NEW STABILITY CONDITIONS FOR NEUTRAL SYSTEMS WITH DISTRIBUTED DELAYS

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ABSTRACT

The certain and uncertain neutral systems with distributed delays are investigated in this paper. The uncertainties under consideration are time-varying, but norm bounded. By the delay-dividing approach, firstly new asymptotic stability conditions for the certain neutral systems with distributed delays are given by Lyapunov method. Then the asymptotic stability conditions for the uncertain case are obtained subsequently. Numerical examples illustrate that the proposed criteria are effective and lead to less conservative results than existing ones.

Keywords: Neutral Systems, Robust Stability, Distributed Delays, Asymptotic Stability

1. INTRODUCTION

The problem of stability of time-delay systems of neutral type has received considerable attention in the last two decades. To start with, the delay-dependent stability criteria can be seen in [4, 8] by the model transformation method. Next, matrix decomposition method is applied to neutral systems in [1, 11]. This method divides the delayed terms into two groups (the stabilizing ones and destabilizing ones) and enables one to consider the stabilizing effect of part of the delayed terms, which caused less conservative results than before. Besides, discrete delay bi-decomposition approach is introduced in [13] on the basis of the results in [12].

However, for some systems, delay phenomena may not be simply considered as delays in the velocity terms or discrete delays in the states. Therefore, it is desirable to extend the system model to include distributed delays and stability analysis for them is of both practical and theoretical importance. In the recent papers by Chen & Zheng [14] and Han [11], a descriptor system approach (see Fridman [2] and [3]) has been used to investigate the stability of neutral systems with discrete and distributed delays. Han [11] rewrites the discrete-delay term and employs a decomposition technique [1]. Different from Han [11], Chen and Zheng [14] rewrite both the discrete-delay and the distributed-delay terms and apply Moon’s inequality [16]. In addition, since a new form of Lyapunov functional including some triple-integral terms, discrete, neutral, and distributed-delay dependent criteria have been proposed, the augmented vector and triple-integral terms play key roles in the reduction of conservativeness [6].

In this paper, asymptotic stability of uncertain neutral systems with discrete and distributed delays are considered. The delay decomposition approach is used and a novel Lyapunov functional is proposed. Not only discrete delay interval but also neutral delay interval are divided, at the same time a triple-integral term is employed for distributed delays. By linear matrix inequality approach, new stability criteria for neutral systems with distributed delays are derived. The resultant stability criteria are less conservative. Numerical examples are given to show the reduced conservativeness.

The remainder of this paper is organized as follows: Section 2 contains the problem statement and preliminaries; Section 3 presents the main results; Section 4 provides two numerical examples to verify the effectiveness of the results; Section 5 draws a brief conclusion.

1.1 Notations

Throughout this paper, * denotes the elements below the main diagonal of a symmetric block matrix, $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is for the set of all $m \times n$ matrices. $A^T$ and $A^{-1}$ denote the transpose and the inverse of a matrix $A$. The notation $X \geq Y$ (or $X > Y$) means...
that $X$ and $Y$ are symmetric matrices, and that $X-Y$ is positive semidefinite (or positive definite).

In this paper, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. If $A$ is a matrix, $\|A\|$ is its spectral norm. i.e.
\[\|A\| = \max \{ |Ax| : |x| = 1 \} = \sqrt{\lambda_{\max}(A^TA)}\]
where $\lambda_{\max}(A)$ (or $\lambda_{\min}(A)$) means the largest (or smallest) eigenvalue of the matrix $A$. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider a class of neutral systems with discrete and distributed delays which are described by the following:
\[
\dot{x}(t) - D(t)\dot{x}(t - \tau) = A(t)x(t) + B(t)x(t - h) + C(t)\int_{t-\tau}^{t} x(s)ds + F(t) + H(t)\chi(t)
\]
\[x(t) = \phi(t), \quad t \in [-\rho, 0]\]
(1)
where $x(t)$ is the state, $\tau > 0$, $h > 0$ and $r \geq 0$ ($r \neq \tau, h$) are constant neutral, discrete and distributed delays.
\[\rho = \max\{\tau, h, r\}\] and the initial condition $\phi(t)$ is a continuously differentiable vector-valued function.

