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ENDOMORPHISM SPECTRUMS AND ENDOMORPHISM TYPES OF GRAPHS $\mathbf{C}_n + \mathbf{y}$

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ABSTRACT

In this paper, the endomorphisms, the half-strong endomorphisms, the locally strong endomorphisms, the quasi-strong endomorphisms, the strong endomorphisms and the automorphisms of a join of cycle C_n and a vertex y are investigated. Some enumerative problems concerning these graphs are solved. In particular, the endomorphism spectrums and the endomorphism types of these graphs are given. It has a wide range of applications in Information Science.

Keywords: *Endomorphism, Endomorphism Spectrum, Endomorphism Type Join Of Graphs*

1. INTRODUCTION

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained (see [2], [5] and their references). The aim of this research is try to establish the relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. Let X be a graph. Denote by $End(X)$, $hEnd(X)$, $lEnd(X)$, $qEnd(X)$, $sEnd(X)$ and $Aut(X)$ the sets of all endomorphisms, half-strong endomorphisms, locally strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms and automorphisms of X, respectively. For a more systematic treatment of different endomorphisms, the endomorphism spectrum and endomorphism type of a graph were introduced in [2]. Hou and Luo characterized endomorphism types of generalized polygons in [6]. In [4] Hou, Luo and Cheng explored the endomorphism monoid of $\overline{P_n}$, the complement of a path P_n with n vertices. It was shown that $End(\overline{P_n})$ is an orthodox monoid. The endomorphism spectrum and the endomorphism type of $\overline{P_n}$ were given. The endomorphism monoids and endomorphism-regularity of graphs were considered by several authors (see [1], [8], [9] and [10]).

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in graph X ,the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. Let X_1 and X_2 be two graphs. The join of X_1 and X_2, denoted by $X_1 + X_2$, is a graph such that $V(X_1 + X_2) = V(X_1) \cup V(X_2)$ and $E(X_1 + X_2) = E(X_1) \cup E(X_2) \cup \{\{x_1, x_2\} \mid x_1 \in V(X_1), x_2 \in V(X_2)\}$. A subgraph H is called an induced subgraph of X if for any $a, b \in H$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph $C_n + y$ is the join of a cycle C_n and a vertex y .In the following, we denote it by $V(C_n) = \{x_1, x_2, \cdots, x_n\}$.

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is called a homomorphism (from X to Y) if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is called a half-strong homomorphism if $\{f(a), f(b)\} \in E(Y)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. A homomorphism f is called a locally strong homomorphism if $\{f(a), f(b)\} \in E(Y)$ implies that for every preimage $x_1 \in V(X)$ of $f(a)$ there exists a preimage $x_2 \in V(X)$ of $f(b)$ such that $\{x_1, x_2\} \in E(X)$ and analogously for every

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preimage of $f(b)$. A homomorphism f is called a quasi-strong homomorphism if $\{f(a), f(b)\} \in E(Y)$ implies that there exists a preimage $x_1 \in V(X)$ of $f(a)$ which is adjacent to every preimage of $f(b)$ and analogously for preimage of $f(b)$. A homomorphism f is called a strong homomorphism if $\{f(a), f(b)\} \in E(Y)$ implies that any preimage of $f(a)$ is adjacent to any preimage of $f(b)$. A homomorphism f is called an it isomorphism if f is bijective and f^{-1} is a homomorphism. A homomorphism (resp. isomorphism) f from X to itself is called an endomorphism (resp. automorphism) of X. The sets of all endomorphisms, half-strong endomorphisms, locally strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms and automorphisms of X are denoted by $End(X)$, $hEnd(X)$, $lEnd(X)$, $qEnd(X)$, $sEnd(X)$ and $Aut(X)$, respectively. Clearly, for any graph X, we always have

$$Aut(X) \subseteq sEnd(X) \subseteq qEnd(X) \subseteq lEnd(X) \subseteq hEnd(X) \subseteq End(X).$$

Various endomorphisms were investigated by many authors (see [2] and its references). To pursue a more systematic treatment of different endomorphisms, Böttcher and Knauer in [2] introduce the concepts of the endomorphism spectrum and the endomorphism type of a graph. For a graph X, the 6-tuple

$$(|EndX|,|hEndX|,|lEndX|,|qEndX|,|sEndX|,|AutX|)$$

is called the endomorphism spectrum of X and is denoted by $\mathrm{EndospecX}$, that is,

$$EndospecX = (|EndX|,|hEndX|,|lEndX|,|qEndX|,|sEndX|,|AutX|).$$

Associate with EndospecX a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$ with $s_i \in \{0,1\}, i = 1,2,3,4,5$, where $s_i = 0$ indicates that the ist coordinate equals to the $(i+1)$st coordinate in EndospecX, $s_i = 1$ otherwise. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the endomorphism type of X and is denoted by EndospecX.

