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SYNCHRONIZATION OF A CLASS OF COMPLEX NETWORKS WITH NETWORKED COUPLING CHANNELS

[1]WEIKE SHANG, [2]XINXIN FU, [3]JIN ZHU, [4]HONGSHENG XI
All the authors are with Department of Auto, University of Science and Technology of China, Hefei, 230027, Anhui, China
E-mail: [1]wufo@mail.ustc.edu.cn

ABSTRACT

The synchronization problem is investigated for a class of discrete-time complex networks, where the networking induced communication constraints between the coupling nodes are considered. By choosing a novel Lyapunov functional, sufficient conditions for the synchronization of the discrete-time complex networks subject to both coupling delays and packet dropouts are given. The derived criteria are presented in the form of linear matrix inequalities (LMIs), which is easy to be solved numerically. Finally, an illustrative example is presented to verify the effectiveness of the results.

Keywords: *Global synchronization, stochastic complex networks, coupling delays, packet dropouts*

1. INTRODUCTION

Complex networks have received considerable attention over the past decades. They are ideal mathematical models for various natural and engineering systems: cellular networks, social systems, the Internet, just to name a few [1]~[16]. The scientific interests in complex networks include, for example, the clustering characteristic (small world effect) and degree distributions, [1]~[3]; the stability and synchronization of complex networks, [4]~[16] and so on.

Due to the finite speeds of transmission and spreading, traffic congestions, as well as bandwidth restriction(for the Internet), information travelling through a complex network is often associated with time delay as well as packet dropouts. This is ubiquitous in biological, physical and engineering networks. Time-delay often causes instability of the system and thus has been investigated extensively in various contexts for complex networks: for deterministic systems the reader is referred to [7], [14]; for stochastic modeling and analysis please refer to [17]~[19]; besides, [20], [21] provide good hints for nonlinear systems. Packet dropouts have also received much attention in recent years, as they are inevitable in imperfect communication networks [22]~[25]. They are firstly treated separately from time-delays, but more recent models intend to combine the two effects together, which is more practical. In most studies, the packet dropout process is modeled as a Bernoulli process with appropriate variations in correspondence to the specific systems.

To the best of the authors' knowledge, the synchronization problem for complex networks subject to the communication constraints is still an open problem, which motivates the present work. The main contributions of the paper are threefold: (1) the imperfect coupling communication channels are considered for the first time in the synchronization context. (2) a stochastic model is utilized to describe the incomplete information phenomenon, taking both coupling delays and data packets dropout into account, and covering the aforementioned models as its special cases. (3) the stochastic analysis methods, the properties of Kronecker product, as well as the free-weight matrix techniques are employed to deal with the synchronization problem, and criterions in terms of linear matrix inequalities (LMIs) are given.

The rest of this paper is organized as follows. In Section 2, the discrete-time complex networks with coupling communication constraints induced by the imperfect networked channels is introduced, and then the problem under consideration is formulated. In Section 3, sufficient conditions are presented to guarantee the synchronization for the considered complex networks. An illustrative example is given in Section 4 to demonstrate the feasibility of the acquired criterion. Finally, concise conclusions are drawn in Section 5.

Notations: Throughout this paper, $\mathbb{R}^n$ is used to denote the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is

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the set of all $m \times n$ real matrices. The superscript 'T' denotes matrix transposition and '*' denotes the transpose of corresponding elements introduced by symmetry. $X > 0$ means that X is real symmetric and positive definite; Moreover, $X > Y$ means $X - Y > 0$. I and 0 represent the identity matrix and zero matrix with compatible dimensions, respectively. Given a matrix A, denote by $\| A \|$ its operator norm, i.e. $\| A \| = sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{max}(A^T A)}$, with $| \ |$ denoting the Euclidean norm on $\mathbb{R}^n$, and , $\lambda_{max}(Q)$ denoting the maximal eigenvalue of square matrix Q. $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k \in N})$ is a complete probability space with a filtration $\{\mathcal{F}_k\}$ satisfying the usual conditions (i.e. $\mathcal{F}_k$ contains all the P −null sets and it is right continuous). $\mathbb{E}\{\cdot\}$ is the mathematical expectation of a random variable with respect to the given probability space, and $\mathbb{E}\{\xi | \chi\}$ denotes the conditional mathematical expectation of a random variable $\xi \in \mathcal{F}$ with respect to the subfield created by χ (i.e. $\sigma(\chi)$). All the matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

2. PROBLEM FORMULATION AND PRELIMINARIES

The following discrete-time complex network with nonlinearity and multiple time-delays is considered

$$x_m(k+1) = Ax_m(k) + Bf\left(x_m(k)\right) + B_d g\left(x_m\left(k - d(k)\right)\right) + \sum_{\alpha=1}^{N} \sum_{p=0}^{r} \omega_{m\alpha}^{(p)} I_p(k) \Gamma_p x_\alpha(k - \tau_p(k)) \qquad m = 1,2,\cdots,N \tag{1}$$

where $x_m(k)$, $x_m\left(k - d(k)\right)$, $x_\alpha(k - \tau_p(k) \in \mathbb{R}^n$ denote the state vector, delayed state vector, and the coupling delayed vector of the mth node of the complex networks at time k respectively. A, B, B_d are all known matrices with compatible dimensions. $f, g: \mathbb{R}^n \to \mathbb{R}^n$ are both vector-valued nonlinear functions to be given later. $d(k)$, $\tau_p(k)$ are integers denoting the state time-delay and all the possible coupling delays which satisfy:

$$\underline{d} \le d(k) \le \overline{d} \tag{2}$$

$$\tau_0(k) = 0 \tag{3}$$

$$\underline{\tau_p} \le \tau_p(k) \le \overline{\tau_p} \quad p = 1,2,\cdots,r \tag{4}$$

where $\underline{d}$, $\overline{d}$, $\underline{\tau_p}$ and $\overline{\tau_p}$ $(p = 1,2,\cdots,r)$ are all known positive integers. Moreover we define that $d_{max} = \max_p(\overline{\tau_p}, \overline{d})$. Γ_p denotes the inner-coupling matrices linking the αth coupling state variable with time-delay $\tau_p(k)$, $W^{(p)} = (\omega_{m\alpha}^{(p)})_{N \times N}$ is the outer-coupling configuration matrix of the network with $\omega_{m\alpha,p} \ge 0 (m \ne \alpha)$, but not all zeros, and the coupling configuration matrix $W^{(p)} (p = 0,1,\cdots,r)$ is assumed to satisfy the diffusive connections:

$$\omega_{m\alpha}^{(p)} = \omega_{\alpha m}^{(p)}, m \ne \alpha$$

$$\sum_{m=1}^{N} \omega_{m\alpha}^{(p)} = \sum_{\alpha=1}^{N} \omega_{m\alpha}^{(p)} = 0,$$

$$m, \alpha = 1,2,\cdots,N; \ p = 0,1,\cdots,r$$

$I_p(k)$ is assumed to be a random variable satisfying certain discrete probabilistic distributions on the interval [0,1] which can be acquired from statistical tests, mutually unrelated with each other (for $p = 0,1,\cdots,r$) with mathematical expectation α_p and variance γ_p^2.