The continuous norm of $\phi(t)$ is defined as
\[\|\phi\| = \max_{t \in [-\rho, 0]}|\phi(t)|.\]

$A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times n}$, $C(t) \in \mathbb{R}^{n \times n}$, $D(t) \in \mathbb{R}^{n \times n}$ are uncertain matrices. We assume that uncertainties are norm-bounded and can be described as:
\[
A(t) = A + \Delta A(t),
B(t) = B + \Delta B(t),
C(t) = C + \Delta C(t),
D(t) = D + \Delta D(t)
\]
(3)
where $A, B, C, D$ are known constant matrices; The admissible uncertainties are assumed to satisfy the following condition:
\[
\begin{pmatrix}
\Delta A(t) & \Delta B(t) & \Delta C(t) & \Delta D(t)
\end{pmatrix}
= H(t)\begin{bmatrix}
E_1 & E_2 & E_3 & E_4
\end{bmatrix}
\]
(4)
where $H, E_1, E_2, E_3, E_4$ are known constant matrices with appropriate dimensions and $F(t)$ is an unknown and time-varying matrix satisfying:
\[F^T(t)F(t) \leq I, \quad \forall t\]
(5)
Throughout this paper, we assume that the matrix $D(t)$ is Schur stable. Considering $F(t) = 0$, we get the nominal neutral system with distributed delays as follows:
\[
\dot{x}(t) - D(t)\dot{x}(t - \tau) = Ax(t) + Bx(t - h) + C(t)\int_{t-\tau}^{t} x(s)ds
\]
(6)

In this paper, we will firstly consider the asymptotic stability for the nominal system (6) and obtain the promising results for it. Then we will extend to analyze the system of (1) and deduce the homologous conclusions. In order to analyze the asymptotic and robust stability of the neutral system with distributed delay, the following inequalities and integral inequalities are required. They are stated in the lemmas given below.

**Lemma 2.1:** [5] Given a symmetric block matrix described by the following:
\[
S = \begin{bmatrix}
S_{11} & S_{12} \\
S_{12}^T & S_{22}
\end{bmatrix}
\]
where $S_{11}, S_{12}$ and $S_{22}$ are proper dimensions. Then the following three conditions are equivalent:

(i) $S < 0$

(ii) $S_{11} < 0$, $S_{22} - S_{12}S_{11}^{-1}S_{12} < 0$

(iii) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12} < 0$

**Lemma 2.2:** [5] Let $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{n \times n}$, then we have
\[x^TUx \leq \|UV\|\|x\|^T, \quad \forall x \in \mathbb{R}^n\]

**Lemma 2.3:** [9] For any constant matrix $H \in \mathbb{R}^{n \times n}$, $H > 0$ and scalar $\gamma > 0$, vector function
\[
\omega : [0, \gamma] \rightarrow \mathbb{R}^n
\] such that the integrations $\int_0^\gamma \omega^T(s)H\omega(s)ds$ and $\int_0^\gamma \omega(s)ds$ are well defined, then the following inequality holds:
\[\frac{1}{\gamma} \int_0^\gamma \omega^T(s)H\omega(s)ds \leq \frac{1}{\gamma} \int_0^\gamma \omega^T(s)ds \leq \frac{1}{\gamma} \int_0^\gamma \omega(s)ds
\]

**Lemma 2.4:** [7] For any constant matrix $H \in \mathbb{R}^{n \times n}$, $H = H^T > 0$
scalar $\gamma > 0$ vector function $\omega : [0, \gamma] \rightarrow \mathbb{R}^n$ such that the integrations in the following are well defined, then the inequality holds:
\[-\int_0^\gamma \omega^T(s)H\omega(s)ds + \int_0^\gamma \omega(s)ds \leq \frac{1}{\gamma} \int_0^\gamma \omega^T(s)ds \leq \frac{1}{\gamma} \int_0^\gamma \omega^T(s)ds \leq \frac{1}{\gamma} \int_0^\gamma \omega(s)ds
\]