There are 32 possibilities, that is, *endotype* 0 up to *endotype* 31. It is known that *Endotype* 0 describes unretractive graphs, *endotype* 0 up to 15 describe *S*-unretractive graphs, *endotype* 16 describes *E*-*S*-unretractive graphs which are not unretractive, *endotype* 31 describes graphs for which all 6 sets are different (see[2] and its references).

A subgraph of X is called the endomorphic image of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c,d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e., for $a,b \in V(X)$, $(a,b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f.

We shall use the standard terminology and notation of semigroup theory as in [7] and of graph theory as in [3].

2. ENDOMORPHISM SPECTRUMS AND ENDOMORPHISM TYPES OF $C_n + y$

In this section, we will character various endomorphisms of wheel graph. The endomorphism spectrum and endomorphism type of $C_n + y$ will be given.

Let $V(C_n + y) = \{x_1, x_2, \cdots, x_n\} \cup \{y\}$. Now we suppose $\{x_1, x_n\} \in E$, $\{x_i, x_{i+1}\} \in E$ for any $i = 1,2,\cdots,n-1$ and $\{x_i, y\} \in E$ for any $i = 1,2,\cdots,n$. In the following, we denote $C_n + y$ by W_n.

To character various endomorphisms of wheel graph, two cases need to be considered separately: n is an odd and n is an even. We first consider graph $C_{2m+1} + y$. The following result are given in [8].

Lemma 2.1[8] $End(X_1 + X_2) = Aut(X_1 + X_2)$ if and only if $End(X_i) = Aut(X_i)$, $i = 1,2$.

Since odd cycle is unretractive, it is easy to give the following Lemma:

Lemma 2.2 $W(2m+1)$ is unretractive, that is $End(W(2m+1)) = Aut(W(2m+1))$.

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The next Lemma characterizes the automorphism group of $W(2m+1)$.

Lemma 2.3

$$Aut(W(n)) \cong \begin{cases} S_4, & if\ n=3, \\ D_n, & if\ n>3. \end{cases}$$

where S_4 is the symmetric group of degree 4 and D_n is the dihedral group of degree n.

Proof If $m=3$, then $W(3)\cong K_4$. Since every permutation of the vertices of the complete graph K_4 is a automorphism, $Aut(W(3))=S_4$. If $n>3$, then $d(x_i)=3$ and $d(y)=n>3$.Thus $f(y)=y$ for any $f\in Aut(W(n)))$. Let $f^{'}\in Aut(C_n))$. Define

$$f(x)=\begin{cases} f^{'}(x), & x=x_i, i=1,2,\cdots,n, \\ y, & x=y. \end{cases}$$

Then $f\in Aut(W(n))$.

Define $\phi: Aut(C_n)\to Aut(W(n))$ by $f^{'}\to f$. Clearly, ϕ is an isomorphism from $Aut(C_n)$ to $Aut(W(n))$.Therefore, $Aut(C_n)\cong Aut(W(n))$.Notice that $Aut(C_n)\cong D_n$. Therefore, $Aut(W(n))\cong D_n$.

Theorem 2.4 $Endospec(W(2m+1))=$

$$\begin{cases} (24,24,24,24,24,24), & if\ m=1, \\ (4m+2,4m+2,4m+2, & \\ 4m+2,4m+2,4m+2), & if\ m\geq 2. \end{cases}$$

Proof It follows directly from Lemmas 2.2 and 2.3.

Lemma 2.5 $End(W(2m))=sEnd(W(2m))$

Proof Let $f\in End(W(2m))\setminus sEnd(W(2m))$. If $f(x_1)=f(x_2)$ for some $x_1,x_2\in V(W(2m))$, then $N(x_1)=N(x_2)$. Notice that $N(x_1)\neq N(x_2)$ for any $x_1,x_2\in V(W(2m))$. A contradiction. Hence $End(W(2m))=sEnd(W(2m))$.

