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# ADAPTIVE FUZZY SLIDING MODE CONTROL FOR X-Z INVERTED PENDULUM

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**ABSTRACT**

X–Z inverted pendulum is a new kind of inverted pendulum and it can move with the combination of the vertical and horizontal forces. In this paper, the control problem of X-Z inverted pendulum with system uncertainties is addressed, and a pair of decoupled adaptive fuzzy sliding mode control method is proposed. The fuzzy logic system is employed to approximate the system uncertainties as well as the complicated intermediate control functions. For updating the parameter of the fuzzy system, a proportional-integral adaptation law is proposed. Finally, simulation studies are carried out to show the stabilization of the X-Z inverted pendulum under the proposed control method.

**Keywords:** *Adaptive Fuzzy Control (AFC), Sliding Mode Control (SMC), X-Z Inverted Pendulum*

## 1. INTRODUCTION

The inverted pendulum is nonminimum phase, nonlinear and underactuated complicated system which makes it a very difficult problem that provides much challenging problem to the controller design. In the last two decades, there are many literatures on stabilization and tracking control for the conventional inverted pendulum [1]. Except the wide research on the conventional inverted pendulum, a lot of researchers pay their attention to the other kinds of inverted pendulums, such as spherical inverted pendulum [2-4], X–Z inverted pendulum [1,5-6]. Compared with the conventional inverted pendulum, the X–Z inverted pendulum, in reality, is more like the real control object. In [5], Maravall established a hybrid fuzzy control system that incorporates a Takagi–Sugeno fuzzy control structure with PD control for stabilizing the X–Z inverted pendulum. In [6], the PID controllers have been used to the tracking control of the X–Z inverted pendulum system. And good tracking control performance is obtained.

As we know, fuzzy logic control is a model-free method, and it can handle ill-defined and complex nonlinear systems, even those with significant uncertainties and unknown dynamics. Fuzzy rule-based control systems have been extensively used in many areas, including cluster analysis, the controller design, and image processing. The fuzzy control methods have been shown to be effective for systems with uncertainties [7-9]. The stability analysis is always an important aspect in the completeness of controller design. With traditional fuzzy control system, the stability of the closed-loop system of a Mamdani fuzzy structure is very hard to be proved. To guarantee the closed-loop stability, some certain hybrid control approach is often used [7,10,11]. Among those schemes, sliding mode control (SMC) is a kind of robust stabilizing control method by driving the system states into a predefined sliding surface. The main advantages of sliding mode control are the system robustness with structured and unstructured uncertainties and satisfactory transient performance can all be preserved [12-13].

Recently, many study results show that corporate the SMC with the fuzzy control methods not only can alleviate the chattering effects in SMC but can decrease the complexity of fuzzy controller with reduced number of fuzzy rules [14]. In this paper, the adaptive fuzzy sliding mode control method is used to control X-Z inverted pendulum. The fuzzy logic system is employed to approximate the system uncertainties and the complicated intermediate control functions.

## 2. PROBLEM DESCRIPTION AND PRELIMINARIES

The X-Z inverted pendulum on a pivot driven by vertical and horizontal control forces can be seen in Fig.1. The control inputs of the system are based on the X-Z vertical and horizontal displacements of the pivot.

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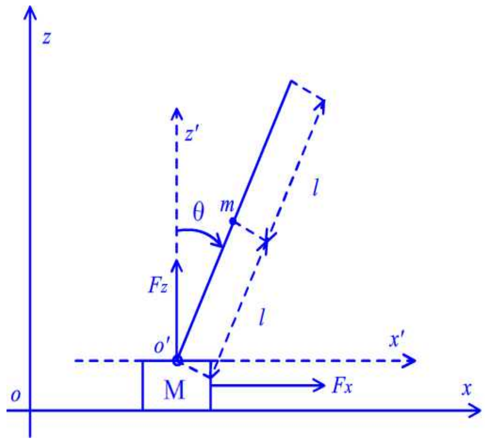


*Fig.1: The Structure Of The X-Z Inverted Pendulum.*

The state equations of the X-Z inverted pendulum were given in [1] which can be described by:

$$\ddot{x}=\frac{Ml\dot{\theta}^2\sin\theta+(M+m\cos^2\theta)F_x-mF_z\sin\theta\cos\theta}{M(M+m)}$$
$$\ddot{z}=\frac{Ml\dot{\theta}^2\cos\theta+(M+m\sin^2\theta)F_z-mF_x\sin\theta\cos\theta}{M(M+m)}-g \quad (1)$$
$$\ddot{\theta}=\frac{-F_x\cos\theta+F_z\sin\theta}{Ml}$$

where $(x,z),(\dot{x},\dot{z}),(\ddot{x},\ddot{z})$ are the position, speed and the acceleration of the pivot respectively. $l$ is the distance form the mass center of the inverted pendulum to the pivot. And $g$ is the acceleration constant of gravity. $M,m$ are the mass of the pivot and the pendulum. $F_z$ is the vertical force, and $F_x$ is the horizontal force.