**Lemma 2.5:** [10] For given matrices $Q = Q^T$, $M$ and $N$ with appropriate dimensions, then $Q + M\delta F(t)N + N^T\delta F^T(t)M^T < 0$
for all $F(t)$ satisfying $F^T(t)F(t) \leq I$, if and only if there exists a scalar $\delta > 0$, such that
\[Q + \delta^2 MM^T + \delta^2 NN^T < 0\]
The objective of this paper is to further reduce the conservatism of the stability conditions for uncertain neutral systems with distributed delays to ensure a larger maximum upper bound on the delay.
3. MAIN RESULTS

In this section, we study the asymptotic stability for the system (1) and (6) based on time domain approach. A new Lyapunov-Krasovskii functional is utilized and the proposed stability criterion is discrete-, distributed- and neutral-delay-dependent.

3.1 Results on The Nominal Neutral System

For the asymptotic stability for system (6), we have the following result.

Theorem 3.1: For given scalars \( \tau, \rho \) and \( \rho \), the neutral system with discrete and distributed delays described by (6) is asymptotically stable, if there exist matrices

\[
P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} > 0, \quad Q = \begin{bmatrix} Q_a \\ Q_b \end{bmatrix} > 0
\]

\[
V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} > 0, \quad R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} > 0, \quad W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} > 0
\]

\[
X_1 > 0, X_2 > 0, \eta_1 > 0, \eta_2 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0 > 0
\]

with appropriate dimensions such that

\[
\Theta + \Lambda_1 \Theta A_1 + \Lambda_2 \Theta A_1 + L^T ML < 0
\]

(7)

where

\[
\Theta = \begin{bmatrix}
\theta_{11} & \cdots & \theta_{1n} \\
\vdots & \ddots & \vdots \\
\theta_{n1} & \cdots & \theta_{nn}
\end{bmatrix}
\]

\[
\Lambda_1 = \begin{bmatrix}
I & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I
\end{bmatrix}
\]

\[
\Lambda_2 = \begin{bmatrix}
A & 0 & \cdots & 0 \\
0 & B & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C
\end{bmatrix}
\]

Proof: Define a legitimate Lyapunov functional candidate as

\[
V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t))
\]

(8)

where

\[
V_1(x(t)) = \eta^T(t)P \eta(t)
\]

\[
V_2(x(t)) = \int_{t-\tau}^{t} \left[ x^T(s) \hat{x}(s) \right] Q \left[ x(s) \right] ds
\]

\[
+ \int_{t-\tau}^{t} \left[ x^T(s) x^T(s - \frac{\tau}{2}) \right] V \left[ x(s) \right] ds
\]

\[
+ \int_{t-\tau}^{t} \int_{t-\tau}^{s} \hat{x}^T(s)(\tau X_1) \hat{x}(s) ds d \theta
\]

\[
+ \int_{t-\tau}^{t} \int_{t-\tau}^{s} \hat{x}^T(s)(\frac{\tau}{2} X_2) \hat{x}(s) ds d \theta
\]

\[
V_3(x(t)) = \int_{t-\rho}^{t} \left[ x^T(s) \hat{x}(s) \right] R \left[ \hat{x}(s) \right] ds
\]

\[
+ \int_{t-\rho}^{t} \left[ x^T(s) x^T(s - \frac{\rho}{2}) \right] W \left[ x(s) \right] ds
\]

\[
+ \int_{t-\rho}^{t} \int_{t-\rho}^{s} \hat{x}^T(s)(h Y_1) \hat{x}(s) ds d \theta
\]

\[
+ \int_{t-\rho}^{t} \int_{t-\rho}^{s} \hat{x}^T(s)(\frac{h}{2} Y_2) \hat{x}(s) ds d \theta
\]

\[
V_4(x(t)) = \int_{t-\rho}^{t} \hat{x}^T(s) Z(s) x(s) ds
\]

\[
+ \int_{t-\rho}^{t} \int_{t-\rho}^{s} \hat{x}^T(s)(r Z_2) x(s) ds d \theta
\]

\[
+ \int_{t-\rho}^{t} \int_{t-\rho}^{s} \hat{x}^T(s)(\frac{r^2}{2} Z_3) \hat{x}(s) ds d \theta
\]

where \( P, Q, R, V, W, X_i, Y_i, Z_i, (i=1,2,3) \) are defined as in Theorem 3.1.
The time derivative of $V(x(t))$ along the trajectory of system (8) is given by

$$
\dot{V}(x(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) + \dot{V}_4(x(t)) \tag{13}
$$

