (20) controlled by

$$u(t) = -Fy(t)$$

Verifies:

$$\dot{x}(t) = \left([A + \Delta A(t)] - [G + \Delta G(t)] K_2^{-1} K_1 \right) x(t) - \left([B + \Delta B(t)] - [G + \Delta G(t)] K_2^{-1} F \right) u(t) + M \left[a + \|B\| \|F\| c + \|G\| \|K_2^{-1}\| \|F\| c + g \|K_2^{-1}\| \|F\| c + g \|K_2^{-1}\| \|F\| \|C\| + g \|K_2^{-1}\| \|K_1\| + b \|F\| \|C\| + b \|F\| c \right] \|x(s)\| e^{-ws} + \int_0^t M e^{-ws} \sum_{i=1}^p \|\delta_i\| \|E_i + e\| \|x(s)\|^2 ds$$

$$[C + \Delta C(t)] x(t) + \sum_{i=1}^p (\delta x(t))_i [E_i + \Delta E_i(t)] x(t)$$

$$\dot{x}(t) = (A - GK_2^{-1}K_1 - B[C + \Delta C(t)] + GK_2^{-1}F[C + \Delta C(t)])x(t) + (\Delta A(t) - \Delta G(t)K_2^{-1}F[C + \Delta C(t)] - \Delta B(t)F[C + \Delta C(t)] + \Delta G(t)K_2^{-1}F[C + \Delta C(t)])x(t) + \sum_{i=1}^p (\delta x(t))_i [E_i + \Delta E_i(t)]x(t)$$

$$\dot{x}(t) = (A - GK_2^{-1}K_1 - BFC + GK_2^{-1}FC)x(t) + [(\Delta A - \Delta GK_2^{-1}K_1 - \Delta BFC - \Delta BF\Delta C) + (\Delta GK_2^{-1}FC + \Delta GK_2^{-1}F\Delta C - BF\Delta C + GK_2^{-1}FC)x(t)] + \sum_{i=1}^p (\delta x(t))_i [E_i + \Delta E_i(t)]x(t)$$

$$x(t) = e^{(A - GK_2^{-1}K_1 - BFC + GK_2^{-1}FC)t} x_0 + \int_0^t e^{(A - GK_2^{-1}K_1 - BFC + GK_2^{-1}FC)(t-s)} (\Delta A(t) - \Delta GK_2^{-1}K_1 - \Delta BFC - \Delta BF\Delta C + \Delta GK_2^{-1}FC + \Delta GK_2^{-1}F\Delta C - BF\Delta C + GK_2^{-1}FC)x(s) ds + \int_0^t e^{(A - GK_2^{-1}K_1 - BFC + GK_2^{-1}FC)(t-s)} \sum_{i=1}^p (\delta x(t))_i [E_i + \Delta E_i(t)]x(s) ds$$

However:

$$\forall t \geq 0 : \left\| e^{((A - GK_2^{-1}K_1) - (B - GK_2^{-1}F)C)t} \right\| \leq M e^{wt}$$

And

$$\|\Delta A(t)\| \leq a; \|\Delta B(t)\| \leq b; \|\Delta C(t)\| \leq c; \|\Delta G(t)\| \leq g \text{ and } \|\Delta e(t)\| \leq e$$

From where:

$$\|x(t)\| \leq M e^{wt} \|x_0\| + \int_0^t M e^{w(t-s)} [a + \|B\| \|F\| c + \|G\| \|K_2^{-1}\| \|F\| c + g \|K_2^{-1}\| \|F\| c + g \|K_2^{-1}\| \|F\| \|C\| + g \|K_2^{-1}\| \|K_1\| + b \|F\| \|C\| + b \|F\| c] \|x(s)\| ds + \int_0^t M e^{w(t-s)} \sum_{i=1}^p \|\delta_i\| \|E_i + e\| \|x(s)\|^2 ds$$

$$\|x(t) e^{-wt}\| \leq M \|x_0\| + \int_0^t M e^{-ws} \sum_{i=1}^p \|\delta_i\| \|E_i + e\| \|x(s)\|^2 ds$$

$$\|x(t) e^{-wt}\| \leq M \|x_0\| + \int_0^t M e^{-ws} \sum_{i=1}^p \|\delta_i\| \|E_i + e\| \|x(s)\|^2 ds$$

$$\|x(t) e^{-wt}\| \leq M \|x_0\| + \int_0^t M e^{-ws} \sum_{i=1}^p \|\delta_i\| \|E_i + e\| \|x(s)\|^2 ds$$

$$\|x(t) e^{-wt}\| \leq M \|x_0\| + \int_0^t M e^{-ws} \sum_{i=1}^p \|\delta_i\| \|E_i + e\| \|x(s)\|^2 ds$$

