

ROBUST PREDICTIVE CONTROL FOR POLYTOPIC TIME SINGULAR SYSTEM WITH INPUT CONSTRAINTS

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ABSTRACT

For polytopic time singular system with inputs constraints, the paper addresses a robust predictive control law using linear matrix inequality (LMI). A piecewise constant control sequence is calculated by minimizing the worst-case linear quadratic objective function. At each sample time, the sufficient conditions on the existence of the model predictive controllers are derived and expressed as linear matrix inequalities. The resulting predictive control law leads to regular, impulse-free and robust stable system, and the performance of this closed-loop system is guaranteed. Finally the numerical imitation shows the effectiveness of the proposed method.

Keywords: *Singular System, Predictive Control, Polytopic Uncertainty, Input Constraints, Linear Matrix Inequality, Time Delay*

1. INTRODUCTION

Model predictive control (MPC) [1,2] is a popular strategy in dealing with multivariable constrained control problems that are encountered in process industries. It has been attracted notable attentions in the control of dynamic systems and plays an important role in control practice. Analysis and synthesis approaches for robust MPC have been extensively studied. [3-5]

In Ref.[3], their main idea is to use infinite horizon control laws to guarantee robust stability for state feedback. Another paper by Vesel et al[4] presented the problem of designing a robust output/state model predictive control for linear polytopic systems with input constraints. All the time demanding computations of state feedback gain matrices were realized off-line. The actual value of the control variable was obtained through simple on-line computation of scalar parameters and the convex combination of the computed matrix gains. Another work considered output feedback robust model predictive control for the quasi-linear parameter varying (quasi-LPV) system with bounded disturbance. An iterative algorithm is proposed for the on-line synthesis of the control law via convex optimization [5]. References [6-8] addressed the robust model predictive control problems, giving the sufficient conditions on the existence of robust predictive control law and analyzing the feasibility and asymptotically stability of the closed-loop uncertain systems with delay.

The singular system model is a natural representation of dynamic system. It describes a larger class of systems than the normal linear system model and has wide applications in process modeling. Robust model predictive control is also essential in the application of singular systems [9-12]. A piecewise constant control sequence in Ref.[9] was calculated by minimizing the worst-case linear quadratic objective function. At each sample time, the sufficient conditions on the existence of the model predictive control were derived and expressed as linear matrix inequalities. Ref.[10] considered the stabilization of linear continuous time singular systems and presented a sampled-data model predictive control scheme. For uncertain singular systems with both state and input delays, the approximate solutions of optimal problems for infinite time interval and with quadratic performance index were calculated in Ref.[11]. The mixed H_2/H_∞ control approach to design of MPC has been proposed in Ref.[12].

The main idea of this paper is to present the robust model predictive control law for polytopic time singular systems with input constraints, to analyze the feasibility of the problem and provide all time demanding computations of state feedback gain matrices guaranteeing the performance robustness and performance (guaranteed cost) over whole uncertainty domain.

The paper is organized as follows. A problem formulation and preliminaries on a predictive state model as a polytopic singular system are given in



the next section. In section 3, the approach of robust state feedback predictive controller design using linear matrix inequality is presented. There is an example to illustrate the effectiveness of the proposed method which is discussed in the section 4. Finally, some conclusions are given in the section 5.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following polytopic singular system with time-delay:

$$\begin{aligned} E\dot{x}(t) &= A(t)x(t) + A_1(t)x(t-h) + B(t)u(t) \\ y(t) &= C(t)x(t) \\ x(t) &= \varphi(t), t \in [-h, 0], \end{aligned} \tag{1}$$

Euclidean norm bounds on the input is given as $\|u(t)\|_2 \leq u_{\max}, t \geq 0$, where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input vector, $y(t) \in R^p$ is the output vector, $\varphi(t)$ is the continuous initial function. E, A_i, A_{1i}, B_i, C and C are real constant matrices with appropriate dimensions, the matrix $E \in R^{n \times n}$ is a real constant matrix with rank $(E) = r < n$, h is positive time-delay constants,

$$\begin{aligned} [A(t) \quad A_1(t) \quad B(t)] &= \sum_{i=1}^m \lambda_i(t) [A_i \quad A_{1i} \quad B_i], \\ \sum_{i=1}^m \lambda_i(t) &= 1, \lambda_i(t) \geq 0 \\ \Omega &= \{ \sum_{i=1}^m \lambda_i(t) [A_i \quad A_{1i} \quad B_i] \mid \sum_{i=1}^m \lambda_i(t) = 1, \\ &\lambda_i(t) > 0, t > 0 \} \end{aligned}$$