In fact, $\dot{V}_1(x(t))$, $\dot{V}_2(x(t))$, $\dot{V}_3(x(t))$ and $\dot{V}_4(x(t))$ can be computed and estimated as follows:

$$
\dot{V}_1(x(t)) = \eta(t)P\eta(t) + \eta(t)P\eta(t)
= \xi_1(t)(\Lambda_1^1PA_1 + \Lambda_2^1PA_2)\xi(t)
$$

where $\Lambda_1$ and $\Lambda_2$ have been defined before, and $\xi(t)$ is given by

$$
\xi(t) = x(t)^T \eta(t) \left[ x(t) \right]
$$

Then we consider the time derivative of $V_2(x(t))$, we have

$$
\dot{V}_2(x(t)) = \int_0^t \left[ x(t)^T \eta(t) \right] \left[ x(t) \right] dt
$$

Using Lemma 2.3 yields

$$
\dot{V}_2(x(t)) \leq \int_0^t \left[ x(t)^T \eta(t) \right] \left[ x(t) \right] dt
$$

Similarly we obtain

$$
\dot{V}_3(x(t)) = \int_0^t \left[ x(t)^T \eta(t) \right] \left[ x(t) \right] dt
$$

Subsequently, we consider $\dot{V}_4(x(t))$ in the following:

$$
\dot{V}_4(x(t)) = x(t)^T Z x(t) - x(t)^T (t) Z x(t) + x(t)^T (t) Z x(t)
$$

By Lemma 2.3 and Lemma 2.4

$$
\dot{V}_4(x(t)) \leq x(t)^T (t) Z x(t) - x(t)^T (t) Z x(t) + x(t)^T (t) Z x(t)
$$

Therefore, if there exist matrices $P = [P_{ij}]_{i,j=1}^n$, $Q = [Q_{ij}]_{i,j=1}^m$, $R = [R_{ij}]_{i,j=1}^m$, $W = [W_{ij}]_{i,j=1}^m$, $Z_1 = [Z_{1ij}]_{i,j=1}^m$, and $Z_2 = [Z_{2ij}]_{i,j=1}^m$, such that

$$
P > 0, Q = [Q_{ij}]_{i,j=1}^m, \quad R = [R_{ij}]_{i,j=1}^m, W = [W_{ij}]_{i,j=1}^m, \quad Z_1 = [Z_{1ij}]_{i,j=1}^m, \quad Z_2 = [Z_{2ij}]_{i,j=1}^m
$$

with appropriate dimensions such that (7), then we have $V(x(t)) < 0$ which guarantees system (8) is asymptotically stable. This completes the proof.  

Remark 3.1: By dividing the neutral delay interval $[-\tau, 0]$ into $[-\tau, -\bar{h}]$ and $[-\bar{h}, 0]$, then different functional were chosen on each subinterval. We have performed a similar trick with discrete delay interval $[-h, 0]$ and we can see that the division provides extra freedom for neutral delay terms and reduces the conservatism.

Furthermore, asymptotic stability conditions for $r = 0$ and $\tau \equiv h$ in the system (6) are obtained on the basis of Theorem 3.1. In this paper we will show them in the following Corollary 3.1 and Corollary 3.2.

Corollary 3.1: For given scalars $\tau$, $h$ and $r = 0$, the neutral system with distributed delays described by (6) is asymptotically stable, if there exist matrices $P = [P_{ij}]_{i,j=1}^n$, $Q = [Q_{ij}]_{i,j=1}^m$, $R = [R_{ij}]_{i,j=1}^m$, $W = [W_{ij}]_{i,j=1}^m$, $Z_1 = [Z_{1ij}]_{i,j=1}^m$, and $Z_2 = [Z_{2ij}]_{i,j=1}^m$, such that

$$
P > 0, Q = [Q_{ij}]_{i,j=1}^m, \quad R = [R_{ij}]_{i,j=1}^m, W = [W_{ij}]_{i,j=1}^m, \quad Z_1 = [Z_{1ij}]_{i,j=1}^m, \quad Z_2 = [Z_{2ij}]_{i,j=1}^m
$$

with appropriate dimensions such that (7), then we have $V(x(t)) < 0$ which guarantees system (8) is asymptotically stable. This completes the proof.  