Assumption 2.1: The nonlinear function $f(x(k)), g(x(k)): \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be satisfying the following sector nonlinearity described as

$$[f(x) - f(y) - K_1(x-y)]^T [f(x) - f(y) - K_2(x-y)] \le 0, \forall x, y \in \mathbb{R}^n \tag{5}$$

$$[g(x) - g(y) - L_1(x-y)]^T [g(x) - g(y) - L_2(x-y)] \le 0, \forall x, y \in \mathbb{R}^n \tag{6}$$

both of which satisfy the zero initial condition, i.e. $f(0) = 0, g(0) = 0$, and K_1, K_2, L_1, L_2 are all known matrices satisfying $K_1 - K_2 < 0, L_1 - L_2 < 0$.

Remark 2.1: The model in(1) includes both delayed (for $p = 1,2,\cdots,r$) and non-delayed coupling (for $p = 0$,), which makes it very general, and can cover most of the existing models.

Remark 2.2: The system dynamics (1) is of much significance due to the following reasons: (1) As far as the authors know, there has been little work on the synchronization for complex networks subject to imperfect coupling channels (taking both coupling time-delays and packet dropouts into account); (2) the sequence of unrelated random variables with discrete probabilistic distributions used to describe the coupling information implies that multiple packets might be transferred between the nodes; it is different from the model presented in [23] and [24], which takes exclusive Bernoulli random variables and allows only one packet at time k, hence the model in this paper is more realistic. (3) the model in [25] is a special case of the model in this paper if we set the assumed

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sequence of random variables with arbitrary discrete probabilistic distributions in this paper defined on [0 1] in this paper to be Bernoulli distributions. The assumption in this paper is reasonable because in practice the transmitted information can be neither completely missing nor completely received, but only a part of the initial information can be transmitted successfully. In that case, the usually assumed Bernoulli distribution which only takes the completely successful case and the completely missing case in account is not quite suitable. Similar ideas can be referred to [26], [27], and [28], which focuses on the missing measurements without taking multiple time-delays into account.

Firstly, for simplicity denote

$$x(k) = col\{x_1(k), x_2(k), \cdots, x_N(k)\}$$

$$f(x(k)) = col\{f(x_1(k)), f(x_2(k)), \cdots, f(x_N(k))\}$$

$$g(x(k)) = col\{g(x_1(k)), g(x_2(k)), \cdots, g(x_N(k))\}$$

$$\alpha_p = E\{I_p(k)\}\ , \qquad \gamma_p^2 = \mathrm{E}\left\{\left|I_p(k) - \alpha_p\right|^2\right\}$$

$$\overline{\Gamma_p} = \alpha_p \times \Gamma_p \quad p = 0,1,\cdots r$$

$$W^{(p)} = (\omega_\alpha^{(p)})_{N\times N}$$

$$W^{(p)} W^{(q)} = W^{(p,q)} = (\omega_{m\alpha}^{(p,q)})_{N\times N} \ p,q = 0,1,\cdots r$$

By the Kronecker product '⊗' of matrix, the complex networks dynamics of (1) can be recast into the following compact form

$$\begin{aligned} x(k+1) = (I\otimes A + \mathrm{I}_0(k) \times W_0 \otimes \Gamma_0) x(k) \\ + (I\otimes B) f(x(k)) \\ + (I\otimes B_d) g\left(x\left(k - d(k)\right)\right) \\ + \sum_{p=1}^{r} \mathrm{I}_p(k) \times \left(W^{(p)} \otimes \Gamma_p\right) x(k \\ - \tau_p(k)) \end{aligned} \tag{7}$$

Moreover, to make the deduction more concise, the augmented complex networks can be denoted in the following form

$$\begin{aligned} x(k+1) = y(k) + \sum_{p=0}^{r} (\mathrm{I}_p(k) - \alpha_p) \\ \times \left(W^{(p)} \otimes \Gamma_p\right) x(k - \tau_p(k)) \end{aligned} \tag{8}$$

where

$$\begin{aligned} y(k) = \left(I\otimes A + W_0 \otimes \overline{\Gamma_0}\right) x(k) + (I\otimes B) f(x(k)) \\ + (I\otimes B_d) g\left(x\left(k - d(k)\right)\right) \\ + \sum_{p=1}^{r} \left(W^{(p)} \otimes \overline{\Gamma_p}\right) x(k - \tau_p(k)) \end{aligned}$$

Definition 2.1: [18],[19] The discrete-time stochastic complex network (1) is said to be asymptotically synchronized in the mean square sense if, for all the addressed communication constraints, it holds that

$$\lim_{k\to\infty} E\{|x_m(k) - x_\alpha(k)|^2\} = 0\ , \qquad 1 \le m < \alpha \le N \tag{9}$$

In this paper, we aim at presenting certain sufficient conditions for the stochastic synchronization problem between the nodes of a class of discrete-time stochastic complex network (1) subject to the aforementioned imperfect coupling channels. With the stochastic analysis method, as well as the free weight matrix technique, we construct a novel Lyapunov functional and develop an LMI approach to ensure the addressed stochastic complex networks to be synchronized in the mean square sense.

3. MAIN RESULTS

Before presenting the main results, we list the following useful lemmas.

