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ON HILBERT-TYPE INTEGRAL OPERATOR INEQUALITY AND APPLICATION

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ABSTRACT

In the paper, by using the way of weight functions and the theory of operators, a Hilbert-type integral operator with the homogeneous kernel of $-\lambda$-degree and its norm are considered. As for applications, two equivalent inequalities with the best constant factors and some particular norms are obtained.

Keywords: *Integral operator, Beta function, Hilbert's type integral inequality*

1. INTRODUCTION

If $p>1, 1/p+1/q=1$, $f(\geq 0)\in L^p(0,\infty)$, $g(\geq 0)\in L^q(0,\infty)$,

$\| f\|_p=\{\int_0^\infty f^p(x)dx\}^{1/p}>0$ and $\| g\|_q>0$,

then we have the following famous Hardy-Hilbert's integral inequality and its equivalent form [1]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\| f\|_p\| g\|_q \quad (1.1)$$

$$\left\{\int_0^\infty[\int_0^\infty \frac{f(x)}{x+y}dx]^p dy\right\}^{1/p}<[\frac{\pi}{\sin(\pi/p)}]^p\| f\|_p, \quad (1.2)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [2]).

In 1934, Hardy et al. [1] gave a basic theorem with the general kernel as follows (see [1], Theorem 319):

Theorem 1.1 Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $k(x,y)$ is a homogeneous function of -1-degree, and $k=\int_0^\infty k(u,1)u^{-1/p}du$ is a positive number. If $k(1,u)u^{-1/p}$ and $k(1,u)u^{-1/q}$ are strictly decreasing functions for $u>0$, $f(x)$, $g(x)\geq 0$, $0<\| f\|_p=\{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}}<\infty$, $0<\| g\|_q=\{\int_0^\infty g^q(x)dx\}^{\frac{1}{q}}<\infty$, then we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy<k\| f\|_p\| g\|_q \quad (1.3)$$

$$\int_0^\infty[\int_0^\infty k(x,y)f(x)dx]^p dy<k^p\| f\|_p^p \quad (1.4)$$

where the constant factors k and k^p are the best possible.

Note. In particular, we find some classical Hilbert-type inequalities as:

(1) for $k(x,y)=\frac{1}{x+y}$, since $k=\pi/\sin(\pi/p)$, (1.3) reduces (1.1);

(2) For $k(x,y)=\frac{1}{\max\{x,y\}}$, (1.3) reduces to (see [3], Theorem 341)

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \| f \|_p \| g \|_q \quad (1.5)$$

In 2006-2008, some authors also considered the operator expressing of (1.3)- (1.4).

Suppose that $k(x,y) \ge 0$ is a symmetric function with $k(x,y) = k(y,x)$, and

$$k_0(p) := \int_0^\infty k(x,y)(\tfrac{x}{y})^{\frac{1}{r}} dy, \ (r = p,q\,; x > 0)$$

is a positive number independent of x. Define an operator $T : L^r(0,\infty) \to L^r(0,\infty)$ ($r = p,q$) as:

For $f \in L^p(0,\infty)$,

$$(Tf)(y) := \int_0^\infty k(x,y) f(x) dx, \quad y \in (0,\infty) \quad (1.6)$$

Or $g \in L^q(0,\infty)$,

$$(Tf)(x) := \int_0^\infty k(x,y) g(y) dy, \quad x \in (0,\infty) \quad (1.7)$$

Then we may define the formal inner product of Tf and g as

$$(Tf,g) = \int_0^\infty \int_0^\infty k(x,y) f(x) g(y) dxdy \quad (1.8)$$

In 2006, Yang [3] proved that if for $\varepsilon \ge 0$ small enough, $k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{r}}$ is strictly decreasing for $y > 0$ the integral $\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{r}} dy = k_\varepsilon(p)$ is also a positive number independent of $x > 0$, $k_\varepsilon(p) = k_0(p) + o(1)$ ($\varepsilon \to 0^+$), and

$$\int_a^\infty k(x,y)(\tfrac{x}{y})^{\frac{1+\varepsilon}{r}} dy = k_\varepsilon(p) + \tilde{O}(1) \quad (1.9)$$

then $\| T \|_p = k_0(p)$, in this case, if $f(x)$, $g(x) \ge 0$, $f \in L^p(0,\infty)$, $g \in L^q(0,\infty)$, $\| f \|_p$, $\| g \|_q > 0$, then we have two equivalent inequalities as:

$$(Tf,g) < \| T \|_p \| f \|_p \| g \|_g$$

$$\| Tf \|_p < \| T \|_p \| f \|_p \quad (1.10)$$

where the constant $\| T \|_p$ is the best possible. In particular,for $k(x,y)$ being-1-degree homogeneous, inequalities (1.10) reduce to (1.3)-(1.4).