Lemma 2.6 Let $f\in End(W(2m))$. If $f(y)\neq y$, then $f(C_{2m})\cong P_2$ or $f(C_{2m})\cong P_3$.

Proof If $f(y)\neq y$, then $f(y)\in V(C_{2m})$. Since y is adjacent to every vertex of C_{2m}, $f(y)$ is adjacent to every vertex of $f(C_{2m})$. Notice that $d(x_i)=3$ for any $x_i\in V(W(2m))$. Therefore, $f(C_{2m})\cong P_2$ or $f(C_{2m})\cong P_3$.

Lemma 2.7 Let $f\in End(W(2m))\setminus Aut(W(2m))$. If $f(y)=y$, then $f(C_{2m})\cong P_t$ for some $t\leq m$.

Proof If $f(y)=y$, then $f(C_{2m})\in V(C_{2m})$. Since every connected subgraphs of C_{2m} are paths, $f(C_{2m})\cong P_{t+1}$ for some $t\leq m$.

Let $f\in lEnd(W(2m))$. Suppose $2m=2kt$ for some positive integer t and some even k. Denote $V(P_{t+2})=\{x_0,x_1,\cdots,x_t,x_{t+1}\}$. If

$f^{-1}(x_0)=\{x_i,x_{i+2t},x_{i+4t},\cdots\cdots,x_{i+2kt-2t}\}$,

for any $l\in V(P_{t+1})$ such that $1\leq l\leq t$,

$f^{-1}(x_l)=\{x_{i+l},x_{i-l},x_{i+2t+l},x_{i-2t-k}\cdots\cdots,x_{i+(k-1)t+l},$
$x_{i-(k-1)t-l}\}$,

$f^{-1}(x_{t+1})=\{x_{i+t},x_{i+3t},x_{i+5t},\cdots\cdots x_{i+(2k-1)t}\}$.

Then we call f is a complete folding from C_{2m} to P_{t+2}.

Lemma 2.8 Let $f\in End(W(2m))$. Then f is locally-strong if and only if $f|_{C_{2m}}$ is a complete folding.

Proof Sufficiency. If $f|_{C_{2m}}$ is a complete folding, then f is locally-strong.

Necessity. Let f be a locally-strong endomorphism of $W(2m)$. Then $f|_{C_{2m}}$ is a homomorphism from C_{2m} to P_t. Suppose $V(P_{t+2})=\{x_0,x_1,\cdots,x_t,x_{t+1}\}$ and $f(i)=0$. Since $i+1,i-1$ is adjacent to i, $f(i+1),f(i-1)$ is adjacent to 0. Thus $f(i+1)=f(i-1)=1$. At the same time, $i-2$ is adjacent to $i-1$, $i+2$ is adjacent to $i+1$. Therefore $f(i-2)=f(i+2)=2$. A similar argument will show that $f(i+k-1)=f(i-k+1)$ for any $k=3,\cdots,t-1,t$ .Therefore, $f|_{C_{2m}}$ is a complete folding.

Lemma 2.9 $|lEnd(W(2m))|=\sum_{t,k}4mt+4mt+8m$, (where $2m=2kt$ and $t\neq 1,2$).

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Proof Let $f \in lEnd(W(2m))$. By Lemma 2.8, $f|_{C_{2m}}$ is a complete folding. Suppose $2m = 2kt$ for some positive integer t and some even k .Then $f|_{C_{2m}}(C_{2m}) \cong P_{t+1}$. If $f(y) = y$, Then $V(P_{t+1}) \subseteq V(C_{2m})$ .It is easy to see that it has $2t$ choice to map $W(2m)$ to P_{t+1} for some positive integer t .Now C_{2m} has $2m$ subgraphs isomorphism to P_{t+1}. Hence there exists exactly $4mt$ endomorphisms in this case.

If $f(y) \neq y$, then $I_f \cong K_3$ or $I_f \cong P_3 + y$. If $I_f \cong K_3$, then $f|_{C_{2m}}(C_{2m}) \cong K_2$ .Now $W(2m)$ has only one ρ_f and $2m$ subgraphs isomorphism to K_3 .Giving one certain I_f, f has 4 choice to map ρ_f to I_f .Hence there exists exactly $8m$ endomorphisms in this case.