### 2.1. Description Of The Fuzzy Logic System

The fuzzy logic systems that employs singleton fuzzification, sum-product inference and center-off-sets defuzzification can be modeled by

$$\alpha(x)=\frac{\sum_{j=1}^{N}\theta_j\Pi_{i=1}^{n}\mu_{F_i^j}(x_i)}{\sum_{j=1}^{N}[\Pi_{i=1}^{n}\mu_{F_i^j}(x_i)]}, \quad (2)$$

where $\alpha(x)$ is the output of the fuzzy system, $x$ is the input vector, $\mu_{F_i^j}(x_i)$ is $x_i$'s membership of $j$th rule and $\theta_j$ is the centroid of the $j$th consequent set. Eq. (2) can be rewritten as:

$$\alpha(x)=\vartheta^T\psi(x) \quad (3)$$

with

$\vartheta=[\vartheta_1,\cdots,\vartheta_N]^T$, $\psi(x)=[p_1(x),p_2(x),\cdots,p_N(x)]^T$,

and the fuzzy basis function can be expressed as:

$$p_j(x)=\frac{\Pi_{i=1}^{n}\mu_{F_i^j}(x_i)}{\sum_{j=1}^{N}[\Pi_{i=1}^{n}\mu_{F_i^j}(x_i)]}.$$

## 3. CONTROL DESIGN OF THE X-Z INVERTED PENDULUM

Define the following transform:

$$x_1=x+l\sin\theta, z_1=z+l\cos\theta-l. \quad (4)$$

Based on system (1), we have

$$\ddot{x}_1=\frac{F_x\sin^2\theta+F_z\sin\theta\cos\theta-Ml\dot{\theta}^2\sin\theta}{M+m}$$
$$\ddot{z}_1=\frac{F_x\sin\theta\cos\theta+F_z\cos^2\theta-Ml\dot{\theta}^2\cos\theta}{M+m}-g \quad (5)$$
$$\ddot{\theta}=\frac{-F_x\cos\theta+F_z\sin\theta}{Ml}$$

Let $F_1=(F_x\sin\theta+F_z\cos\theta-Ml\dot{\theta}^2)/(M+m)$, $F_2=(-F_x\cos\theta+F_z\sin\theta)/Ml$, we can obtain

$$\ddot{x}_1=F_1\sin\theta$$
$$\ddot{z}_1=F_1\cos\theta-g \quad (6)$$
$$\ddot{\theta}=F_2$$

and

$$F_x=(M+m)F_1\sin\theta-MlF_2\cos\theta,$$
$$F_z=(M+m)F_1\cos\theta+MlF_2\sin\theta. \quad (7)$$

If we define the change of coordinates as $\tan\theta=y$, $u_1=F_1\cos\theta$, $u_2=(F_2+2\dot{\theta}^2\tan\theta)\sec^2\theta$, and take the system uncertainties into consideration, yields

$$\ddot{x}_1=u_1y$$
$$\ddot{z}_1=u_1-g \quad (8)$$
$$\ddot{y}=u_2+f(\xi)$$

where $\xi=[x_1,\dot{x}_1,z_1,\dot{z}_1,y,\dot{y}]$, and $f(\xi)$ is unknown system uncertainty with unknown bound.

Let us define the sliding mode surface as

$$s_1=\dot{x}_1+\lambda_1x_1,$$
$$s_2=\dot{z}_1+\lambda_2z_1, \quad (9)$$

with $\lambda_i>0, i=1,2$ such that the roots of the polynomial $H_i(s)=\lambda_i+s_i$ related to the characteristic equation of $H_i(s)=0$ are all in the open-half plane.

From (8) and (9) we have

$$\dot{s}_2=u_1-g+\lambda_2\dot{z}_1,$$
$$\dot{s}_1=u_1y+\lambda_1\dot{x}_1. \quad (10)$$

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Then the transformed control input $u_1$ can be constructed as

$$u_1 = g - \lambda_2 \dot{z}_1 - k_2 s_2, \tag{11}$$

where $k_2 > 0$ is a design parameter. If we choose $\lambda_2$ and $k_2$ small enough then we can get $u_1 > 0$.