In this paper, use the way of weight function and the theory of operators. A new Hilbert-type integral operator is considered which an extension of the result is in [3]. As for applications, an extended Hilbert-type integral inequality and the equivalent form are given, and some particular norms are obtained.

2. MAIN RESULT

If $k_\lambda(x,y)$ is a measurable function, satisfying for λ, u, x, $y > 0$, $k_\lambda(ux,uy) = u^{-\lambda} k_\lambda(x,y)$, then we call $k_\lambda(x,y)$ the homogeneous function of $-\lambda$-degree.

Lemma 2.1. If $r > 1, \frac{1}{r} + \frac{1}{s} = 1, \lambda > 0$, $k(x,y) \ge 0$ is a homogeneous function of $-\lambda$-degree, and $k_\lambda(r) := \int_0^\infty k(1,u) u^{-\frac{1}{r}} du$ a positive number, define the weight functions $\omega(r,x)$ and $\omega(s,y)$ as

$$\omega(r,x) = \int_0^\infty k_\lambda(x,y)(\tfrac{x}{y})^{\frac{1}{r}} dy \quad (2.1)$$

then we have

(I) $\int_0^\infty k(1,u) u^{-\frac{1}{s}} du = k_\lambda(r)$;

(II) $\omega(r,x) = \omega(s,y) = k_\lambda(r)$.

Proof: (I) Setting $v = \dfrac{1}{u}$, by the assumption, we obtain $\int_0^\infty k(1,u) u^{-\frac{1}{r}} du = \int_0^\infty k(v,1) v^{-\frac{1}{s}} dv = k_\lambda(r)$

(ii) Setting $u = y/x$ in the integrals $\omega(r,x)$, in view of (i), we still find that $\omega(r,x) = k_\lambda(r)$.

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Similarly we have $\omega(s,y)=k_\lambda(r)$, The lemma is proved.

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we set $\phi(x)=x^{\frac{p}{r}-1}$, $\psi(x)=x^{\frac{q}{s}-1}$, $\psi^{1-p}(x)=x^{\frac{p}{r}-1}, x\in(0,\infty)$.

Define the real space as

$$L_\phi^p(0,\infty):=\left\{ f, \| f\|_{p,\phi}:=[\int_0^\infty \phi(x)|f(x)|^p dx]^{1/p}<\infty \right\}$$

, and then we may also define the spaces $L_\psi^q(0,\infty)$ and $L_{\psi^{1-p}}^p(0,\infty)$, $k(x,y)$ is continuous in $(0,\infty)\times(0,\infty)$,satisfying $k(x,y)=k(y,x)>0$, for $x,y\in(0,\infty)$. Define the integral operator $T: L_\phi^p(0,\infty)\to L_{\psi^{1-p}}^p(0,\infty)$ as:

For $f\in L_\phi^p(0,\infty)$

$$(Tf)(y):=\int_0^\infty k(x,y)f(x)dx, \quad y\in(0,\infty) \quad (2.2)$$

or $g\in L_\psi^q(0,\infty)$,

$$(Tg)(x):=\int_0^\infty k(x,y)g(y)dy, \quad x\in(0,\infty) \quad (2.3)$$

For $\varepsilon(\geq 0)$ small enough and $x>0$, setting $\overline{k}_l(\varepsilon,x)$ as

$$\overline{k}_{l_1,l_2}(\varepsilon,x)=\int_0^\infty k(x,y)\left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$

$(l_1=p,q, l_2=r,s)$,

We have the following theorem:

Theorem 2.2

(1)If $\overline{k}_{l_1,l_2}(0,x)=\int_0^\infty k(x,y)\left(\frac{x}{y}\right)^{\frac{1}{l_2}}dy=k_r$ ($l_2=r,s$; $x>0$), and k_r is a constant independent of x, then $T\in B$ ($L_\phi^p(0,\infty)\to L_{\psi^{1-p}}^p(0,\infty)$), and $\|T\|_l\leq k_r$ ($l=p,q$);

(2) If $\overline{k}_{l_1,l_2}(\varepsilon,x)=k_r(\varepsilon)$ ($l_1=p,q, l_2=r,s$; $x>0$) is independent of x, and $k_r(\varepsilon)=k_r+o(1)$ ($\varepsilon\to 0^+$), then $\|T\|_l=k_r$ ($l=p,q$).

