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NECESSARY OPTIMALITY AND DUALITY FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING

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ABSTRACT

The aim of this paper is to deal with a class of multiobjective semi-infinite programming problem. For such problem, several necessary optimality conditions are established and proved using the powerful tool of $K-$subdifferential and the generalized convexity namely generalized uniform $K-(F,\alpha,\rho,d)-$convexity. We also formulate the Wolf type dual models for the semi-infinite programming problem and establish the corresponding duality theorems.

Keywords: *$K-$Subdifferential, Semi-Infinite Programming, Necessary Optimality Condition, Wolf Duality*

1. INTRODUCTION

The so-called semi-infinite programming problems is characterized by the optimization of an objective function in finitely many variables over a feasible region defined by an infinite number of constraints . Recently, many authors have been interested in semi-infinite programming problems since this model plays a key role in a particular physical or social science situation, i.e., control of robots, mechanical stress of materials, and air pollution abatement etc. To date, many authors are developing interesting results on the optimality conditions and duality results for semi-infinite programming problems. In particular, Qingxiang zhang[1] obtained the necessary and sufficient optimality conditions for the nondifferentiable nonlinear semi-infinite programming involving B-arcwise connected functions. In [2, 3, 4], the optimality conditions and duality results under various constraints qualification for semi-infinite programming problems were established.

On the other hand, optimality conditions and duality results in generalized convex multiobjective optimization are also a very important research topic. For example, we can see in [5, 6], the sufficient optimality conditions and duality results were obtained under the generalized convex functions. For details, the readers are advised to consult [7, 8].

In this paper, motivated by the above work, we first define a kind of generalize convex functions about the local cone approximation, K-directional derivative and K-subdifferential. Then, the necessary optimality conditions are obtained for a class of multiobjective semi-infinite programming problem involving the new generalized convexity. Further, we formulate the Wolf type dual model for the semi-infinite programming problem and establish the weak and strong duality theorems relating to the semi-infinite programming problem and the corresponding duality problem.

2. DEFINITIONS AND PRELIMINARIES

Let X be a nonempty set of R^n .The epigraph of a real-valued function $f: X \to R$ is the following subset of $X \times R$:

$$epi\ f = \{(x,r) \in X \times R \mid f(x) \le r\}$$

Definition2.1. Let $K(\cdot,\cdot)$ be a local cone approximation. Then, $f^k(x;\cdot): X \times X \to R \bigcup \{+\infty\}$ is said to be K-directional derivative at x, where

$$f^K(x;y) = \inf\{\xi \in R \mid (y,\xi) \in K(epi\ f,(x, f(x)))\}$$

Definition2.2. [9] $f: X \to R$ is said to be K-subdifferentiable, if there exists convex compact set $\partial^K f(x)$, such that

$$f^K(x;y) = \max_{\xi \in \partial^K f(x)} \langle \xi, y \rangle, \forall y \in R^n,$$

Where, $\partial^K f(x) = \{x^* \in X^* \mid \langle y, x^* \rangle \le f^K(x;y), \forall y \in R^n\}$ is K-subdifferential of f at x.

Definition2.3. A functional $F: X \times X \times R^n \to R$ ($X \subset R^n$) is said to be sublinear about the third variable, if for $\forall(x_1,x_2) \in X \times X$, it satisfies

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(i) $F(x_1, x_2; \alpha_1 + \alpha_2) \le F(x_1, x_2; \alpha_1), \forall \alpha_1, \alpha_2 \in R^n$.

(ii) $F(x_1, x_2; r\alpha) = rF(x_1, x_2; \alpha_1), \forall r \in R_+, \alpha \in R^n$.

We suppose that $X \subset R^n$ is nonempty; $f : X \to R$ is local lipschitz function; $F : X \times X \times R^n \to R$ is sublinear; $\phi : R \to R$; $b : X \times X \times [0,1] \to R_+$, $\lim\limits_{\lambda \to 0^+} b(x, x_0; \lambda) = b(x, x_0); \alpha : X \times X \to R_+ \setminus \{0\}$; $d : X \times X \to R^n$ is a pseudometric on R^n, $\rho \in R$.