Remark 3.1: By dividing the neutral delay interval $[-\tau, 0]$ into $[-\tau, -\bar{h}]$ and $[-\bar{h}, 0]$, then different functional were chosen on each subinterval. We have performed a similar trick with discrete delay interval $[-h, 0]$ and we can see that the division provides extra freedom for neutral delay terms and reduces the conservatism.

Furthermore, asymptotic stability conditions for $r = 0$ and $\tau \equiv h$ in the system (6) are obtained on the basis of Theorem 3.1. In this paper we will show them in the following Corollary 3.1 and Corollary 3.2.
with appropriate dimensions such that
\[ \tilde{\Theta} + \tilde{\Lambda}_1' P \tilde{\Lambda}_2 + \tilde{\Lambda}_2' P \tilde{\Lambda}_1 + \tilde{E} \tilde{M} \tilde{L} < 0 \] (18)
where
\[
\tilde{\Lambda}_1 = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 \\
\end{bmatrix}
\]
\[
\tilde{\Lambda}_2 = \begin{bmatrix}
A & 0 & 0 & 0 & B & D & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
I & 0 & -I & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\tilde{\mathcal{L}} = \begin{bmatrix}
A & 0 & 0 & 0 & B & D & 0 & 0 \\
\end{bmatrix}
\]
\[
\tilde{M} = Q_{22} + R_{22} + \tau^2 X_1 + \frac{\tau^2}{4} X_2 + h^2 Y_1 + \frac{h^2}{4} Y_2
\]

\[
\tilde{\Theta} = \begin{bmatrix}
\Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & 0 & 0 \\
* & \Theta_{22} & \Theta_{23} & \Theta_{24} & \Theta_{25} & \Theta_{26} & 0 & 0 \\
* & * & \Theta_{33} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Theta_{44} & \Theta_{45} & 0 & 0 & 0 \\
* & * & * & * & \Theta_{55} & 0 & \Theta_{57} & 0 \\
* & * & * & * & * & \Theta_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Theta_{77} & 0 \\
\end{bmatrix}
\]

\[
\Theta_{11} = Q_{11} + Q_{12} A + A^T Q_{21} + R_{11} + R_{12} A + A^T R_{21} + V_{11} - X_1 - X_2 + W_{11} - Y_1 - Y_2 \\
\Theta_{12} = V_{12} + X_2 \\
\Theta_{13} = X_1 \\
\Theta_{14} = W_{14} + Y_2 \\
\Theta_{15} = Y_1 + Q_{12} B + R_{12} B \\
\Theta_{16} = Q_{22} D + R_{22} D \\
\Theta_{17} = V_{22} - V_{11} - X_2 \\
\Theta_{21} = -V_{12} \\
\Theta_{22} = -Q_{11} - V_{22} - X_1 \\
\Theta_{23} = -Q_{12} \\
\Theta_{24} = W_{22} - W_{11} - Y_2 \\
\Theta_{25} = -W_{12} \\
\Theta_{26} = -R_{11} - W_{22} - Y_1 \\
\Theta_{27} = -R_{12} \\
\Theta_{33} = -Q_{22} \\
\Theta_{34} = -W_{22} - W_{11} - Y_2 \\
\Theta_{35} = -R_{22} \\
\Theta_{36} = -Q_{22} \\
\Theta_{37} = -R_{22} \\
\Theta_{44} = -W_{22} - W_{11} - Y_2 \\
\Theta_{45} = -W_{12} \\
\Theta_{46} = -R_{11} - W_{22} - Y_1 \\
\Theta_{47} = -R_{12} \\
\Theta_{55} = -Q_{22} \\
\Theta_{56} = -Q_{22} \\
\Theta_{57} = -R_{22} \\
\Theta_{66} = -Q_{22} \\
\Theta_{77} = -R_{22}
\]