Assume that model predictive control for (1) will be considered over an infinite horizon. Let T be the fixed sampling interval. At sampling time kT for $k = 0, 1, \dots$, plant measurements are obtained, then a predictive model is used to predict future behaviors of the system. $x(kT + \sigma, kT)$ denote the predicted state at time $kT + \sigma$, based on the measurements at sampling time kT , $x(kT, kT)$ refers to the state measured at sampling time kT , $u(kT + \sigma, kT)$ is the control action for time $kT + \sigma$ obtained by an optimization problem over the infinite prediction horizon.

For the polytopic singular system with delay (1), the rolling optimization performance index in the infinite horizon is considered as follows:

$$\min_{u(kT + \sigma, kT), \sigma \geq 0} \max_{A_i(k) \quad B_i(k) \in \Omega} J_{\infty}(k) \tag{2}$$

$$\begin{aligned} J_{\infty}(k) &= \int_0^{\infty} (x(kT + \sigma, kT)^T R_1 x(kT + \sigma, kT) \\ &+ u(kT + \sigma, kT)^T R_2 u(kT + \sigma, kT)) d\sigma \end{aligned} \tag{3}$$

where $R_1 > 0, R_2 > 0$ are the weighted matrices.

The problem studied in this paper can be summarized as follows. Design the robust model predictive controller with state feedback and input constraints in the form

$$\begin{aligned} u(kT + \sigma, kT) &= Kx(kT + \sigma, kT), \sigma \geq 0. \\ \|u(kT + \tau, kT)\|_2 &\leq u_{\max}, \tau \geq 0 \end{aligned} \tag{4}$$

such that the system (1) is regular, impulse-free and robust stable and meets the performance index (2) and (3).

Definition 1. Singular system $E\dot{x}(t) = Ax(t) + Bu(t)$ is stabilizable if there exists control law $u(t) = K(t)x(t)$ such that the closed-loop system is regular, impulse-free and asymptotically stable.

Definition 2. Singular system $E\dot{x}(t) = Ax(t) + A_1x(t-h)$ is regular, impulse-free and asymptotically stable if there exists matrix Q, P such that $E^T P = P^T E \geq 0$, and $AP^T + P^T A + P^T A_1 Q^{-1} A_1 P + Q < 0$

Lemma 1. Let orthogonal matrices $U = [U_1 \ U_2]$, $V = [V_1 \ V_2]$ be such that $E = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T$, from which it can be seen that $EV_2 = 0, U_2^T E = 0$. the following items are true.

(1) All Z satisfying $Z E^T = E Z^T \geq 0$ can be parameterized as $Z = EV_1 W^T V_1^T + SV_2^T$, where $W \geq 0 \in R^{r \times r}, S \in R^{m \times (n-r)}$

(2) Furthermore, when $Z = EV_1 W^T V_1^T + SV_2^T$ is nonsingular and $W > 0$, then there exists \hat{W} such that $(EV_1 W^T V_1^T + SV_2^T)^{-T} = U_1 W U_1^T E + U_2 \hat{S}$ with $\hat{W} = \Sigma_r^{-1} W^{-1} \Sigma_r^{-1}$ and $\hat{S} = U_2^T (EV_1 W^T V_1^T + SV_2^T)^{-T}$

3. MAIN RESULTS

To solve the robust MPC problem, the key is to solve the optimization problem (2)(3). We first need to compute $J_{\infty}(k)$ by a maximization over the whole uncertainty domain $[A(k) \ A_1(k) \ B(k)] \in \Omega$. However, this maximization is not numerically tractable. Hence, in Ref.[9], by imposing an inequality constraint, an upper bound for $J_{\infty}(k)$ is derived, and then the upper bound is minimized. Consider a quadratic function:

$$V(x(t)) = x(t)^T E^T P x(t) + \int_{t-h}^t x^T(s) Q x(s) ds > 0 \tag{5}$$



with $Q > 0, E^T P = P^T E \geq 0$ and P is nonsingular .