Lemma 3.1: [18],[19] Let $U = (\alpha_{ij})_{N\times N}$ be a symmetric matrix satisfying that the sum of entries in each row of U is zero. $x = col\{x_1, x_2, \cdots, x_N\}$, $y = col\{y_1, y_2, \cdots, y_N\}$, $x_i, y_i \in \mathbb{R}^n, i = 1,2,\cdots, N$. $P \in \mathbb{R}^{n\times n}$. Then the following equality holds

$$x(U\otimes P)y = -\sum_{1\le i<j\le N} \alpha_{ij}(x_i - x_j)^T P (y_i - y_j)$$

Lemma 3.2: Let α be real scalar and A, B, C, D be matrices with compatible dimensions. Then the following properties of Kronecker product hold

$$\alpha(A\otimes B) = (\alpha A)\otimes B = A\otimes(\alpha B)$$

$$(A\otimes B)^T = A^T \otimes B^T$$

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$$

$$(A+B)\otimes(C+D) = A\otimes C + A\otimes D + B\otimes C + B\otimes D$$

Lemma 3.3: Let $A \in \mathbb{R}^{m\times p}, B \in \mathbb{R}^{p\times n}$ and $= AB$. If each column sum of A is zero or each row sum of B is zero, then each column sum or row sum of C is zero.

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Lemma 3.4: [29] For any random variable $\xi \in \mathcal{F}$ satisfying $\mathbb{E}\{|\xi|\} < +\infty$ and σ −field $\mathcal{g} \subset \mathcal{F}$, then it always holds that

$$\mathbb{E}\{\mathbb{E}\{\xi|\mathcal{g}\}\} = \mathbb{E}\{\xi\}$$

Now, we are ready to present the main results of the paper in the following.

Theorem 3.1: Under assumption (2.1), the stochastic complex networks described in (1) can be synchronized in the mean square if there exist positive matrices $P, Q, R_p, p = 1,2,\cdots r$, positive real numbers ρ_1, ρ_2, such that the following $\frac{N(N-1)}{2}$ LMIS hold.

$$\Upsilon_{m\alpha} = \begin{pmatrix} \Upsilon_{m\alpha,11} & \Upsilon_{m\alpha,12} \\ * & \Upsilon_{m\alpha,22} \end{pmatrix} < 0 \quad (1 \le m < \alpha \le N) \tag{10}$$

where

$$\Upsilon_{m\alpha,11} = \begin{pmatrix} \Theta_{m\alpha,11} & \Theta_{m\alpha,12} & \Theta_{m\alpha,13} & \Theta_{m\alpha,14} \\ * & \Theta_{m\alpha,22} & 0 & \Theta_{m\alpha,24} \\ * & * & \Theta_{m\alpha,33} & 0 \\ * & * & * & \Theta_{m\alpha,44} \end{pmatrix}$$

$$\begin{aligned}\Theta_{m\alpha,11} = {} & A^T PA - N\omega_{m\alpha}^{(0)}\left(A^T P\overline{\Gamma_0} + \overline{\Gamma_0}^T PA\right) \\ & - N\omega_{m\alpha}^{(0,0)}\overline{\Gamma_0}^T P\overline{\Gamma_0} \\ & - \gamma_0^2 N\omega_{m\alpha}^{(0,0)}\Gamma_0{}^T P\Gamma_0 - P \\ & + \sum_{p=1}^{r}\left(1 + \overline{\tau_p} - \underline{\tau_p}\right)R_p \\ & - \rho_1(K_1^T K_2 + K_2^T K_1) \\ & - \rho_2(L_1^T L_2 + L_2^T L_1)\end{aligned}$$

$$\Theta_{m\alpha,12} = A^T PB - N\omega_{m\alpha}^{(0)}\overline{\Gamma_0}^T PB + \rho_1(K_1^T + K_2^T)$$

$$\Theta_{m\alpha,13} = \rho_2(L_1^T + L_2^T)$$

$$\Theta_{m\alpha,14} = A^T PB_d - N\omega_{m\alpha}^{(0)}\overline{\Gamma_0}^T PB_d$$

$$\Theta_{m\alpha,22} = B^T PB - 2\rho_1 \times I$$

$$\Theta_{m\alpha,24} = B^T PB_d$$

$$\Theta_{m\alpha,33} = (1 + \overline{d} - \underline{d})Q - 2\rho_2 \times I$$

$$\Theta_{m\alpha,44} = B_d{}^T PB_d - Q$$

$$\Upsilon_{m\alpha,12} = \begin{pmatrix} \Phi_{m\alpha,11} & \Phi_{m\alpha,12} & \cdots & \Phi_{m\alpha,1r} \\ \Phi_{m\alpha,21} & \Phi_{m\alpha,22} & \cdots & \Phi_{m\alpha,2r} \\ 0 & 0 & \cdots & 0 \\ \Phi_{m\alpha,41} & \Phi_{m\alpha,42} & \cdots & \Phi_{m\alpha,4r} \end{pmatrix}$$

$$\Phi_{m\alpha,1p} = -N\omega_{m\alpha}^{(p)}A^T P\overline{\Gamma_p} - N\omega_{m\alpha}^{(0,p)}\overline{\Gamma_0}^T P\overline{\Gamma_p} \qquad p = 1,2,\cdots,r$$

$$\Phi_{m\alpha,2p} = -N\omega_{m\alpha}^{(p)}B^T P\overline{\Gamma_p} \qquad p = 1,2,\cdots,r$$

$$\Phi_{m\alpha,4p} = -N\omega_{m\alpha}^{(p)}B_d{}^T P\overline{\Gamma_p} \qquad p = 1,2,\cdots,r$$

$$\Upsilon_{m\alpha,22} = \begin{pmatrix} \Psi_{m\alpha,11} & \Psi_{m\alpha,12} & \cdots & \Psi_{m\alpha,1r} \\ * & \Psi_{m\alpha,22} & \cdots & \Psi_{m\alpha,2r} \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & \Psi_{m\alpha,rr} \end{pmatrix}$$

$$\Psi_{m\alpha,pp} = -R_p - N\omega_{m\alpha}^{(p,p)}\overline{\Gamma_p}^T P\overline{\Gamma_p} - \gamma_p^2 N\omega_{m\alpha}^{(p,p)}\Gamma_p{}^T P\Gamma_p \qquad p = 1,2,\cdots,r$$

$$\Psi_{m\alpha,pq} = -N\omega_{m\alpha}^{(p,q)}\overline{\Gamma_p}^T P\overline{\Gamma_q} \qquad 1 \le p < q \le r$$

Proof: We denote

$x_{m\alpha}(k) = x_m(k) - x_\alpha(k)$,

$f_{m\alpha}(x(k)) = f(x_m(k)) - f(x_\alpha(k))$,

$g_{m\alpha}(x(k)) = g(x_m(k)) - g(x_\alpha(k))$,

$g_{m\alpha}(x(k - d(k))) = g(x_m(k - d(k))) - g(x_\alpha(k - d(k)))$,

$x_{m\alpha}(k - \tau_p(k)) = x_m(k - \tau_p(k)) - x_\alpha(k - \tau_p(k))$, $\quad p = 1,2,\cdots,r$.