**Corollary 3.2:** For given scalars \( \tau \equiv \tau \) and \( r \), the neutral system with distributed delays described by (6) is asymptotically stable, if there exist matrices
\]
\[
P = [P_{k}]_{k=1} > 0, Q = [Q_{k}]_{k=2} > 0, V = [V_{k}]_{k=2} > 0 \\
X_1 > 0, X_2 > 0, Y_1 > 0, Y_2 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0
\]
with appropriate dimensions such that
\[ \tilde{\Theta} + \tilde{\Lambda}_1' P \tilde{\Lambda}_2 + \tilde{\Lambda}_2' P \tilde{\Lambda}_1 + \tilde{E} \tilde{M} \tilde{L} < 0 \] (19)
where
\[
\tilde{\mathcal{L}} = \begin{bmatrix}
A & 0 & 0 & B & D & 0 & 0 & C \\
\end{bmatrix}
\]
\[
\tilde{M} = Q_{22} + R_{22} + \tau^2 X_1 + \frac{\tau^2}{4} X_2 + \frac{\tau^4}{4} Z_3 \\
\tilde{\Lambda}_2 = \begin{bmatrix}
A & 0 & 0 & B & D & 0 & 0 & C \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
\end{bmatrix}
\]
\[
\tilde{\Theta} = \begin{bmatrix}
\hat{\Theta}_{11} & \hat{\Theta}_{12} & \hat{\Theta}_{13} & \hat{\Theta}_{14} & \hat{\Theta}_{15} & \hat{\Theta}_{16} & 0 & 0 \\
* & \hat{\Theta}_{22} & \hat{\Theta}_{23} & \hat{\Theta}_{24} & \hat{\Theta}_{25} & \hat{\Theta}_{26} & 0 & 0 \\
* & * & \hat{\Theta}_{33} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \hat{\Theta}_{44} & \hat{\Theta}_{45} & 0 & 0 & 0 \\
* & * & * & * & \hat{\Theta}_{55} & 0 & \Theta_{57} & 0 \\
* & * & * & * & * & \hat{\Theta}_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Theta_{77} & 0 \\
\end{bmatrix}
\]

\[
\hat{\Theta}_{11} = Q_{11} + Q_{12} A + A^T Q_{21} + V_{11} - X_1 - X_2 + W_{11} - Y_1 - Y_2 \\
\hat{\Theta}_{12} = V_{12} + X_2 \\
\hat{\Theta}_{13} = X_1 + Q_{12} B + R_{12} B \\
\hat{\Theta}_{14} = Q_{22} D + R_{22} D \\
\hat{\Theta}_{15} = V_{22} - V_{11} - X_2 \\
\hat{\Theta}_{16} = -V_{12} \\
\hat{\Theta}_{17} = -Q_{11} - V_{22} - X_1 \\
\hat{\Theta}_{21} = -Q_{12} \\
\hat{\Theta}_{22} = W_{22} - W_{11} - Y_2 \\
\hat{\Theta}_{23} = -W_{12} \\
\hat{\Theta}_{24} = -R_{11} - W_{22} - Y_1 \\
\hat{\Theta}_{25} = -R_{12} \\
\hat{\Theta}_{26} = -Q_{22} \\
\hat{\Theta}_{27} = -R_{22} \\
\hat{\Theta}_{33} = -Q_{22} \\
\hat{\Theta}_{34} = -W_{22} - W_{11} - Y_2 \\
\hat{\Theta}_{35} = -R_{22} \\
\hat{\Theta}_{36} = -Q_{22} \\
\hat{\Theta}_{37} = -R_{22} \\
\hat{\Theta}_{44} = -W_{22} - W_{11} - Y_2 \\
\hat{\Theta}_{45} = -W_{12} \\
\hat{\Theta}_{46} = -R_{11} - W_{22} - Y_1 \\
\hat{\Theta}_{47} = -R_{12} \\
\hat{\Theta}_{55} = -Q_{22} \\
\hat{\Theta}_{56} = -Q_{22} \\
\hat{\Theta}_{57} = -R_{22} \\
\hat{\Theta}_{66} = -Q_{22} \\
\hat{\Theta}_{77} = -R_{22}
\]