Definition2.4. f is said to be generalized uniform $K-(F, \alpha\ , \rho,\ d)-$ convex at $x_0 \in X$, if for all $x \in X$, there exists local cone approximation K , such that

$$b(x, x_0)\phi[f(x) - f(x_0)]$$
$$\ge F(x, x_0, \alpha(x, x_0)\xi) + \rho d^2(x, x_0), \forall \xi \in \partial^K f(x_0)$$

Definition2.5. f is said to be strict generalized uniform $K-(F, \alpha,\ \rho, d)-$ convex at $x_0 \in X$, if for all $x \in X, x \ne x_0$, there exists local cone approximation K , such that

$$b(x, x_0)\phi[f(x) - f(x_0)]$$
$$> F(x, x_0, \alpha(x, x_0)\xi) + \rho d^2(x, x_0), \forall \xi \in \partial^K f(x_0)$$

Assumption A_0 Let the local cone approximation K be one among the tangent cone, arrival directional cone, Clarke tangent cone, and feasible directional cone.

Lemma2.1.[10] (i)The $f^K(x, \cdot)$ is positively homogeneous and subadditive function.

(ii) $f^K(x, \cdot)$ is convex function.

Lemma2.2.[10] $0 \in \partial^K f(\bar{x}) \Leftrightarrow f^K(\bar{x}; y) \ge 0$, $\forall y \in R^n$.

Theorem2.1. If $\bar{x}$ is a local minimum of $f(x)$ on X , and satisfies the assumption A_0 , then there exists $0 \in \partial^K f(\bar{x})$.

The result can be obtained easily.

3. NECESSARY CONDITIONS

In this paper, we consider the following multi-objective semi-infinite programming problem:

$$\text{(SIMP)} \quad \begin{aligned} &\min\ f(x) = (f_1(x), f_2(x), \cdots, f_p(x))^T \\ &\text{s.t. } g_t(x) \square 0, t \in T, \\ &\qquad x \in X. \end{aligned}$$

Where $X \subset R^n$ is a nonempty open subset, $f : x \to R^n$, $g_t : x \to R, t \in T$ and T is an infinite compact index set. We put $X^0 = \{x \in X \,|\, g_t(x) \square 0, t \in T\}$ for the feasible set of problem (SIMP). Then we define

$$T(\bar{x}) = \{t \in T \,|\, g_t(\bar{x}) = 0\},\ R_+^{(T)} = \{\mu : T \to R_+ \,|\, t \in T\}$$

Where $T(\bar{x})$ is active constraint set; $R_+^{(T)}$ means that for all $t \in T$, $\mu_t \square 0$ and only finitely many are strictly positive.

Now we give the following single objective semi-infinite programming problem:

$$(SIMP)_\lambda \quad \begin{aligned} &\min\ F(x) = \sum_{i=1}^{p} \lambda_i f_i(x) = \lambda^T f(x) \\ &s.t.\ g_t(x) \le 0, t \in T, \\ &\qquad x \in X. \end{aligned}$$

Where $(\lambda_1, \lambda_2, \cdots, \lambda_p)^T \in \Lambda^+ \ or \Lambda^{++}$, we define

$$\Lambda^+ = \{\ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)^T \,|\, \lambda_i \square 0, i = 1, 2, \cdots p, \sum_{i=1}^{p} \lambda_i = 1\},$$

$$\Lambda^{++} = \{\ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)^T \,|\, \lambda_i > 0, i = 1, 2, \cdots p, \sum_{i=1}^{p} \lambda_i = 1\},$$

$$\Omega^0 = \begin{cases} \{y \in R^n \,|\, g_t^{K_t}(\bar{x}; y) \le 0, t \in T(\bar{x})\}, & T(\bar{x}) \ne \Phi \\ R^n, & T(\bar{x}) = \Phi \end{cases},$$

$$\Omega_-^0 = \begin{cases} \{y \in R^n \,|\, g_t^{K_t}(\bar{x}; y) < 0, t \in T(\bar{x})\}, & T(\bar{x}) \ne \Phi \\ R^n\ , & T(\bar{x}) = \Phi \end{cases}.$$

Denote by R_+^P the nonnegative orthant of R^P .

Definition3.1. We say that $(SIMP)_\lambda$ satisfies the constraint qualification C_0 at $\bar{x} \in X$, if $\Omega_-^0 \ne \Phi$ always holds.

Lemma3.1. Let $X \in R^n$ be a nonempty set. If

(i) $\psi_i, i \in (1, 2, \cdots, p)$ is generalized uniform $K_i - (F, \alpha, \rho_i, d)$ − convex function on X with

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respect to $F, \phi, b, \alpha, \rho_i, d$ and local cone approximation K_i;

(ii) $\rho_i \square 0; a < 0 \Rightarrow \phi(a) < 0; b > 0;$

(iii) For all $x \in X$, $0 \in \partial^{K_i}\psi_i(x)$, $i \in \{1, 2, \cdots, p\}$. Then the following system has no solution $x \in X$.