Let $\mathfrak{X}(k) \triangleq \{x(k), x(k-1), \cdots, x(k - d_{max})\}$, and consider the following Lyapunov candidate for the augmented system (7)

$$V(\mathfrak{X}(k)) = \sum_{i=1}^{5} V_i(\mathfrak{X}(k))$$

Where

$$V_1(\mathfrak{X}(k)) = x^T(k)(U \otimes P)x(k)$$

$$V_2(\mathfrak{X}(k)) = \sum_{v=k-d(k)}^{k-1} g^T(x(v))(U \otimes Q)g(x(v))$$

$$V_3(\mathfrak{X}(k)) = \sum_{l=\underline{d}}^{\overline{d}-1}\sum_{v=k-l}^{k-1} g^T(x(v))(U \otimes Q)g(x(v))$$

$$V_4(\mathfrak{X}(k)) = \sum_{p=1}^{r}\sum_{v=k-\tau_p(k)}^{k-1} x^T(v)(U \otimes R_p)x(v)$$

$$V_5(\mathfrak{X}(k)) = \sum_{p=1}^{r}\sum_{l=\underline{\tau_p}}^{\overline{\tau_p}-1}\sum_{v=k-l}^{k-1} x^T(v)(U \otimes R_p)x(v)$$

with $P, Q, R_p, p = 1,2,\cdots,r$ being positive definite matrices to be determined, and U being defined as

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$$U=\begin{pmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{pmatrix} \quad (11)$$

By calculating the difference of $V(\mathfrak{X}(k))$ along the solutions of the augmented complex networks (7) and taking the mathematical expectation condition $\mathfrak{X}(k)$, we have

$$\mathbb{E}\{\Delta V(\mathfrak{X}(k))|\mathfrak{X}(k)\}=\sum_{i=1}^{5}\mathbb{E}\{\Delta V_i(\mathfrak{X}(k))|\mathfrak{X}(k)\} \quad (12)$$

Then, one has

$$\begin{aligned}&\mathbb{E}\{\Delta V_1(\mathfrak{X}(k))|\mathfrak{X}(k)\}\\&=\mathbb{E}\{V_1(\mathfrak{X}(k+1))|\mathfrak{X}(k)\}-V_1(\mathfrak{X}(k))\\&=\mathbb{E}\{x^T(k+1)(U\otimes P)x(k+1)|\mathfrak{X}(k)\}\\&\qquad -x^T(k)(U\otimes P)x(k)\\&=\mathbb{E}\left\{\left[y(k)+\sum_{p=0}^{r}(I_p(k)-\alpha_p)(W^{(p)}\otimes\Gamma_p)x(k-\tau_p)\right]^T(U\otimes P)\left[y(k)+\sum_{p=0}^{r}(I_p(k)-\alpha_p)(W^{(p)}\otimes\Gamma_p)x(k-\tau_p)\right]|\mathfrak{X}(k)\right\}-x^T(k)(U\otimes P)x(k)\end{aligned}$$

It is worth pointing out that $y(k), I_p(k)-\alpha_p$ are both measurable with respect to $\sigma(\mathfrak{X}(k))$. Then it follows

$$\begin{aligned}&\mathbb{E}\{\Delta V_1(\mathfrak{X}(k))|\mathfrak{X}(k)\}\\&=\mathbb{E}\left\{y^T(k)(U\otimes P)y(k)+\sum_{p=0}^{r}\gamma_p^2\times\left[(W^{(p)}\otimes\Gamma_p)x(k-\tau_p)\right]^T(U\otimes P)\left[(W^{(p)}\otimes\Gamma_p)x(k-\tau_p)\right]-x^T(k)(U\otimes P)x(k)|\mathfrak{X}(k)\right\}\end{aligned} \quad (13)$$

It is noted that the former deduction has used the fact that $I_p(k), p=0,1,\cdots,r$ are mutually unrelated with each other.

$$\begin{aligned}&\mathbb{E}\{\Delta V_2(\mathfrak{X}(k))|\mathfrak{X}(k)\}\\&=\mathbb{E}\{V_2(\mathfrak{X}(k+1))|\mathfrak{X}(k)\}-V_2(\mathfrak{X}(k))\\&=\mathbb{E}\{\Big(\sum_{v=k+1-d(k+1)}^{k}-\sum_{v=k-d(k)}^{k-1}\Big)\times g^T(x(v))(U\otimes Q)g(x(v))|\mathfrak{X}(k)\}\\&=\mathbb{E}\{g^T(x(k))(U\otimes Q)g(x(k))\\&\qquad -g^T\big(x(k-d(k))\big)(U\otimes Q)g\big(x(k-d(k))\big)\\&\qquad +\sum_{v=k+1-d(k+1)}^{k-d(k)}g^T(x(v))(U\otimes Q)g(x(v))|\mathfrak{X}(k)\}\\&\le\mathbb{E}\{g^T(x(k))(U\otimes Q)g(x(k))\\&\qquad -g^T\big(x(k-d(k))\big)(U\otimes Q)g\big(x(k-d(k))\big)\\&\qquad +\sum_{v=k+1-\bar{d}}^{k-\underline{d}}g^T(x(v))(U\otimes Q)g(x(v))|\mathfrak{X}(k)\}\end{aligned} \quad (14)$$