### 3.2 Results on The Uncertain Neutral System

For the asymptotic stability for system (6), we have the following result.
Theorem 3.2: For given scalars $\tau, h$ and $r$, the neutral system described by (1) with uncertainty described by (3), (4), (5) is asymptotically stable, if there exist matrices
\[
P = [P_{ij}]_{n \times n} > 0, \quad Q = [Q_{ij}]_{n \times n} > 0
\]
\[
V = [V_{ij}]_{n \times n} > 0, \quad R = [R_{ij}]_{n \times n} > 0, \quad W = [W_{ij}]_{n \times n} > 0
\]
\[
X_1 > 0, X_2 > 0, Y_1 > 0, Y_2 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0, Z_4 > 0
\]
with appropriate dimensions such that
\[
\begin{align*}
\begin{bmatrix}
\Theta + \Lambda_i^T P A_1 + \Lambda_i^T P A_1 + \frac{1}{\varepsilon} \chi_i^T H H^T \chi_i + \varepsilon 
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
L^T M + \frac{1}{\varepsilon} \chi_i^T H H^T M
\end{bmatrix}
\]
\[
\begin{bmatrix}
0
\end{bmatrix}
\]
\[
< 0
\]
(20)
where
\[
X_1 = \begin{bmatrix}
E_d & 0 & 0 & 0 & E_d & 0 & 0 & 0 & E_c
\end{bmatrix}
\]
\[
X_2 = \begin{bmatrix}
Q_{21} + R_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
and other notations have been defined in Theorem 3.1.

Proof: We replace $\Theta, \Lambda_i, L$ with $\Theta(t), \Lambda_i(t), L(t)$ on the basis of result of Theorem 3.1, thus we have the inequality by Lemma 2.1.
\[
\begin{align*}
\begin{bmatrix}
\Theta(t) + \Lambda_i^T P A_1 + \Lambda_i^T P A_1 + \frac{1}{\varepsilon} \chi_i^T H H^T \chi_i + \varepsilon 
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
L^T M + \frac{1}{\varepsilon} \chi_i^T H H^T M
\end{bmatrix}
\]
\[
\begin{bmatrix}
0
\end{bmatrix}
\]
\[
< 0
\]
(21)
Considering the uncertainty described by (3), (4), (5), we obtain the following matrix inequality
\[
\begin{align*}
\begin{bmatrix}
\Theta + \Lambda_i^T P A_1 + \Lambda_i^T P A_1 + \frac{1}{\varepsilon} \chi_i^T H H^T \chi_i + \varepsilon 
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
L^T M + \frac{1}{\varepsilon} \chi_i^T H H^T M
\end{bmatrix}
\]
\[
\begin{bmatrix}
0
\end{bmatrix}
\]
\[
< 0
\]
(22)
By Lemma 2.5, we have
\[
\begin{align*}
\begin{bmatrix}
\Theta + \Lambda_i^T P A_1 + \Lambda_i^T P A_1 + \frac{1}{\varepsilon} \chi_i^T H H^T \chi_i + \varepsilon 
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
L^T M + \frac{1}{\varepsilon} \chi_i^T H H^T M
\end{bmatrix}
\]
\[
\begin{bmatrix}
0
\end{bmatrix}
\]
\[
< 0
\]
Therefore, we obtain the linear matrix inequality (20) which guarantees the uncertain neutral system with distributed delay is asymptotically stable. This completes the proof.

We also obtain the corollaries when $r=0$ and $\tau \equiv h$ in the system (1) on the basis of Theorem 3.2.

Corollary 3.3: For given scalars $\tau, h$ and $r=0$, the neutral system described by (1) with uncertainty described by (3), (4), (5) is asymptotically stable, if there exist matrices
\[
P = [P_{ij}]_{n \times n} > 0, \quad Q = [Q_{ij}]_{n \times n} > 0
\]
\[
V = [V_{ij}]_{n \times n} > 0, \quad R = [R_{ij}]_{n \times n} > 0, \quad W = [W_{ij}]_{n \times n} > 0
\]
\[
X_1 > 0, X_2 > 0, Y_1 > 0, Y_2 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0, Z_4 > 0
\]
with appropriate dimensions such that
\[
\begin{align*}
\begin{bmatrix}
\Theta + \Lambda_i^T P A_1 + \Lambda_i^T P A_1 + \frac{1}{\varepsilon} \chi_i^T H H^T \chi_i + \varepsilon 
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
L^T M + \frac{1}{\varepsilon} \chi_i^T H H^T M
\end{bmatrix}
\]
\[
\begin{bmatrix}
0
\end{bmatrix}
\]
\[
< 0
\]
(22)
where
\[
X_1 = \begin{bmatrix}
E_d & 0 & 0 & 0 & E_d & 0 & 0 & 0 & E_c
\end{bmatrix}
\]
\[
X_2 = \begin{bmatrix}
Q_{21} + R_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
and other notations have been defined in Corollary 3.1.