Let's put:

$$M \|x_0\| = K; f(t) = a + \|B\| \|F\| c + \|G\| \|K_2^{-1}\| \|F\| c + g \|K_2^{-1}\| \|F\| c + g \|K_2^{-1}\| \|F\| \|C\| + g \|K_2^{-1}\| \|K_1\| + b \|F\| \|C\| + b \|F\| c$$

$$g(t) = M \left(\sum_{i=1}^p \|\delta_i\| \|E_i + e\| \right) e^{-ws}$$

$$m(t) = \|x(t) e^{-wt}\|$$

Therefore the previous inequality can be written under the following form:

$$\Rightarrow m(t) \leq K + \int_0^t f(s) m(s) ds + \int_0^t g(s) m^2(s) ds$$

(23)

Let us apply generalization of the Gronwall-Bellman lemma [5, 9]; for that, let us notice initially that the hypothesis (H) is verified. i.e.:



$1 - K \int_0^{+\infty} g(t) \cdot \exp. (\int_0^t f(s) ds) dt > 0$ That is equivalent to:

$$1 - M^2 \|x_0\| \int_0^t \left[\sum_{i=1}^p \|E_i\| \|C_i + c\| \right] e^{(w+\sigma)t} > 0$$

What returns while using (H1) to:

$$1 + \frac{M^2 \|x_0\| \int_0^t \left[\sum_{i=1}^p \|E_i\| \|C_i + c\| \right] e^{(w+\sigma)t}}{w + \sigma} > 0$$

What is verified by (H2).

Consequently, this generalization makes it possible to have:

$$\|x(t)e^{-wt}\| \leq M_0 \|x_0\| e^{\sigma t}$$

$$\|x(t)\| \leq M_0 \|x_0\| e^{(w+\sigma)t}$$

Where:

$$M_0 = 1 + \frac{M^2 \|x_0\| \int_0^t \left[\sum_{i=1}^p \|E_i\| \|C_i + c\| \right] e^{(w+\sigma)t}}{w + \sigma}$$

This result shows well that the control law elaborate continues to stabilize the system in spite of the variations of its parameters. What ensures its robustness relative to uncertainties.

5 ROBUSTNESS RELATIVE TO INPUT NOISE:

We consider the non linear perturbed system representing the satellite:

$$\dot{x}(t) = A_2 x(t) + B_2 u(t) + \sum_1^3 \left(\int_0^t u_i(s) ds \right) (E_i x(t)) + D.p(t) \tag{24}$$

Where:

$$x(0) = x_0$$

And :

$$A_2 = A - GK_2^{-1} K_1$$

$$B_2 = B - GK_2^{-1}$$

$x(t) \in \mathfrak{R}^n$ denotes the state vector ;

$u(t) = (u_1(t), u_2(t), u_3(t))$ the control.

$p(t)$ is the input noise. A, B, C, E_i, G and D are constant known matrices of appropriate dimensions.

Consider again the control by static output feedback

$$u(t) = -Fy$$

Thus, the system (24) can be written under the form:

$$\dot{x}(t) = F_0 x(t) + \sum_1^3 (\delta x(t))_i E_i x(t) + D.p(t)$$

(25)

Where:

$$F_0 = (A_2 - B_2 FC)$$

and :

$$\delta_i = (-K_2^{-1} K_1 + K_2^{-1} FC)_i$$

Theorem 5:

For the input noise such that:

$$\int_0^{+\infty} \|p(t) dt\| \leq -\frac{w}{M \|D\| \tau} \left(1 + \frac{M \|x_0\|}{w} \right)$$

(26)

The system (25) controlled by

$$u(t) = -Fy(t)$$

satisfies for all $t \geq 0$

$$\int_0^{+\infty} \|x(t)\| dt \leq -\frac{1}{\tau} \ln \left[1 + \frac{M \|x_0\| \tau}{w} + \frac{M \|D\| \tau}{w} \left(\int_0^{+\infty} \|p(t)\| dt \right) \right]$$

(27)

And then is stable.

Where: $\tau = M \sum_{i=1}^p \|\delta_i\| \|E_i\|$

And $\delta_i = (-K_2^{-1} K_1 + K_2^{-1} FC)_i$

Ln: denotes the Neperian logarithm.

Proof:

The solution of system (25) is given by:

$$x(t) = e^{F_0 t} x_0 + \int_0^t e^{F_0(t-s)} \left(\sum_{i=1}^p (\delta_i x(t))_i E_i x(t) \right) ds + \int_0^t e^{F_0(t-s)} D.p(s) ds$$

(28)

$$\Rightarrow \|x(t)\| \leq M e^{wt} \|x_0\| + \int_0^t (M e^{w(t-s)}) \left(\sum_{i=1}^p \|\delta_i\| \|E_i\| \|x(s)\|^2 ds + \int_0^t e^{w(t-s)} \|D\| \|p(s)\| ds \right)$$

$$\|x(t)e^{-wt}\| \leq M \|x_0\| + \int_0^t (M e^{ws}) \left(\sum_{i=1}^p \|\delta_i\| \|E_i\| \|x(s)e^{-ws}\|^2 ds + \int_0^t e^{-ws} \|D\| \|p(s)\| ds \right)$$

$$\|x(t) \cdot e^{-\alpha t}\| \leq l(t) + \int_0^t \tau \|x(t)\| \|x(s) e^{-\alpha s}\| ds$$