At sampling time kT , suppose that $V(x(t))$ satisfies the following robust stability constraint:

$$\dot{V}(x(kT + \tau, kT)) \leq -(x^T(kT + \tau, kT)R_1x(kT + \tau, kT) + u^T(kT + \tau, kT)R_2u(kT + \tau, kT)) \quad (6)$$

For all $[A(k) \ A_1(k) \ B(k)] \in \Omega$, $\tau \geq 0$ with control law (4) and $J_\infty(k)$ to be finite, we must have $x(\infty, kT) = 0, V(\infty, kT) = 0$ under the control law (4). Hence, integrating both sides of the inequality (6) from $\tau = 0$ to ∞ , we obtain

$$J_\infty(k) \leq V(x(kT)) \quad (7)$$

thus, the robust MPC problem at time kT can be solved by minimizing $V(x(kT))$ subject to the imposed constraint (6), namely

$$\max_{[A(k) \ A_1(k) \ B(k)] \in \Omega} J_\infty(k) \leq V(x(kT)) \leq \gamma,$$

this gives an upper bound on the robust performance objectives.

The goal of robust MPC algorithm has been redefined to synthesize, at each time step k , a constant state-feedback control law $u(kT + \sigma, kT) = Kx(kT + \sigma, kT), \sigma \geq 0$. to minimize this upper bound, only the first computed input $u(kT, kT) = Kx(kT, kT)$ is implemented. At the next sampling time, the state $x((k+1)T)$ is measured, and the optimization is repeated to recompute K . The following theorem gives the LMI conditions for the feasibility of the optimization problem (2)(3) and the expression of the state feedback matrix K .

Theorem 1. Let $x(kT)$ be the state of uncertain system (1) measured at sampling time kT . The state feedback matrix K in the controller (4) that minimizes $V(x(kT))$ is given by

$$K = Y^T (EV_1WV_1^T + SV_2^T)^{-T} \quad (8)$$

where $X_1 > 0, W > 0, Y, S$ and a scalar γ are obtained from the following convex programming problem:

$$\min_{\gamma, X_1, M_1, S, Y} \gamma + tr(M_1) \quad (9)$$

$$s.t. \begin{bmatrix} \gamma I & x^T(kT)V_1 \\ V_1^T x(kT) & W \end{bmatrix} \geq 0 \quad (10)$$

$$\begin{bmatrix} M_1 & N^T \\ N & X_1 \end{bmatrix} > 0 \quad (11)$$

$$\begin{bmatrix} E_1 & A_l X_1 & I & Y & Z & Z \\ * & -X_1 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -R_2^{-1} & 0 & 0 \\ * & * & * & * & -X_1 & 0 \\ * & * & * & * & * & -R_2^{-1} \end{bmatrix} < 0 \quad (12)$$

$$\begin{bmatrix} u_{\max} I & Y^T V_1 \\ * & W \end{bmatrix} \geq 0 \quad (13)$$

where, $E_l = ZA_l^T + YB_l^T + A_l Z^T + B_l Y^T, l = 1, 2, \dots, m$.

$$Z = EV_1WV_1^T + SV_2^T.$$

$\int_{-h}^0 x^T(kT + \tau, kT)x(kT + \tau, kT)d\tau = N^T N \cdot V_1, V_2$, can be obtained by Lemma 1.

Remark 1. Notice that K in (8) and the solutions X_1, W, Y, S, γ to LMIs (9)–(13) depend only on the current state $x(kT)$ at sampling time kT , hence, X_1, W, Y, S, γ remain constant in a certain interval $[kT, (k+1)T)$, but in different intervals, X_1, W, Y, S, γ can be different with the change of $x(kT)$.

Proof. At sampling interval $[kT, (k+1)T)$, define Lyapunov-krasovskii functions as follows:

$$V(x(kT)) = x(kT)^T E^T P x(kT) + \int_{-h}^0 x^T(kT + \tau, kT) Q x(kT + \tau, kT) d\tau \quad (16)$$

where $Q > 0, E^T P = P^T E \geq 0$ and P is nonsingular.