Let treat $y = y^*$ as an intermediate control function, and form the second equation of (10) we have

$$u_1 y^* = -\lambda_1 \dot{x}_1 - k_1 s_1, \tag{12}$$

where $k_1 > 0$ is controller design parameter. Note $u_1 > 0$, the intermediate control input $y^*$ can be described as

$$y^* = \frac{-\lambda_1 \dot{x}_1 - k_1 s_1}{u_1}. \tag{13}$$

To realize $y$ converges to $y^*$, define $e = y - y^*$ and the sliding surface

$$s_3 = \lambda_3 e + \dot{e}, \tag{14}$$

then we have

$$\dot{s}_3 = \lambda_3 \dot{y} - \lambda_3 \dot{y}^* + u_2 + f(\xi) - \ddot{y}^*. \tag{15}$$

Since $\dot{y}$ and $\ddot{y}^*$ have complicated structure, in this paper, we employ the fuzzy logic system to approximate the unknown function $f(\xi)$ incorporated with $\dot{y}$ and $\ddot{y}^*$. Let us define

$$\alpha(\xi, u_1) = -\lambda_3 \dot{y}^* + f(\xi) - \ddot{y}^*, \tag{16}$$

then we can approximate the unknown nonlinear function $\alpha(\xi, u_1)$, through the fuzzy logic system (3), as

$$\hat{\alpha}(\xi, u_1, \vartheta) = \vartheta^T \psi(x, u_1). \tag{17}$$

Let us define the ideal parameter of $\vartheta$ as

$$\vartheta^* = \arg\min_{\vartheta} \left[ \sup \left| \alpha(\xi, u_1) - \hat{\alpha}(\xi, u_1, \vartheta) \right| \right], \tag{18}$$

and define the parameter estimation error and the fuzzy system approximation error as

$$\begin{aligned} &\tilde{\vartheta} = \vartheta - \vartheta^*, \\ &\varepsilon(\xi, u_1) = \alpha(\xi, u_1) - \hat{\alpha}(\xi, u_1, \vartheta^*). \end{aligned} \tag{19}$$

As in literature [8,9], it is reasonable for us to assume that the fuzzy logic system approximation error is bounded, i.e., there exists some positive constant $\bar{\varepsilon}$, such that

$$\left| \varepsilon(\xi, u_1) \right| < \bar{\varepsilon}. \tag{20}$$

From above analysis, we can obtain

$$\begin{aligned} &\hat{\alpha}(\xi, u_1, \vartheta) - \alpha(\xi, u_1) = \\ &\hat{\alpha}(\xi, u_1, \vartheta) - \hat{\alpha}(\xi, u_1, \vartheta^*) + \hat{\alpha}(\xi, u_1, \vartheta^*) - \alpha(\xi, u_1) \\ &= \tilde{\vartheta}^T \psi(\xi, u_1) - \varepsilon(\xi, u_1). \end{aligned} \tag{21}$$

Then the controller $u_2$ can be chosen as

$$u_2 = -\lambda_3 \dot{y} - \hat{\alpha}(\xi, u_1, \vartheta) - k_3 s_3 - k_4 sign(s_3), \tag{22}$$

where $k_3, k_4 > 0$ are design parameters. The fuzzy system parameter is updated by the following adaptation PI law:

$$\vartheta = \int_0^t \left[ \sigma \gamma_1 \mid s_3 \mid \vartheta + \gamma_1 s_3 \psi(\xi, u_1) \right] d\tau - \gamma_2 \delta \tag{23}$$

with

$$\delta = \sigma \mid s_3 \mid \vartheta - s_3 \psi(\xi, u_1), \tag{24}$$

where $\sigma, \gamma_1, \gamma_2 > 0$ are design parameters. `The update law (24) this paper designed has a nice performance as the statement of the following theorem.

**Theorem 1.** The update law (24) can guarantee that the fuzzy system parameter $\vartheta \in L_\infty$ for bounded initial $\vartheta(0)$.