Where:

$$l(t) = K + \int_0^t M e^{-\alpha s} \|D\| \|p(s)\| ds$$

$$K = M \|x_0\|$$

$$\tau = M \sum_{i=1}^p \|\delta_i\| \|E_i\|$$

Using the Bellman-Gronwall's lemma [5, 9], we obtain:



$$\begin{aligned} \|x(t)e^{-\omega t}\| &\leq l(t) \exp\left[\int_0^t \tau \|x(s)\| ds\right] \\ \|x(t)\| \exp\left[-\int_0^t \tau \|x(s)\| ds\right] &\leq l(t)e^{\omega t} \\ \frac{d}{dt}\left(\exp\left[\int_0^t -\tau \|x(s)\| ds\right]\right) &\geq -\tau.l(t).e^{\omega t} \end{aligned}$$

(29)

Integrating from 0 to t:

$$\exp\left(\int_0^t -\tau \|x(s)\| ds\right) \geq 1 - \tau \int_0^t l(s)e^{\omega s} ds$$

$$so : g(t) = 1 - \tau \int_0^t l(s).e^{\omega s} ds. \quad t \in \mathbb{R}^+$$

g(t) Decreases from g(0)=1 to

$$\beta = 1 - \delta \int_0^{+\infty} e^{\omega s} l(s) ds;$$

Show that hypothesis (26) implies $\beta > 0$. then:

$$\begin{aligned} \int_0^{+\infty} e^{\omega s} l(s) ds &= \int_0^{+\infty} e^{\omega s} \left[M \|x_0\| + \int_0^s M e^{-\omega \alpha} \|D\| \|p(\alpha)\| d\alpha \right] ds \\ &= \int_0^{+\infty} e^{\omega s} M \|x_0\| ds + M \|D\| \int_0^{+\infty} e^{\omega s} \left(\int_0^s e^{-\omega \alpha} \|p(\alpha)\| d\alpha \right) ds \\ &= M \|x_0\| \int_0^{+\infty} e^{\omega s} ds + M \|D\| \int_0^{+\infty} \int_0^s e^{\omega s} . e^{-\omega \alpha} \|p(\alpha)\| d\alpha ds \end{aligned}$$

Using the theorem of Fubini, one obtains [9]:

$$\begin{aligned} \int_0^{+\infty} e^{\omega s} l(s) ds &= M \|x_0\| \int_0^{+\infty} e^{\omega s} ds + M \|D\| \int_0^{+\infty} e^{-\omega \alpha} \|p(\alpha)\| \left(\int_{\alpha}^{+\infty} e^{\omega s} ds \right) d\alpha \\ &= \frac{-M \|x_0\|}{\omega} - \frac{M \|D\|}{\omega} \left[\int_0^{+\infty} \|p(\alpha)\| d\alpha \right] \end{aligned}$$

(30)

$$So \beta = 1 - \tau \int_0^{+\infty} e^{\omega s} l(s) ds;$$

Then:

$$\beta = 1 + \frac{M \|x_0\| \tau}{\omega} + \frac{M \|D\| \tau}{\omega} \left[\int_0^{+\infty} \|p(\alpha)\| d\alpha \right]$$

It follows that hypothesis (20) that: $\beta > 0$.

And as: $g(t) = 1 - \tau \int_0^t l(s).e^{\omega s} ds$. is an increasing function. From $g(0)=1$ to $\beta > 0$, then :

$$\forall t \geq 0 \quad 1 - \tau \int_0^t l(s)e^{\omega s} ds > 0$$

Therefore:

$$\exp\left(\int_0^t \tau \|x(s)\| ds\right) \leq \frac{1}{1 - \tau \int_0^t e^{\omega s} l(s) ds}$$

Using (30), one obtains:

$$\begin{aligned} \|x(s)\| &\leq l(t)e^{\omega t} \exp\left[\int_0^t \tau \|x(s)\| ds\right] \\ \|x(t)\| &\leq \frac{l(t).e^{\omega t}}{1 - \tau \int_0^t e^{\omega s} l(s) ds} = -\frac{1}{\tau} \frac{d}{dt} \left[\ln\left(1 - \tau \int_0^t e^{\omega s} l(s) ds\right) \right] \end{aligned}$$

(30) Implies:

$$\int_0^{+\infty} \|x(t)\| dt \leq -\frac{1}{\tau} \ln \left[1 + \frac{M \|x_0\| \tau}{\omega} + \frac{M \|D\| \tau}{\omega} \left[\int_0^{+\infty} \|p(\alpha)\| d\alpha \right] \right].$$

So the static output feedback law $u(t) = -Fy$ is a robust control for the perturbed system (24).

6 CONCLUSION

We have obtained the necessary and sufficient condition for output feedback stabilizability for a micro satellite and an iterative algorithm has been presented to compute the stabilizing gain.

In the first part, we simplified the system by treating only its linear part; then we proposed a control law by static output feedback.

We generalized this control (SOF) to the non linear model of the micro satellite, and we studied the robustness of the control related to uncertainties of the parameters and to external variation of the disturbance.

The synthesis of the control suggested is based on the solution of the state space equation and on the Bellman-Gronwall lemma.

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