Proof (I)

$$(\int_0^\infty k(x,y)f(x)dx)^p=$$

$$\{\int_0^\infty k(x,y)[\frac{x^{\frac{1}{qr}}}{y^{\frac{1}{ps}}}f(x)][\frac{y^{\frac{1}{ps}}}{x^{\frac{1}{qr}}}]dx\}^p$$

$$\leq\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{s}}x^{\frac{p}{r}-1}f^p(x)dx$$

$$[\int_0^\infty k(x,y)(\frac{y}{x})^{\frac{1}{r}}y^{\frac{q}{s}-1}dx]^{p-1}$$

$$=k_r^{p-1}y^{1-\frac{p}{r}}\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{s}}x^{\frac{p}{r}-1}f^p(x)dx$$

$$\|Tf\|_{p,\psi^{1-p}}=(\int_0^\infty(\int_0^\infty k(x,y)f(x)dx)^p dy)^{\frac{1}{p}}$$

$$\leq k_r^{\frac{1}{q}}(\int_0^\infty\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{s}}x^{\frac{p}{r}-1}f^p(x)dxdy)^{\frac{1}{p}}$$

$$=k_r^{\frac{1}{q}}(\int_0^\infty[\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{s}}dy]x^{\frac{p}{r}-1}f^p(x)dx)^{\frac{1}{p}}$$

$$=k_r^{\frac{1}{q}}(\int_0^\infty \omega(r,x)x^{\frac{p}{r}-1}f^p(x)dx)^{\frac{1}{p}}$$

$$=k_r\|f\|_{p,\phi}<\infty$$

It follows that $Tf\in L_{\psi^{1-p}}^p(0,\infty)$ and $\|T\|_p\leq k_r$ (cf. [7]). By the same way, one has $Tf\in L_{\psi^{1-q}}^q(0,\infty)$ and $\|T\|_q\leq k_r$.

(II) It is obvious that condition (2) covers condition (1). By condition (2), it follows that

$$\int_a^\infty k(y,x)\left(\frac{y}{x}\right)^{\frac{1+\varepsilon(r/p)}{r}}dx=k_r(\varepsilon)+\tilde{o}(1) \quad (2.4)$$

$$(a\to 0^+)$$

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For any a, $\varepsilon > 0$, set $f_\varepsilon(x) = 0$, $x \in (0,a)$; $f_\varepsilon(x) = (\varepsilon a^\varepsilon)^{1/p} x^{-\frac{1+\varepsilon(r/p)}{r}}$, $x \in [a,\infty)$, then $\| f_\varepsilon \|_p = 1$, and by (2.4),

$$\| T \|_p \geq \| Tf_\varepsilon \|_p = \{\int_0^\infty (\int_0^\infty k(x,y) f_\varepsilon(x)dx)^p dy\}^{\frac{1}{p}}$$

$$\geq (\varepsilon a^\varepsilon)^{1/p} \{\int_a^\infty (\int_a^\infty k(x,y) x^{-\frac{1+\varepsilon(r/p)}{r}} dx)^p dy\}^{\frac{1}{p}}$$

$$= (\varepsilon a^\varepsilon)^{1/p} \{\int_a^\infty y^{-1-\varepsilon} (\int_a^\infty k(y,x)(\frac{y}{x})^{-\frac{1+\varepsilon(r/p)}{r}} dx)^p dy\}^{\frac{1}{p}}$$

$$= (\varepsilon a^\varepsilon)^{1/p} \{\int_a^\infty y^{-1-\varepsilon} (k_r(\varepsilon) + \tilde{o}(1))^p dy\}^{\frac{1}{p}}$$

$$= k_r(\varepsilon) + \tilde{o}(1)$$

In virtue of condition (2), it follows that $\| T \|_p \geq k_r$ (for a, $\varepsilon > 0$). Hence, combining with $\| T \|_p \leq k_r$ in (1), one has $\| T \|_p = k_r$, by the same way, one has $\| T \|_q = k_r$. The theorem is proved.