**Proof**. Define the Lyapunov candidate function as

$$V_1 = \frac{1}{2\gamma_1} \left( \vartheta + \gamma_2 \delta \right)^T \left( \vartheta + \gamma_2 \delta \right). \tag{25}$$

Then we have

$$\begin{aligned} \dot{V}_1 &= \frac{1}{\gamma_1} \left( \vartheta + \gamma_2 \delta \right)^T \left( \dot{\vartheta} + \gamma_2 \dot{\delta} \right) \\ &= \left( \vartheta + \gamma_2 \delta \right)^T \left( -\sigma \left| s_3 \right| \vartheta + s_3 \psi(\xi, u_1) \right). \end{aligned} \tag{26}$$

If $\delta$ is chosen as (24), one can obtain

$$\begin{aligned} \dot{V}_1 &\le -\sigma \left| s_3 \right| \left\| \vartheta \right\|^2 + \left| s_3 \right| \left\| \vartheta \right\| \left\| \psi(\xi, u_1) \right\| - \gamma_2 \left\| \delta \right\|^2 \\ &\le -\sigma \left| s_3 \right| \left\| \vartheta \right\| \left( \left\| \vartheta \right\| - c / \sigma \right). \end{aligned} \tag{27}$$

Noting $c = \sup \left\| \psi(\xi, u_1) \right\|$. Then we can conclude that if $\left\| \vartheta \right\| > c / \sigma$, $\dot{V}_1 < 0$. Thus we know that $\vartheta \in L_\infty$. This ends the proof of theorem 1.

From above discussion, now we are ready to give the following results.

**Theorem 2.** Consider system (1) or the equivalent system (8). The sliding mode surfaces are defined as (10) and (14), the parameter adaptation law is given by (23) and (24), the controller is defined by (11)-(13) and (22), then we have the following results:

I. All signals in the closed loop system remain bounded.

II. The system states and their derivatives asymptotically converge to zero.

**Proof.** Let define the following Lypunov function:

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$$V=\frac{1}{2}\sum_{i=1}^{3}s_i^2+\frac{1}{2\gamma_1}\left(\tilde{\vartheta}+\gamma_2\delta\right)^T\left(\tilde{\vartheta}+\gamma_2\delta\right). \quad (28)$$

From (10) and (11), we have

$$s_2\dot{s}_2=-k_2s_2^2.$$

According to (15) and (22) we get

$$s_3\dot{s}_3=-\tilde{\vartheta}^T\psi(\xi,u_1)s_3+\varepsilon(\xi,u_1)s_3 -k_3s_3^2-k_4|s_3|. \quad (29)$$

Let $V_2=\frac{1}{2\gamma_1}\left(\tilde{\vartheta}+\gamma_2\delta\right)^T\left(\tilde{\vartheta}+\gamma_2\delta\right)$, substituting the fuzzy system adaptation law (23) and (24) into (25) and by using the inequality $-2\tilde{\vartheta}^T\vartheta\le-\|\tilde{\vartheta}\|^2+\|\vartheta^*\|^2$, one can get

$$\begin{aligned}\dot{V}_1&=\left(\vartheta+\gamma_2\delta\right)^T\left(-\sigma|s_3|\vartheta+s_3\psi(\xi,u_1)\right)\\&\le\tilde{\vartheta}\psi(\xi,u_1)s_3-\gamma_2\|\delta\|^2\\&-\frac{1}{2}\sigma|s_3|\|\tilde{\vartheta}\|^2+\frac{1}{2}\sigma|s_3|\|\tilde{\vartheta}^*\|^2\end{aligned} \quad (30)$$

Then if we choose that $k_4>\bar{\varepsilon}+0.5\sigma\|\tilde{\vartheta}^*\|^2$, form (30) and (31) we know $s_3\dot{s}_3+\dot{V}_2\le-k_3s_3^2$ which means that $s_3$ converges to zero, i.e. $e\to0$ as time $t\to\infty$. Then from (10)-(12), we know

$$\dot{s}_1=u_1(y^*+e)+\lambda_1\dot{x}_1. \quad (31)$$

According to (13) and $e\to0$, one can obtain that

$$s_1\dot{s}_1=-k_1s_1^2 \quad (32)$$

From above discussion, we can obtain that

$$\dot{V}\le-k_1s_1^2-k_2s_2^2-k_3s_3^2. \quad (33)$$

So, $\dot{V}$ is always negative, which means that the signals $s_i, i=1,2,3$ and $\tilde{\vartheta}+\gamma_2\delta$ are bounded. Then from (23), we can easily know $\vartheta,\tilde{\vartheta}\in L_\infty$. Integrating (34) yields:

$$\int_0^\infty\sum_{i=1}^{3}k_is_i^2dt\le V(0)-V(\infty)<\infty \quad (34)$$

which implies that $s_i\in L_2$. Then form (10) and (15), we can easily conclude that $\dot{s}_i\in L_\infty$. At last by using Barbalat's lemma [8], we know $s_i\to0$, and all the signals in the closed-loop system is bounded, and the states of the system converge to zero as $t\to\infty$. This ends the proof of the theorem.