If there exist a scalar γ satisfying $x^T(kT)E^T P x(kT) \leq \gamma$, then $x^T(kT)E^T P x(kT) \leq \gamma$ is equivalent to (10) by the Schur complement and ref.[9]. Furthermore, an invariant ellipsoid $\mathcal{X} = \{z \mid z^T V_1 W^{-1} V_1^T z \leq 1\}$ for the predicted states of the uncertain system (1) is obtained. The second item in (14) may be reduced to

$$\begin{aligned} & \int_{-h}^0 x^T(kT + \tau, kT) Q x(kT + \tau, kT) d\tau \\ &= \int_{-h}^0 tr(x^T(kT + \tau, kT) X_1^{-1} x(kT + \tau, kT)) d\tau \quad (15) \\ &= tr(N^T N X_1^{-1}) = tr(N^T X_1^{-1} N) \end{aligned}$$

where $X_1^{-1} = Q$, assume there exist a matrix M_1 such that $tr(N^T X_1^{-1} N) < tr(M_1)$, then (11) holds by the Schur complement. So $V(x(kT)) < \gamma + tr(M_1)$ and the problem (7) is implied $\min \gamma + tr(M_1)$. From (1) and (2), (4) is implied for $\tau \geq 0$:



$$\begin{aligned} & \dot{V}(x(kT + \tau)) \\ &= \dot{x}^T(kT + \tau)E^T P x(kT + \tau) + \\ & x^T(kT + \tau)P^T E \dot{x}(kT + \tau) + \\ & x^T(kT + \tau)Q x(kT + \tau) - \\ & x^T(kT + \tau - h)Q x(kT + \tau - h) \\ &= M^T P x(kT + \tau) + x^T(kT + \tau)P^T M + \\ & x^T(kT + \tau)Q x(kT + \tau) - \\ & x^T(kT + \tau - h)Q x(kT + \tau - h) \\ &\leq -x^T(kT + \tau)(R_1 + K^T R_2 K)x(kT + \tau) \end{aligned} \tag{16}$$

where $M = Ax(kT + \tau) + A_1 x^T(kT + \tau - h) + BKx(kT + \tau)$.

Furthermore, (16) is equivalent to

$$\begin{bmatrix} x^T(kT + \tau) & x^T(kT + \tau - h) \end{bmatrix} \begin{bmatrix} \Phi_1 & P^T A_1 \\ * & -Q \end{bmatrix} \begin{bmatrix} x(kT + \tau) \\ x(kT + \tau - h) \end{bmatrix} \leq 0 \tag{17}$$

where $\Phi_1 = (A + BK)^T P + P^T (A + BK) + Q + R_1 + K^T R_2 K$,

by the Schur complement lemma, we have

$$\begin{bmatrix} \Phi_2 & P^T A_1 & P^T & K^T & I & I \\ * & -Q & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -R_2^{-1} & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & -R_1^{-1} \end{bmatrix} < 0 \tag{18}$$

where $\Phi_2 = (A + BK)^T P + P^T (A + BK)$.

Multiplying by $diag\{P^{-T}, X_1, I, I, I, I\}$, on the left, $diag\{P^{-1}, X_1, I, I, I, I\}$, on the right, and defining $Z = P^{-T} > 0, Y = ZK^T$, by Lemma 1, Z can be reconstructed by $Z = EV_1 W V_1^T + S V_2^T$, we have (21).

The inequality (18) is affine in $[A(k + i) \ A_1(k + i) \ B(k + i)]$, hence it is satisfied for all $[A(k + i) \ A_1(k + i) \ B(k + i)] \in \Omega$ if and only if there exist $X_1 > 0, Y, W > 0, S$ at sampling time kT such that (12) is hold.

$$\begin{bmatrix} \Psi & A_1 X_1 & I & Y & Z & Z \\ * & -X_1 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -R_2^{-1} & 0 & 0 \\ * & * & * & * & -X_1 & 0 \\ * & * & * & * & * & -R_2^{-1} \end{bmatrix} < 0 \tag{19}$$

where $\Psi = ZA^T + YB^T + AZ^T + BY^T$.

Now, we will reduce the input constraints to LMI, at sampling time kT ,

$\|u(kT + \tau, kT)\|_2 \leq u_{\max}, \tau \geq 0$, furthermore

$$\begin{aligned} \max_{\tau \geq 0} \|u(kT + \tau, kT)\|_2^2 &= \max \|Y^T Z^{-T} x(kT + \tau, kT)\|_2^2 \\ &\leq \max_{z \in Z} \|Y^T V_1 W^{-1} V_1^T z\|_2^2 \\ &= \lambda_{\max}(W^{-1/2} V_1^T Y Y^T V_1 W^{-1/2}) \end{aligned} \tag{20}$$

By the Schur complement lemma, the input constraints is equivalent to (13). The state feedback predictive controller $K = Y^T (E V_1 W V_1^T + S V_2^T)^{-T}$ at sample time $[kT, (k + 1)T]$.