Theorem 2.3: Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\overline{k}_{l_1,l_2}(\varepsilon, x)$ ($l_1 = p,q$, $l_2 = r,s$; $x > 0$) satisfy condition (1) of Theorem 2.2. If f, $g \geq 0$, and $f \in L^p_\phi(0,\infty)$, $g \in L^q_\psi(0,\infty)$, then one has the following two equivalent inequalities:

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y) dxdy \leq k_r \| f \|_{p,\phi} \| g \|_{q,\psi} \quad (2.5)$$

$$\{\int_0^\infty y^{\frac{p}{r}-1} (\int_0^\infty k(x,y) f(x) dx)^p dy\}^{\frac{1}{p}} \leq k_r \| f \|_{p,\phi} \quad (2.6)$$

where the constant factor $k_r = \int_0^\infty k(x,y) \left(\frac{x}{y}\right)^{\frac{1}{r}} dy$ is independent of x.

Proof By Holder's inequality with weight and condition (1), one has

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y) dxdy$$

$$= \int_0^\infty \int_0^\infty k(x,y)[\frac{x^{\frac{1}{qr}}}{y^{\frac{1}{ps}}} f(x)][\frac{y^{\frac{1}{ps}}}{x^{\frac{1}{qr}}} g(y)] dxdy$$

$$\leq \{\int_0^\infty \int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{s}} x^{\frac{p}{r}-1} f^p(x) dxdy\}^{\frac{1}{p}} \{\int_0^\infty \int_0^\infty k(x,y)(\frac{y}{x})^{\frac{1}{r}} y^{\frac{q}{s}-1} g^q(y) dxdy\}^{\frac{1}{q}} \quad (2.7)$$

$$= \{\int_0^\infty [\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{s}} dy] x^{\frac{p}{r}-1} f^p(x) dx\}^{\frac{1}{p}} \{\int_0^\infty [\int_0^\infty k(x,y)(\frac{y}{x})^{\frac{1}{r}} dx] y^{\frac{q}{s}-1} g^q(y) dy\}^{\frac{1}{q}}$$

$$= k_r \{\int_0^\infty x^{\frac{p}{r}-1} f^p(x) dx\}^{\frac{1}{p}} \{\int_0^\infty y^{\frac{q}{s}-1} g^q(y) dy\}^{\frac{1}{q}}$$

and (2.5) is valid.

Set $g(y) = y^{\frac{p}{r}-1} \{\int_0^\infty k(x,y) f(x) dx\}^{p-1}$ ($y \in (0,\infty)$) and use (2.5) to obtain

$$0 < \int_0^\infty y^{\frac{q}{s}-1} g^q(y) dy = \int_0^\infty y^{\frac{p}{r}-1} [\int_0^\infty k(x,y) f(x) dx]^p dy = \int_0^\infty \int_0^\infty k(x,y) f(x) g(y) dxdy \leq k_r \| f \|_{p,\phi} \{\int_0^\infty y^{\frac{q}{s}-1} g^q(y) dy\}^{\frac{1}{q}} \quad (2.8)$$

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$$\{\int_0^\infty y^{\frac{q}{s}-1} g^q(y)dy\}^{\frac{1}{p}}$$

$$= \{\int_0^\infty (\int_0^\infty k(x,y) f(x)dx)^p dy\}^{\frac{1}{p}} \quad (2.9)$$

$$\leq k_r \| f \|_{p,\phi}$$

Hence (2.6) is valid, and one shows that (2.5) implies (2.6).

If (2.6) is valid, by Holder's inequality, one has

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y) dxdy$$

$$= \int_0^\infty (\int_0^\infty k(x,y) f(x)dx) g(y)dy$$

$$\leq \{\int_0^\infty (\int_0^\infty k(x,y) f(x)dx)^p dy\}^{\frac{1}{p}} \| g \|_{q,\psi}$$

(2.10)

Then by (2.6), one has (2.5). It follows that (2.5) is equivalent to (2.6). The theorem is proved.