$$\begin{aligned}&\mathbb{E}\{\Delta V_3(\mathfrak{X}(k))|\mathfrak{X}(k)\}\\&=\mathbb{E}\{V_3(\mathfrak{X}(k+1))|\mathfrak{X}(k)\}-V_3(\mathfrak{X}(k))\\&=\mathbb{E}\{\sum_{l=\underline{d}}^{\bar{d}-1}\Big(\sum_{v=k+1-l}^{k}-\sum_{v=k-l}^{k-1}\Big)\times g^T(x(v))(U\otimes Q)g(x(v))|\mathfrak{X}(k)\}\\&=\mathbb{E}\{(\bar{d}-\underline{d})g^T(x(k))(U\otimes Q)g(x(k))\\&\qquad -\sum_{l=\underline{d}}^{\bar{d}-1}g^T(k-l)(U\otimes Q)g(k-l)|\mathfrak{X}(k)\}\\&=\mathbb{E}\{(\bar{d}-\underline{d})g^T(x(k))(U\otimes Q)g(x(k))\\&\qquad -\sum_{v=k+1-\bar{d}}^{k-\underline{d}}g^T(x(v))(U\otimes Q)g(x(v))|\mathfrak{X}(k)\}\end{aligned} \quad (15)$$

Similarly, one has

$$\begin{aligned}&\mathbb{E}\{\Delta V_4(\mathfrak{X}(k))|\mathfrak{X}(k)\}\\&=\mathbb{E}\{V_4(\mathfrak{X}(k+1))|\mathfrak{X}(k)\}-V_4(\mathfrak{X}(k))\\&\le\mathbb{E}\{\sum_{p=1}^{r}(x^T(k)(U\otimes R_p)x(k)\\&\qquad -x^T(k-\tau_p(k))(U\otimes R_p)x(k-\tau_p(k)))\end{aligned}$$

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$$+\sum_{p=1}^{r}\sum_{v=k+1-\overline{\tau_p}}^{v=k-\underline{\tau_p}} x^T(v)(U\otimes R_p)x(v)|\,\mathfrak{X}(k)\} \quad (16)$$

$$\mathbb{E}\{\Delta V_5(\mathfrak{X}(k))|\mathfrak{X}(k)\}$$
$$=\mathbb{E}\{V_5(\mathfrak{X}(k+1))|\mathfrak{X}(k)\}-V_5(\mathfrak{X}(k))$$
$$=\mathbb{E}\{\sum_{p=1}^{r}\sum_{l=\underline{\tau_p}}^{\overline{\tau_p}-1}(x^T(k)(U\otimes R_p)x(k) -x^T(k-l)(U\otimes R_p)x(k-l))|\mathfrak{X}(k)\}$$
$$=\mathbb{E}\{\sum_{p=1}^{r}(\overline{\tau_p}-\underline{\tau_p})x^T(k)(U\otimes R_p)x(k) -\sum_{p=1}^{r}\sum_{v=k+1-\overline{\tau_p}}^{v=k-\underline{\tau_p}} x^T(v)(U\otimes R_p)x(v)\,|\mathfrak{X}(k)\} \quad (17)$$

It is easy deduced that $W^{(p)}U=UW^{(p)}=NW^{(p)}$, hence it follows

$$(W^{(p)}\otimes\overline{\Gamma_p})^T(U\otimes P)(W^{(q)}\otimes\overline{\Gamma_q}) = NW^{(p,q)}\otimes\overline{\Gamma_p}^T P\overline{\Gamma_q} \qquad p,q=0,1,\cdots,r$$

In view of lemma (3.1), when (13)~(17) are substituted into (12), one can have that

$$\mathbb{E}\{\Delta V(\mathfrak{X}(k))|\mathfrak{X}(k)\}$$
$$=\mathbb{E}\{V(\mathfrak{X}(k+1))|\mathfrak{X}(k)\}-V(\mathfrak{X}(k))$$
$$\le \sum_{1\le m<\alpha\le N}\mathbb{E}\{x_{m\alpha}^T(k)[(A^TPA - N\omega_{m\alpha}^{(0)}\left(A^TP\overline{\Gamma_0}+\overline{\Gamma_0}^TPA\right) -N\omega_{m\alpha}^{(0,0)}\overline{\Gamma_0}^TP\overline{\Gamma_0} -\gamma_0^2N\omega_{m\alpha}^{(0,0)}\overline{\Gamma_0}^TP\overline{\Gamma_0}-P +\sum_{p=1}^{r}\left(1+\overline{\tau_p}-\underline{\tau_p}\right)R_p)x_{m\alpha}(k)$$
$$+2(A^TPB-N\omega_{m\alpha}^{(0)}\overline{\Gamma_0}^TPB)f_{m\alpha}(x(k))$$
$$+2(A^TPB_d-N\omega_{m\alpha}^{(0)}\overline{\Gamma_0}^TPB_d)g_{m\alpha}(x(k-d(k)))$$
$$+2\sum_{p=1}^{r}\left(-N\omega_{m\alpha}^{(p)}A^TP\overline{\Gamma_p}-N\omega_{m\alpha}^{(0,p)}\overline{\Gamma_0}^TP\overline{\Gamma_p}\right)x_{m\alpha}(k-\tau_p(k))]$$
$$+f_{m\alpha}^T(x(k))[B^TPBf_{m\alpha}(x(k)) + 2B^TPB_dg_{m\alpha}\left(x(k-d(k))\right) + 2\sum_{p=1}^{r}\left(-N\omega_{m\alpha}^{(p)}B^TP\overline{\Gamma_p}\right)x_{m\alpha}(k-\tau_p(k))]$$
$$+g_{m\alpha}^T(x(k))[(1+\overline{d}-\underline{d})Q]g_{m\alpha}(x(k))$$
$$+g_{m\alpha}^T(x(k-d(k)))[(B_d{}^TPB_d - Q)g_{m\alpha}\left(x(k-d(k))\right) + 2\sum_{p=1}^{r}\left(-N\omega_{m\alpha}^{(p)}B_d{}^TP\overline{\Gamma_p}\right)x_{m\alpha}(k-\tau_p(k))]$$
$$+\sum_{p=1}^{r}\sum_{q=1}^{r}x_{m\alpha}^T\left(k-\tau_p(k)\right)(-N\omega_{m\alpha}^{(p,q)}\overline{\Gamma_p}^TP\overline{\Gamma_q}) \times x_{m\alpha}(k-\tau_q(k))$$
$$+\sum_{p=1}^{r}x_{m\alpha}^T\left(k-\tau_p(k)\right)\left(-R_p-\gamma_p^2N\omega_{m\alpha}^{(p,p)}\overline{\Gamma_p}^TP\overline{\Gamma_p}\right) \times x_{m\alpha}\left(k-\tau_p(k)\right)|\mathfrak{X}(k)\} \quad (18)$$