If $I_f \cong P_3 + y$, then $f|_{C_{2m}}(C_{2m}) \cong P_3$ .Now $W(2m)$ has only t different ρ_f and $2m$ subgraphs isomorphism to $P_3 + y$ .Giving one ρ_f and one I_f , f has 2 choice to map ρ_f to I_f .Hence there exists exactly $4mt$ endomorphisms in this case.

Therefore, $|lEnd(W(2m))| = \sum_{t,k} 4mt + 4mt + 8m$.

Lemma 2.10 $qEnd(W(2m)) = sEnd(W(2m))$ for any $m > 2$.

Proof Let $f \in qEnd(W(2m)) \setminus sEnd(W(2m))$. Then there exist $x_1, x_2 \in V(C_{2m})$ with $N(x_1) \neq N(X_2)$ such that $f(x_1) = f(x_2)$ .Since any endomorphism make an edge to an edge, $f(C_{2m})$ contains at least one edge. Suppose $\{f(x_i), f(x_1)\}$ is an edge of $f(C_{2m})$ .Now we claim that $|f^{-1}(f(x_1))| = 2$. Otherwise, there exists $x_j \in f^{-1}(f(x_i))$ such that x_j is adjacent to every vertex of $f^{-1}(f(x_1))$. It follows from $\{x_j, y\} \in E$ that $d(x_j) \geq 4$. A contradiction. A similar argument will show that $|f^{-1}(f(x_j))| \leq 2$.

If there exists $x_p \in C_{2m}$ such that $f(x_p) \neq f(x_i)$ and $\{f(x_p), f(x_1)\} \in E(W(2m))$. By the definition of endomorphism, there exists $x_q \in f^{-1}(f(x_p))$ such that $\{x_1, x_q\} \in E$ and $\{x_2, x_q\} \in E$. Thus x_1, x_2, x_j, x_q form a 4-cycle in C_{2m} , A contradiction.

If $\{f(x_p), f(x_1)\} \notin E(W(2m))$ for any $f(x_p) \neq f(x_i)$. Since $f(x_t) = f(x_i)$ for any $x_t \in C_{2m}$ such that $\{x_t, x_j\} \in E$ for some $x_j \in f^{-1}(f(x_1))$, $|f^{-1}(f(x_i))| = 2$. Without loss of generality, suppose $f^{-1}(f(x_i)) = \{x_i, x_j\}$. Since f is quasi-strong, there exist at least one vertex in $f^{-1}(f(x_1))$ adjacent to every vertex of $f^{-1}(f(x_j))$. Similarly, there exist at least one vertex in $f^{-1}(f(x_i))$ adjacent to every vertex of $f^{-1}(f(x_1))$. Without loss of generality, suppose $\{x_1, x_t\} \in E$ for any $x_t \in f^{-1}(f(x_i))$ and $\{x_i, x_t\} \in E$ for any $t = 1,2$ .Now since $|N(x_2) \cap C_{2m}| = 2$, $\{x_2, x_j\} \in E$. Thus x_1, x_2, x_i, x_j form a 4-cycle in C_{2m} , A contradiction.

Therefore, $qEnd(W(2m)) = sEnd(W(2m))$ for any $m > 2$.

Lemma 2.11 $End(W(2m)) = hEnd(W(2m))$.

Proof Let $f \in End(W(2m)) \setminus hEnd(W(2m))$. Since any endomorphism map a connected subgraph of a graph G to a connected subgraph of itself, $f(C_{2m})$ is connected. Since y is adjacent to every vertex of C_{2m}, $f(y)$ is adjacent to every vertex of $f(C_{2m})$. If $f(y) = y$, then $f(C_{2m}) = C_{2m}$ or $f(C_{2m}) \cong P_t$ for some $t < 2m$; If $f(y) \neq y$, then $f(C_{2m}) \cong P_2$ or $f(C_{2m}) \cong P_3$.

Let $\{a, b\} \in f(C_{2m})$ such that $\{x_1, x_2\} \notin E$ for any $x_1 \in f^{-1}(a)$ and $x_2 \in f^{-1}(b)$. Since $x_1, x_2 \in V(C_{2m})$,there exists a path P from x_1 to x_2. By the definition of endomorphism, there also exist a path p' from $f(x_1)$ to $f(x_2)$. Notice that $\{f(x_1), f(x_2)\} \in E$. Then the subgraph of $W(2m)$ induced by $V(p')$ is a cycle less than C_{2m} . A contradiction.