Note 1 Since $\| T \|_q \leq k_r$, by the same way, one still can show that

$$\{\int_0^\infty x^{\frac{q}{s}-1} (\int_0^\infty k(x,y) g(y)dy)^q dx\}^{\frac{1}{q}} \leq k_r \| g \|_{q,\psi}$$

(2.11)

and (2.11) is equivalent to (2.5). It follows that (2.5), (2.6) and (2.11) are equivalent.

Theorem 2.4 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $\overline{k}_{l_1,l_2}(\varepsilon,x)$ ($l_1 = p,q$, $l_2 = r,s$; $x>0$) satisfy condition (2) of Theorem 2.2. If f, $g \geq 0$, and $f \in L_\phi^p(0,\infty)$, $g \in L_\psi^q(0,\infty)$, and $\| f \|_{p,\phi}$, $\| g \|_{q,\psi} > 0$, T is defined by (2.2) (or (2.3)), and the formal inner product of Tf and g is defined by

$$(Tf, g) = \int_0^\infty \int_0^\infty k(x,y) f(x) g(y) dxdy$$

then one has the following two equivalent inequalities:

$$(Tf, g) < \| T \|_p \| f \|_{p,\phi} \| g \|_{q,\psi} \quad (2.12)$$

$$\| Tf \|_p \leq \| T \|_p \| f \|_{p,\phi} \quad (2.13)$$

where the constant factor $\| T \|_p = \int_0^\infty k(x,y) \left(\frac{x}{y}\right)^{\frac{1}{r}} dy$ in the above inequalities is the best possible.

Proof If (2.7) takes the form of equality, then there exist real numbers A and B such that they are not all zero, and (see [8])

$$A(\frac{x}{y})^{\frac{1}{s}} x^{\frac{p}{r}-1} f^p(x) = B(\frac{y}{x})^{\frac{1}{r}} y^{\frac{q}{s}-1} g^q(y) \quad \text{a.e.}$$

in $(0,\infty) \times (0,\infty)$.

It follows that $Axx^{\frac{p}{r}-1} f^p(x) = Byy^{\frac{q}{s}-1} g^q(y)$ a.e. in $(0,\infty)\times(0,\infty)$. Then there exists a constant C such that $Axx^{\frac{p}{r}-1} f^p(x) = C$ a.e. in $(0,\infty)$, $Byy^{\frac{q}{s}-1} g^q(y) = C$ a.e. in $(0,\infty)$.

Assume that $A \neq 0$ and then one has $x^{\frac{p}{r}-1} f^p(x) = \frac{C}{Ax}$ a.e. in $(0,\infty)$ which contradicts the fact that $f \in L_\phi^p(0,\infty)$. Hence (2.7) takes the form of strict inequality and in view of $\| T \|_p = k_r$ in the result of (2) in Theorem 2.2, one has (2.12).

Since $\| f \|_{p,\phi} > 0$, by (2.8) and (2.9), one has $g \in L_\psi^q(0,\infty)$ and $\| g \|_{q,\psi} > 0$. Hence by using (2.12), (2.8) takes the form of strict inequality and $k_r = \| T \|_p$, so does (2.9), and then (2.13) is valid.

By the same way of Theorem 2.3, (2.12) and (2.13) are obviously equivalent. In view of the fact that the constant factor $\| T \|_p$ in (2.13) is the best possible, one can conclude that the constant factor $\| T \|_p$ in (2.12) is the best possible. Otherwise, by (2.8) and (2.9), one can get a contradiction that the constant factor in (2.13) is not the best possible. The theorem is proved.

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Note 2 By the same way and in view of $\|T\|_p = \|T\|_q$, one has

$$\|Tg\|_q \le \|T\|_p \|g\|_{q,\psi} \qquad (2.14)$$

Where the constant factor $\|T\|_p$ is the best possible, and (2.14), (2.12) and (2.13) are equivalent

For giving some particular cases of Theorems 2.4, one needs the formula of the Beta function $B(u,v)$ as (see [9]):

$$B(u,v) = \int_0^1 (1-t)^{u-1} t^{v-1} dt = B(v,u) \quad (u,v>0)$$

(2.15)