Owing to assumption (2.1), we have the following inequalities

$$\begin{bmatrix} x_m(k)-x_\alpha(k) \\ f(x_m(k))-f(x_\alpha(k))\end{bmatrix}^T \times\begin{bmatrix} -(K_1^TK_2+K_2^TK_1) & K_1^T+K_2^T \\ K_1+K_2 & -2I\end{bmatrix} \times\begin{bmatrix} x_m(k)-x_\alpha(k) \\ f(x_m(k))-f(x_\alpha(k))\end{bmatrix}\ge 0 \quad (19)$$

and

$$\begin{bmatrix} x_m(k)-x_\alpha(k) \\ g(x_m(k))-g(x_\alpha(k))\end{bmatrix}^T \times\begin{bmatrix} -(L_1^TL_2+L_2^TL_1) & L_1^T+L_2^T \\ L_1+L_2 & -2I\end{bmatrix} \times\begin{bmatrix} x_m(k)-x_\alpha(k) \\ g(x_m(k))-g(x_\alpha(k))\end{bmatrix}\ge 0 \quad (20)$$

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Note that(19) and (20) respectively imply

$$\begin{bmatrix} x_{m\alpha}(k) \\ f_{m\alpha}(x(k)) \end{bmatrix}^T \times \begin{bmatrix} -(K_1^T K_2 + K_2^T K_1) & K_1^T + K_2^T \\ K_1 + K_2 & -2I \end{bmatrix} \times \begin{bmatrix} x_{m\alpha}(k) \\ f_{m\alpha}(x(k)) \end{bmatrix} \geq 0 \quad (21)$$

and

$$\begin{bmatrix} x_{m\alpha}(k) \\ g_{m\alpha}(x(k)) \end{bmatrix}^T \times \begin{bmatrix} -(L_1^T L_2 + L_2^T L_1) & L_1^T + L_2^T \\ L_1 + L_2 & -2I \end{bmatrix} \times \begin{bmatrix} x_{m\alpha}(k) \\ g_{m\alpha}(x(k)) \end{bmatrix} \geq 0 \quad (22)$$

Then Multiplying (21) and (22) with ρ_1 and ρ_2 and substituting them into (18) yields

$$\begin{aligned} &\mathbb{E}\{\Delta V(\mathfrak{X}(k))|\mathfrak{X}(k)\} \\ &\quad = \mathbb{E}\{V(\mathfrak{X}(k+1))|\mathfrak{X}(k)\} - V(\mathfrak{X}(k)) \\ &\quad \leq \sum_{1\leq m<\alpha\leq N} \xi_{m\alpha}^T(k) \Upsilon_{m\alpha}\, \xi_{m\alpha}(k) \end{aligned} \quad (23)$$

where $\xi_{m\alpha}(k)$ is defined as

$$\begin{aligned} \xi_{m\alpha}(k) = [&x_{m\alpha}^T(k), f_{m\alpha}^T(x(k)), g_{m\alpha}^T(x(k)), \\ &g_{m\alpha}^T\left(x(k-d(k))\right), x_{m\alpha}^T(k-\tau_1(k)), \\ &x_{m\alpha}^T(k-\tau_2(k)), \cdots, x_{m\alpha}^T(k-\tau_r(k))]^T \end{aligned}$$

From lemma (3.4), it follows readily that

$$\begin{aligned} &\mathbb{E}\{V(\mathfrak{X}(k+1)) - V(\mathfrak{X}(k))\} \\ &\leq c_0 \sum_{1\leq m<\alpha\leq N} \mathbb{E}\{|\xi_{m\alpha}(k)|^2\} \end{aligned} \quad (24)$$

where $c_0 = \max_{1\leq m<\alpha\leq N}\{\lambda_{max}(\Upsilon_{m\alpha})\} < 0$. Note that $\|\xi_{m\alpha}(k)\|^2 \geq \|x_{m\alpha}(k)\|^2$, then it can be deduced that

$$\begin{aligned} &\mathbb{E}\{V(\mathfrak{X}(k+1)) - V(\mathfrak{X}(k))\} \\ &\leq c_0 \sum_{1\leq m<\alpha\leq N} \mathbb{E}\{\|x_{m\alpha}(k)\|^2\} \end{aligned} \quad (25)$$

For any positive integer n, add both the sides of the inequality (25) from 0 to n, we have

$$\begin{aligned} &\mathbb{E}\{V(\mathfrak{X}(n+1)) - V(\mathfrak{X}(0))\} \\ &\leq c_0 \sum_{k=0}^{n} \sum_{1\leq m<\alpha\leq N} \mathbb{E}\{\|x_{m\alpha}(k)\|^2\} \end{aligned} \quad (26)$$

Hence it follows

$$\sum_{k=0}^{n} \sum_{1\leq m<\alpha\leq N} \mathbb{E}\{\|x_{m\alpha}(k)\|^2\} \leq \frac{V(\mathfrak{X}(0))}{-c_0} < +\infty \quad (27)$$

Let $n \to +\infty$, it can be concluded that the positive series

$$\sum_{k=0}^{+\infty} \sum_{1\leq m<\alpha\leq N} \mathbb{E}\{\|x_{m\alpha}(k)\|^2\}$$

is convergent. Hence

$$\lim_{k\to+\infty} \sum_{1\leq m<\alpha\leq N} \mathbb{E}\{\|x_{m\alpha}(k)\|^2\} = 0$$

which obviously implies

$$\lim_{k\to+\infty} \mathbb{E}\{\|x_{m\alpha}(k)\|^2\} = \lim_{k\to+\infty} \mathbb{E}\{|x_m(k) - x_\alpha(k)|^2\} = 0 \qquad (1 \leq m < \alpha \leq N)$$

This completes the proof.

Remark 3.1: The results presented in this work are preliminary. For example, we have assumed the linear coupling between the nodes and that all the communication channels obey the same packet dropout distribution. In this sense much work is still to be done, for example, channels with completely different communication conditions, nodes with nonlinear coupling, and so forth.