Corollary 3.4: For given scalars $\tau \equiv h$ and $r$, the neutral system described by (1) with uncertainty described by (3), (4), (5) is asymptotically stable, if there exist matrices
\[
P = [P_{ij}]_{n \times n} > 0, \quad Q = [Q_{ij}]_{n \times n} > 0
\]
\[
V = [V_{ij}]_{n \times n} > 0, \quad R = [R_{ij}]_{n \times n} > 0, \quad W = [W_{ij}]_{n \times n} > 0
\]
\[
X_1 > 0, X_2 > 0, Y_1 > 0, Y_2 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0, Z_4 > 0
\]
with appropriate dimensions such that
\[
\begin{align*}
\begin{bmatrix}
\Theta + \Lambda_i^T P A_1 + \Lambda_i^T P A_1 + \frac{1}{\varepsilon} \chi_i^T H H^T \chi_i + \varepsilon 
\end{bmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
L^T M + \frac{1}{\varepsilon} \chi_i^T H H^T M
\end{bmatrix}
\]
\[
\begin{bmatrix}
0
\end{bmatrix}
\]
\[
< 0
\]
(23)
where
\[
X_1 = \begin{bmatrix}
E_d & 0 & 0 & 0 & E_d & 0 & 0 & 0 & E_c
\end{bmatrix}
\]
\[
X_2 = \begin{bmatrix}
Q_{21} + R_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
and other notations have been defined in Corollary 3.2.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are given to illustrate the effectiveness and the improvement of the proposed method over some previous ones.

Example 1: Consider the uncertain system
\[
A = \begin{bmatrix}
-0.9 & 0.2 \\
0.1 & -0.9
\end{bmatrix} \quad B = \begin{bmatrix}
-1.1 & -0.2 \\
-0.1 & -1.1
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
-0.12 & -0.12 \\
-0.12 & 0.12
\end{bmatrix} \quad D = \begin{bmatrix}
-0.2 & 0 \\
0.2 & -0.1
\end{bmatrix}
\]
\[
H = I, \quad E_A = E_B = E_C = E_D = 0.1 I, \quad \tau = 0.1
\]
When $h=0.1$, by Corollary 3.4, the upper bound of $r$ that guarantees the asymptotic stability of system (1) calculated by the method in this paper is 6.7.
Example 2: Consider the uncertain system

$$A = \begin{bmatrix} -3.4 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1.1 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$H = I, \quad E_A = E_B = E_C = E_D = 0.2I, \quad \tau = 0.2$$

When $r=1.0, d=0.25$, applying the criteria in Li & Zhu [15], Sun et al. [6] and Theorem 3.2 in this paper we can obtain the allowed upper bound of $h$. We show the results in the following Table 2:

<table>
<thead>
<tr>
<th>$r=0.2$, $\tau=0.0$</th>
<th>$r=0.2$, $\tau=1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li and Zhu</td>
<td>0.70</td>
</tr>
<tr>
<td>Sun et al.</td>
<td>0.72</td>
</tr>
<tr>
<td>Theorem 3.2 in paper</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Table 2: The Allowed Upper Bound Of $h$

Through these examples, it can be seen that our method is less conservative than the previous ones.

5. CONCLUSIONS

This paper is concerned with the stability of linear certain and uncertain neutral systems with distributed delays. Applying the delay-dividing approach, new discrete-, distributed- and neutral-delay-dependent stability criteria are presented based on the Lyapunov theory. Two numerical examples are given to respectively show that our method is less conservative than the previous ones.

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