3.Some particular cases

(a)Setting $k(x,y) = \frac{|x-y|^{\lambda-1}}{(\min\{x,y\})^{\lambda}}$, $0<\lambda<\min\{\frac{1}{p},\frac{1}{q}\}$ $0 \le \varepsilon < \min\{p(\frac{1}{s}-\lambda), q(\frac{1}{r}-\lambda)\}$ one obtains from (2.15) that

$$\bar{k}_{l_1,l_2}(\varepsilon,x) = \int_0^\infty \frac{|x-y|^{\lambda-1}}{(\min\{x,y\})^{\lambda}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$= \int_0^x \frac{(x-y)^{\lambda-1}}{y^{\lambda}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy + \int_x^\infty \frac{(y-x)^{\lambda-1}}{x^{\lambda}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$= \int_0^1 (1-u)^{\lambda-1} u^{(1-\frac{1+\varepsilon(l_2/l_1)}{l_2}-\lambda)-1} du + \int_0^1 (1-u)^{\lambda-1} u^{(\frac{1+\varepsilon(l_2/l_1)}{l_2}-\lambda)-1} du$$
$$\to B(\lambda,\tfrac{1}{r}-\lambda) + B(\lambda,\tfrac{1}{s}-\lambda) = k_r \quad (\varepsilon \to 0^+, l_2 = r,s)$$

Hence by Theorem 2.2

$\|T\|_p = k_r = [B(\lambda,\frac{1}{r}-\lambda) + B(\lambda,\frac{1}{s}-\lambda)]$ and by Theorem 2.4, one has

Corollary3.1. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, f , $g \ge 0$, $f \in L^p_\phi(0,\infty)$, and $g \in L^q_\psi(0,\infty)$, $\|f\|_{p,\phi}$, $\|g\|_{q,\psi} > 0$, then $0<\lambda<\min\{\frac{1}{r},\frac{1}{s}\}$, one has the following two equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda-1}}{(\min\{x,y\})^{\lambda}} f(x)g(y)dxdy < [B(\lambda,\tfrac{1}{r}-\lambda)+B(\lambda,\tfrac{1}{s}-\lambda)] \|f\|_{p,\phi}\|g\|_{q,\psi} \qquad (3.1)$$

$$\{\int_0^\infty y^{\frac{p}{r}-1}(\int_0^\infty \frac{|x-y|^{\lambda-1}}{(\min\{x,y\})^{\lambda}} f(x)dx)^p dy\}^{\frac{1}{p}} \le [B(\lambda,\tfrac{1}{r}-\lambda)+B(\lambda,\tfrac{1}{s}-\lambda)]\|f\|_{p,\phi} \qquad (3.2)$$

where the constant factor $[B(\lambda,\frac{1}{r}-\lambda)+B(\lambda,\frac{1}{s}-\lambda)]$ is the best possible.

(b)Setting $k(x,y) = \frac{|x-y|^{\lambda-1}}{(\max\{x,y\})^{\lambda}}$ $\lambda>0$, $0 \le \varepsilon < \min\{\frac{p}{s},\frac{q}{r}\}$

$$\bar{k}_{l_1,l_2}(\varepsilon,x) = \int_0^\infty \frac{|x-y|^{\lambda-1}}{(\max\{x,y\})^{\lambda}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$= \int_0^x \frac{(x-y)^{\lambda-1}}{x^{\lambda}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy + \int_x^\infty \frac{(y-x)^{\lambda-1}}{y^{\lambda}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$= \int_0^1 (1-u)^{\lambda-1} u^{(1-\frac{1+\varepsilon(l_2/l_1)}{l_2})-1} du + \int_0^1 (1-u)^{\lambda-1} u^{\frac{1+\varepsilon(l_2/l_1)}{l_2}-1} du$$
$$\to B(\lambda,\tfrac{1}{r}) + B(\lambda,\tfrac{1}{s}) = k_r \quad (\varepsilon \to 0^+, l_2 = r,s)$$

Hence by Theorem 2.2

$\|T\|_p = k_r = B(\lambda,\frac{1}{r}) + B(\lambda,\frac{1}{s})$ and by Theorem 2.4, one has