Remark 3.2: This paper only talks about the synchronization criteria of the synchronization problems with the given complex networks subject to imperfect networked coupling channels. For the time-delays and data dropouts between the coupling nodes, how to design the compensation technique and adaptive filters and controllers to ensure the synchronization of the networks is under investigation in our future work.

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4. NUMERICAL EXAMPLES

In this section, we shall give a numerical example to show the effectiveness of the criteria derived in this paper.

Consider a discrete-time complex networks with 3 nodes which is described by the following dynamics

$$A = \begin{pmatrix} 0.29 & -0.26 \\ 0.5 & 0.23 \end{pmatrix}$$
$$B = \begin{pmatrix} 0.25 & 0.75 \\ 0.35 & 0.25 \end{pmatrix}$$
$$B_d = \begin{pmatrix} 0.13 & 0.14 \\ 0.44 & 0.23 \end{pmatrix}$$

$d(k) = 2 + \sin(\frac{\pi}{2}k)$ is the state time-delay, which implies $\overline{d} = 3$ and $\underline{d} = 1$. There exist 3 possible coupling delayed states, i.e. $r = 3$, and all the possible coupling delays are described as $\tau_1(k) = 2 + \sin(\frac{\pi}{2}k)$, $\tau_2(k) = 3 + \cos(\frac{\pi}{2}k)$, $\tau_3(k) = 3 + 2\,sin(\frac{\pi}{2}k)$, with $\underline{\tau_1} = 1$, $\overline{\tau_1} = 3$; $\underline{\tau_2} = 2$, $\overline{\tau_2} = 4$; $\underline{\tau_3} = 1$, $\overline{\tau_3} = 5$.

It is also assumed that

$$W_0 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, W_1 = W_2 = W_3 = 0.5W_0$$

$$\Gamma_0 = \begin{pmatrix} 0.28 & 0 \\ 0 & 0.22 \end{pmatrix}, \Gamma_1 = \Gamma_2 = \Gamma_3 = 0.4\Gamma_0$$

$$f(x_m(k)) = g(x_m(k)) = \begin{pmatrix} 0.2x_m^{(1)}(k) + \tanh(0.1x_m^{(1)}(k)) \\ 0.3x_m^{(2)}(k) - \tanh(0.1x_m^{(2)}(k)) \end{pmatrix}$$

where $x_m(k) = [x_m^{(1)}(k), x_m^{(2)}(k)]^T$. Hence, it is easy to verify that $f(x_m(k))$ and $g(x_m(k))$ all satisfy the sector nonlinearity assumption with

$$K_1 = L_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, K_2 = L_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}$$

To describe the multiple packet dropouts phenomenon, $I_P(k), p = 0,1,2,3$ is assumed to be the following discrete distributions:

$$Prob(I_0(k)) = \begin{cases} 0.5 & I_0(k) = 0 \\ 0.5 & I_0(k) = 1 \end{cases}$$

$$Prob(I_p(k)) = \begin{cases} 0.4 & I_p(k) = 0 \\ 0.2 & I_p(k) = 0.5 \\ 0.4 & I_p(k) = 1 \end{cases} \quad p = 1,2,3$$

Then by theorem 3.1, using the Matlab LMI tool box, we can find a feasible solution with the solved parameters listed as follows

$$P = \begin{pmatrix} 907.5793 & -95.8237 \\ -95.8237 & 267.2214 \end{pmatrix}$$
$$Q = \begin{pmatrix} 253.2414 & 39.7640 \\ 39.7640 & 163.4829 \end{pmatrix}$$
$$R_1 = \begin{pmatrix} 44.7353 & -6.8574 \\ -6.8574 & 13.7844 \end{pmatrix}$$
$$R_2 = \begin{pmatrix} 44.7353 & -6.8574 \\ -6.8574 & 13.7844 \end{pmatrix}$$
$$R_3 = \begin{pmatrix} 32.7381 & -4.6748 \\ -4.6748 & 9.8146 \end{pmatrix}$$
$$\rho_1 = 1.2439e + 003, \quad \rho_2 = 961.0925$$

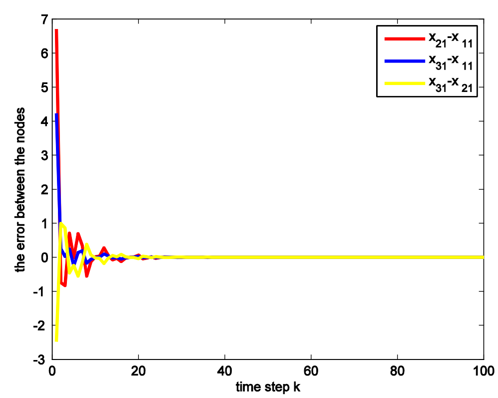

Figure 1: The Synchronization Errors Between The Nodes (The First Entry Of The State)

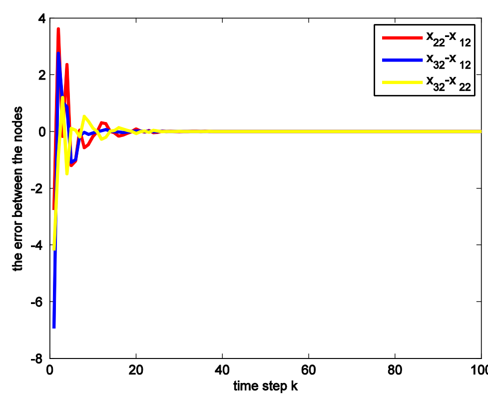

Figure 2: The Synchronization Errors Between The Nodes (The Second Entry Of The State)

It is shown that the investigated complex network subjected to communication constraints satisfies the conditions of theorem3.1, hence can reach asymptotical synchronization in mean square. This is clearly demonstrated in Fig.1 and Fig.2.

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5. CONCLUSIONS

Efforts have been made to investigate the synchronization problems of an array of coupled discrete-time complex networks subject to nonlinearity, mixed time-delays as well as communication constraints. By choosing a new Lyapunov function, we have derived the criteria under which the investigated complex networks can reach global synchronization in mean square. A numerical example illustrates the effectiveness of the results. Future works will focus on, for example, channels in completely differently adverse communication environments, nodes with complex nonlinear coupling, adaptive controllers design and so on.