## 4. SIMULATION RESULTS

The parameters of the X-Z inverted pendulum are chosen as in Table 1.

***Table 1**. The parameters of the X-Z inverted pendulum.*

| $M$ (kg) | $m$ (kg) | $l$ (m) | $g$ (m/s$^2$) |
|---|---|---|---|
| 1 | 0.1 | 0.5 | 9.8 |

The parameters of the sliding mode control are chosen as $\lambda_1=\lambda_3=2,\lambda_2=0.5,k_1=k_3=1,k_2=0.2.$

The fuzzy logic system uses $\xi$ and $u_1$ as the inputs. For each variable of $\xi$, we define three Gaussian membership functions uniformly distributed on the interval $[-1,1]$. And with respect to $u_1$, we define five Gaussian membership functions uniformly distributed on the interval $[-40,40]$. The Gaussian membership functions of $\xi$ are shown in Fig.2.

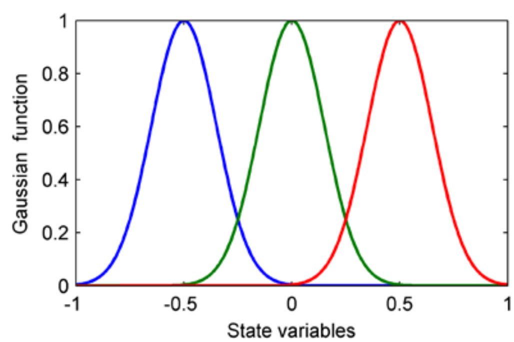


***Fig.2**: Gaussian Membership Functions Of $\xi$*

The initial values of the system are chosen as $x(0)=0.5$, $\dot{x}(0)=0$, $z(0)=0.7$, $\dot{z}(0)=0.2$, $\theta(0)=\pi/4$, $\dot{\theta}(0)=0$. The initial values of the fuzzy system are chosen as $\vartheta(0)=0$. The system uncertainties in (8) are assumed to be:

$$f(\xi)=0.1\theta+\sin t. \quad (35)$$

The simulation results are shown in Fig.3-Fig.6. From the simulation results we can conclude that the stabilization of the X-Z inverted system is achieved and the system performance is good.

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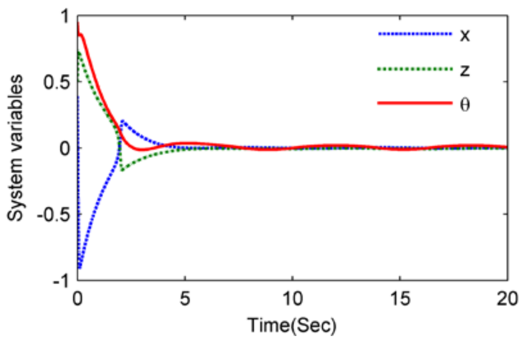

*Fig.3: Stabilization Of The X-Z Inverted Pendulum.*

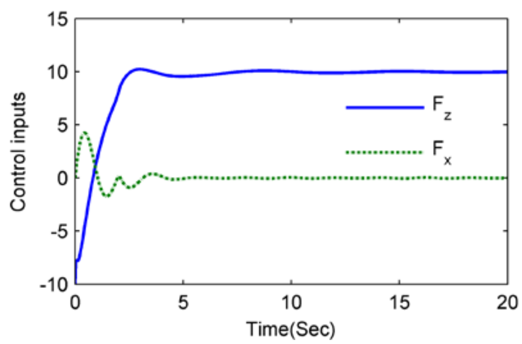

*Fig.4: The Control Inputs.*

## 5. CONCLUSIONS

In this paper, the equivalent transform of the X-Z inverted pendulum is given and the adaptive fuzzy sliding mode controller is designed for X-Z inverted pendulum. The major contributions of our work can be summarized as the following points. Firstly, we give the equivalent transform of the inverted pendulum. Secondly, the fuzzy system is employed to approximate the unknown system uncertainties as well as the intermediate control input functions. The controller we designed can guarantee the stabilization of the system and all the signals in the closed-loop system keep bounded. The simulation results show that good control performance is achieved.

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