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Corollary 3.2 If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1, \frac{1}{r}+\frac{1}{s}=1$, f, $g\geq 0$, $f\in L_{\phi}^{p}(0,\infty)$, and $g\in L_{\psi}^{q}(0,\infty)$, $\| f\|_{p,\phi}$, $\| g\|_{q,\psi}>0$, then $\lambda>0$, one has the following two equivalent inequalities:

$$\int_0^\infty\int_0^\infty \frac{|x-y|^{\lambda-1}}{(\max\{x,y\})^{\lambda}} f(x)g(y)dxdy < [B(\lambda,\tfrac{1}{r})+B(\lambda,\tfrac{1}{s})]\| f\|_{p,\phi}\| g\|_{q,\psi} \quad (3.3)$$

$$\{\int_0^\infty y^{\frac{p}{r}-1}(\int_0^\infty \frac{|x-y|^{\lambda-1}}{(\max\{x,y\})^{\lambda}} f(x)dx)^p dy\}^{\frac{1}{p}} \leq [B(\lambda,\tfrac{1}{r})+B(\lambda,\tfrac{1}{s})]\| f\|_{p,\phi} \quad (3.4)$$

The constant factor $[B(\lambda,\frac{1}{r})+B(\lambda,\frac{1}{s})]$ is the best possible.

(c)Setting $k(x,y)=\frac{|x^{\lambda-1}-y^{\lambda-1}|}{(\max\{x,y\})^{\lambda}}, \lambda>0$,

$\lambda\neq 1, \frac{1}{l_2}, \frac{2}{l_2}(1-\frac{1}{l_2})$ for any $0\leq\varepsilon<\min\{\frac{p}{s},\frac{q}{r}\}$, one obtains that

(1) if $0<\lambda<1, \lambda\neq 1, \frac{1}{l_2}, \frac{2}{l_2}(1-\frac{1}{l_2})$ ($l_2=r,s$), then

$$\bar{k}_{l_1,l_2}(\varepsilon,x)=\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|}{(\max\{x,y\})^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$=\int_0^x \frac{y^{\lambda-1}-x^{\lambda-1}}{x^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$+\int_x^\infty \frac{x^{\lambda-1}-y^{\lambda-1}}{y^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$=\int_0^1 (u^{\lambda-1}-1)u^{-\frac{1+\varepsilon(l_2/l_1)}{l_2}}du$$
$$+\int_0^1 (u^{\lambda-1}-1)u^{\frac{1+\varepsilon(l_2/l_1)}{l_2}-1}du$$
$$\to\int_0^1 (u^{\lambda-1}-1)(u^{-\frac{1}{l_2}}+u^{\frac{1}{l_2}-1})du$$
$$=\frac{(\lambda rs-2)(1-\lambda)rs}{(\lambda r-1)(\lambda s-1)}=k_r$$
$$(\varepsilon\to 0^+, l_2=r,s)$$

(2) If $\lambda>1$, then

$$\bar{k}_{l_1,l_2}(\varepsilon,x)=\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|}{(\max\{x,y\})^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1+\varepsilon(l_2/l_1)}{l_2}} dy$$
$$\to -\frac{(\lambda rs-2)(1-\lambda)rs}{(\lambda r-1)(\lambda s-1)}$$
$$=k_r \quad (\varepsilon\to 0^+, l_2=r,s)$$

Hence by Theorem 2.2

$\| T\|_p=k_r=\left|\frac{(\lambda rs-2)(1-\lambda)rs}{(\lambda r-1)(\lambda s-1)}\right|$ and by Theorem 2.4, one has

Corollary 3.3..

$$\int_0^\infty\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|}{(\max\{x,y\})^{\lambda}} f(x)g(y)dxdy < \left|\frac{(\lambda rs-2)(1-\lambda)rs}{(\lambda r-1)(\lambda s-1)}\right|\| f\|_{p,\phi}\| g\|_{q,\psi} \quad (3.5)$$

$$\{\int_0^\infty y^{\frac{p}{r}-1}(\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|}{(\max\{x,y\})^{\lambda}} f(x)dx)^p dy\}^{\frac{1}{p}} \leq \left|\frac{(\lambda rs-2)(1-\lambda)rs}{(\lambda r-1)(\lambda s-1)}\right|\| f\|_{p,\phi} \quad (3.6)$$

where the constant factor $\left|\frac{(\lambda rs-2)(1-\lambda)rs}{(\lambda r-1)(\lambda s-1)}\right|$ is the best possible.

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