REFRENCES:

[1] D.J. Watts, S.H. Strogatz, "Collective dynamics of 'small-world' network", *Nature*, Vol. 393, 1998, pp. 440-442.

[2] A.L. Barabási, R. Albert, "Emergence of scaling in random networks", *Science*, Vol. 286, 1999, pp. 509-512.

[3] C.G. Li, G.R. Chen, "A comprehensive weighted evolving network model", *Physica A*, Vol. 343, 2004, pp. 288-294.

[4] X.F. Wang, G.R. Chen, "Synchronization in small-world dynamical networks", *Int. J. Bifurcat. Chaos*, Vol. 12, 2002, pp. 187-192.

[5] X.F. Wang, G.R. Chen, "Synchronization in scale-free dynamical networks: robustness and fragility", *IEEE Trans. CAS-I*, Vol. 49, 2002, pp. 54-62.

[6] Y.Q. Yang, X.F. Wu, "Global finite-time synchronization of a class of the non-autonomous chaotic systems", *Nonlinear Dyn*, Vol. 70, 2012, pp. 197-208.

[7] C.G. Li, G.R. Chen, "Synchronization in general complex dynamical networks with coupling delays", *Physica A*, Vol. 343, 2004, pp. 263-278.

[8] W.L. Lu and T.P. Chen, "Synchronization of Coupled Connected Neural Networks With Delays", *IEEE Trans. Circuits Syst. I*, Vol. 51, 2004, pp. 2491-2503.

[9] W.L. Lu and T.P. Chen, "Synchronization analysis of linearly coupled networks of discrete time systems", *Physica D*, Vol. 198, 2004, pp. 148-168.

[10] X.D. Li, X.L. Fu, "Lag synchronization of chaotic delayed neural networks via impulsive control", *IMA J. Math. Control Inform.*, Vol. 29, 2012, pp. 133-145.

[11] J.L. Wang, H.N. Wu, "Global finite-time synchronization of a class of the non-autonomous chaotic systems", *Nonlinear Dyn*, Vol. 70, 2012, pp. 13-24.

[12] J.Y. Cao, G.Z. Wanga, Q.Y. Jianga, Z.X. Han, "A neighbourhood evolving network model", *Phys. Lett. A*, Vol. 349, 2006, pp. 462-466.

[13] J.D. Cao, W.W. Yu, Y.Z. Qu, "New complex network model and convergence dynamical for reputation computation in virtual organizations", *Phys. Lett. A*, Vol. 356, 2006, pp. 414-425.

[14] H.J. Gao, J. Lam, G. Chen, "New criteria for synchronization stability of general complex dynamical networks with coupling delays", *Phys. Lett. A*, Vol. 360, 2006, pp. 263-273.

[15] C.P. Li, W.G. Sun, J. Kurths, "Synchronization of complex dynamical networks with time delays", *Physica A*, Vol. 361, 2006, pp. 24-34.

[16] Z.S. Duan, G.R. Chen, L. Huang, "Synchronization of weighted networks and complex synchronized regions", *Phys. Lett. A*, Vol. 372, 2008, pp. 3741-3751.

[17] Y. Wang, Z.D. Wang, J.L. Liang, "A delay fractioning approach to global synchronization of delayed complex networks with stochastic disturbances", *Phys. Lett. A*, Vol. 372, 2008, pp. 6066-6073.

[18] Y.R. Liu, Z.D. Wang, J.L. Liang, X.H. Liu, "Stability and synchronization of discrete-time Markovian jumping neural networks with mixed mode-dependent time delays", *IEEE Trans. Neural Networks*, Vol. 20, 2009, pp. 1102-1116.

[19] Z.D. Wang, Y. Wang, Y.R. Liu, "Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays", *IEEE Trans. Neural Networks*, Vol. 21, 2010, pp. 11-25.

[20] C.C. Hua, Q.G. Wang, X.P. Guan, "Global adaptive synchronization of complex networks with nonlinear delay coupling", *Phys. Lett. A*, Vol. 368, 2007, pp. 281-288.

[21] J. Zhou, J.A. Lu, J.h. Lü, "Adaptive Synchronization of an Uncertain Complex Dynamical Network", *IEEE Trans. Automat. Contr.*, Vol. 51, 2005, pp. 652-656.

[22] G.P. Liu, Y.Q. Xia, D. Rees, W. S. Hu, "Design and Stability Criteria of Networked Predictive Control Systems With Random Network Delay in the Feedback Channel", *IEEE Trans. Syst.*,

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Man, Cybern. C, Appl. Rev., Vol. 37, 2007, pp. 173-184.

[23] X. He, Z.D. Wang, and D.H. Zhou, "Robust H1 filtering for networked systems with multiple state delays", *Int. J. Control*, Vol. 80, 2007, pp. 1217-1232.

[24] G.L. Wei, Z.D. Wang, X. He, H.S. Shu, "Filtering for networked stochastic time-delay systems with sector nonlinearity", *IEEE Trans. Circuits Syst. II, Exp. Briefs*, Vol. 56, 2009, pp. 71-75.

[25] J.L. Liang, B. Shen, H.L. Dong, L. James, "Robust distributed state estimation for sensor networks with multiple stochastic communication delays", *Int. J. Sys. Sci.*, Vol. 9, 2011, pp. 1459-1471.

[26] G.L. Wei, Z.D. Wang, H.S. Shu, "Robust filtering with stochastic nonlinearities and multiple missing measurements", *Automatica*, Vol. 45, 2009, pp. 836-841.

[27] H.L. Dong, Z.D. Wang, D. W. C. Ho, H.J. Gao, "Variance-constrained H1 filtering for a class of nonlinear time-varying systems with multiple missing measurements: the finite-horizon case", *IEEE Trans. Signal Process.*, Vol. 58, 2010, pp. 2534-2543.

[28] L. Ma, F.P Da, K.J. Zhang, "Exponential H_∞ filter design for discrete time-delay stochastic systems with Markovian jump parameters and missing measurements", I*EEE Trans. CAS-I*, Vol. 58, 2011, pp. 994-1007.

[29] K. Chung. "A Course in Probability Theory", third edition, San Diego, Academic Press, 2011.