Therefore, $End(W(2m)) = hEnd(W(2m))$.

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Lemma 2.12

$$|End(W(2m))| = \sum_{k=1}^{2m} (2cos\frac{k\pi}{m})^{2m} + m \cdot 2^{m+3} - 8m$$

Proof If $f(C_{2m}) \subseteq C_{2m}$, then $f|_{C_{2m}}$ is an endomorphism of C_{2m}. Since y is adjacent to every vertex of C_{2m}, $f(y) = y$ .Thus the number of endomorphisms of $W(2m)$ is equal to the number of endomorphisms of C_{2m}. By [3](page 188),the numbers of endomorphisms of C_{2m} is the sum of the $2mth$ powers of the eigenvalues of C_{2m}. It is $\sum_{k=1}^{2m} (2cos\frac{k\pi}{m})^{2m}$.

If $f(C_{2m}) \not\subseteq C_{2m}$, then $f(C_{2m}) = P_2$ or $f(C_{2m}) = P_3$ .If $f(C_{2m}) = P_2$, then $\rho_f = \{[y],[x_1, x_3, \cdots, x_{2m-1}],[x_2, x_4, \cdots, x_{2m}]\}$ and $I_f \cong K_3$. Notice that there exist $2m$ subgraphs in $W(2m)$ isomorphism to K_3. Now it has 4 choice to map ρ_f to I_f. Hence there exists exactly $8m$ endomorphisms in this case.

If $f(C_{2m}) = P_3$, then $I_f \cong P_3 + y$ and there are $2m$ different I_f in $W(2m)$ .At the same time, there are $2(2^m - 2)$ different ρ_f. Now it has 2 choice to map ρ_f to I_f . Hence there exists exactly $m \cdot 2^{m+3} - 16m$ endomorphisms in this case. Therefore,

$$|End(W(2m))| = \sum_{k=1}^{2m} (2cos\frac{k\pi}{m})^{2m} + m \cdot 2^{m+3} - 8m$$

Theorem 2.13

$$EndospecW(2m) = \left(\sum_{k=1}^{2m} (2cos\frac{k\pi}{m})^{2m} + m \cdot 2^{m+3} - 8m, \sum_{k=1}^{2m} (2cos\frac{k\pi}{m})^{2m} + m \cdot 2^{m+3} - 8m, \sum_{t,k} 4mt + 4mt + 8m, 4m, 4m, 4m\right).(m > 2)$$

Proof It follows directly from Lemmas 2.5, 2.9, 2.10, 2.11 and 2.12.

Theorem 2.14

$$EndospecW(4) = (48, 48, 48, 48, 48, 8).$$

Proof Let $f \in End(W(4)) \square Aut(W(4))$. Then

$$\rho_f = \begin{cases} \{[1,3],[2],[4],[y]\}, & or, \\ \{[1],[3],[2,4],[y]\}, & or, \\ \{[1,3],[2,4],[y]\}. \end{cases}$$

If $\rho_f = \{[1,3],[2],[4],[y]\}$, then $I_f \cong P_3 + y$. Now it is easy to see that there are 4 subgraphs in $W(4)$ isomorphism to I_f .It has 4 choice to map ρ_f to I_f . Hence there exists exactly 16 endomorphisms in this case. Similarly, if $\rho_f = \{[1],[3],[2,4],[y]\}$, then there exists exactly 16 endomorphisms. If $\rho_f = \{[1,3],[2,4],[y]\}$, then $I_f \cong K_3$. Now it is easy to see that there are 4 subgraphs in $W(4)$ isomorphism to I_f .Now it has 6 choice to map ρ_f to I_f . Thus there exists exactly 24 endomorphisms in this case. Notice that $|Aut(W(4))| = 8$. Therefore, $|End(W(4))| = 48$. Clearly, $End(W(4)) = sEnd(W(4))$. Hence $EndspecW(4) = (48, 48, 48, 48, 48, 8)$.

Theorem 2.15

$$EndotypeW(n) = \begin{cases} 0, & if \;\; n \;\; is \;\; odd, \\ 15, & if \;\; n = 4, \\ 25, & if \;\; n \;\; is \;\; even \;\; and \;\; n > 4. \end{cases}$$

Proof It follows directly from Theorems 2.4, 2.13 and 2.14.

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