Lemma 2^[3]. (Feasibility). Any feasible solution of the optimization (9)-(13) at time kT is also feasible for all times $t > k$. Thus if the optimization problem (9) is feasible at time k then it is feasible for all times $t > k$.

Theorem 2. If the optimization problems (9)-(13) exist feasible solutions in the moment kT , thus (i) there also exist feasible solutions in the NT moment $NT (N \geq k)$. (ii) We get a piecewise state feedback control sequence $\{K_k\}_{k=0}^{\infty}$ when k change from 0 to ∞ . Therefore, the closed-loop system which is composed of piecewise state feedback control sequence $\{K_k\}_{k=0}^{\infty}$ is regular, impulse-free and asymptotically stable.

Proof. First, we show that the close-loop system is regular and impulse-free. At time interval $t \in [kT, (k + 1)T]$, by(17) and the Schur complement lemma, the following is guaranteed,

$$\begin{bmatrix} (A + BK)^T P + P^T (A + BK) + Q & P^T A_1 \\ * & -Q \end{bmatrix} < 0 \tag{21}$$

or equivalently, $(A + BK)P^T + P^T (A + BK) + P^T A_1 Q^{-1} A_1 P + Q < 0$. By Definition 2, the close-loop system is regular, impulse-free. Now, we show the asymptotic stability of the close-loop system. At time interval $t \in [kT, (k + 1)T]$, Lyapunov--krasovskii function of the close-loop system as follows:

$$\begin{aligned} V(x(kT)) &= x(kT)^T E^T P x(kT) + \\ & \int_{-h}^0 x^T(kT + \tau, kT) Q x(kT + \tau, kT) d\tau \end{aligned} \tag{22}$$

From (16), it follows that

$$\frac{d}{d\tau}(V(x(t))) \leq -(x(t)^T (R_1 + K^T R_2 K)x(t)) \tag{23}$$

because $R_1 > 0, R_2 > 0$, so $\frac{d}{d\tau}(V(x(t))) < 0$ is derived, furthermore, $V(x(t))$ is strictly decreasing, the close-loop system is asymptotically stable.

4. ILLUSTRATIVE EXAMPLE

Consider the polytopic uncertain singular system with delay both in state equation with parameters as follows:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & -1 \\ 2 & -3 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} -0.3 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.3 & -0.1 \\ 0.1 & 0.3 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 0.3 & 0.1 \\ -0.1 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2.5 \end{bmatrix}$$

where $\gamma = 1, h = 0.5, x(0) = [1 \quad -1]^T, R_1 = R_2 = I, T=0.2$.

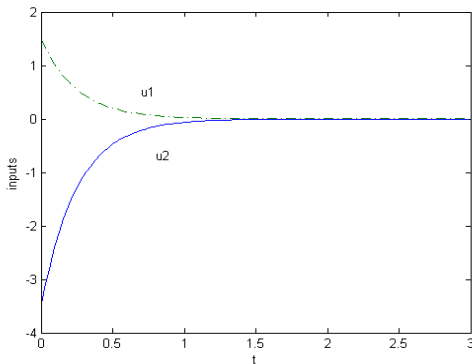


Figure 1: Inputs Of The Close-Loop System

An MPC is designed and the simulation result is shown in Fig. 1 and the simulated inputs of the resulting closed-loop system are also illustrated. It is shown that the closed-loop system is stable and impulse-free.

5. CONCLUSIONS

This paper addresses the problem of designing state feedback robust model predictive controller with input constraints for a class of time-delay singular systems with polytopic uncertainty. The existing sufficient conditions of the robust predictive controller are presented using Lyapunov stability theory and linear matrix inequality (LMI) method. At each sample time, the controller could be determined when these conditions have feasible solutions. We get a piecewise state feedback control sequence $\{K_k\}_{k=0}^{\infty}$ when k changes from 0 to ∞ . The closed-loop system which is composed of piecewise state feedback control sequence $\{K_k\}_{k=0}^{\infty}$ is regular, impulse-free and asymptotically stable. Finally, a numerical example demonstrates the applicability of the proposed approach.

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