(I) There exists $x \in X$ such that $\psi_i(x) < 0$, $i \in \{1, 2, \cdots, p\}$;

(II)There exists $\lambda \in R_+^P \setminus \{0\}$ such that $\sum_{i=1}^{p} \lambda_i \psi_i \square 0$, for all $x \in X$.

Proof: If (I) has a solution, that is, there exists $x \in X$, such that $\psi_i(x) < 0, i \in \{1, 2, \cdots, p\}$, then for every $\lambda \in R_+^P \setminus \{0\}$, we have $\sum_{i=1}^{p} \lambda_i \psi_i < 0$, that is, (II) does not hold.

Assume that (I) has no solution.

Let $M = -\psi(x) - R_+^P$, $\psi = (\psi_1, \psi_2, \cdots \psi_p)^T$, then M is convex.

In fact, if $y_1, y_2 \in M$, then exists $x_1, x_2 \in X$ and $p_1, p_2 \in R_+^P$, such that

$$y_1 = -\psi(x_1) - p_1, y_2 = -\psi(x_2) - p_2$$

For any $t \in (0,1)$, we have

$$(1-t)y_1 + ty_2 = \\ -((1-t)\psi(x_1) + t\psi(x_2)) - ((1-t)p_1 + tp_2)$$

By (i), for any $\bar{x} \in X$, we have

$$b(x_1, \bar{x})\phi[\psi_i(x_1) - \psi_i(\bar{x})] \\ \square F(x_1, \bar{x}; \alpha(x_1, \bar{x})\xi_i) + \rho_i d^2(x_1, \bar{x}),$$

$$b(x_2, \bar{x})\phi[\psi_i(x_2) - \psi_i(\bar{x})] \\ \square F(x_2, \bar{x}; \alpha(x_2, \bar{x})\xi_i) + \rho_i d^2(x_2, \bar{x}).$$

Where $\forall \xi_i \in \partial^{K_i}\psi_i(\bar{x}), i \in \{1, 2, \cdots, p\}$.

By $\rho_i \square 0$ and (iii), we have

$$F(x_1, \bar{x}; \alpha(x_1, \bar{x})0) + \rho_i d^2(x_1, \bar{x}) \square 0$$
$$F(x_2, \bar{x}; \alpha(x_2, \bar{x})0) + \rho_i d^2(x_2, \bar{x}) \square 0$$

Using (ii), we get

$$\psi_i(x_1) \square \psi_i(\bar{x}), \psi_i(x_2) \square \psi_i(\bar{x}), i \in \{1, 2, \cdots, p\}$$

That is

$$(1-t)\psi_i(x_1) + t\psi_i(x_2) \square \psi_i(\bar{x}), i \in \{1, 2, \cdots, p\}$$

Therefore

$$(1-t)\psi(x_1) + t\psi(x_2) \square \psi(\bar{x})$$

Then we have

$$(1-t)y_1 + ty_2 \square -\psi(\bar{x}) - ((1-t)p_1 + tp_2)$$

So, there exists $\bar{p} \in R_+^P$, such that

$$(1-t)y_1 + ty_2 = -\psi(\bar{x}) - ((1-t)p_1 + tp_2 + \bar{p})$$

Now since R_+^P is close convex cone, we obtain

$$((1-t)p_1 + tp_2 + \bar{p}) \in R_+^P$$

Then we get

$$(1-t)y_1 + ty_2 \in M$$

Because (I) has no solution, and $\operatorname{int} R_+^P + R_+^P \subset \operatorname{int} R_+^P$, so we get

$$M \bigcap \operatorname{int} R_+^P = \Phi$$

From the convex set separated theorem, there exists $\lambda \in R^P$, $\lambda \neq 0$, such that

$$\sup \lambda(M) \square \inf \lambda(R_+^P)$$

Since R_+^P is close convex cone, we get

$$\inf \lambda(R_+^P) = 0$$

Now we obtain $\lambda \in R_+^P \setminus \{0\}$ and $\lambda^T \psi(x) \square 0$, that is

$$\sum_{i=1}^{p} \lambda_i \psi_i(x) \square 0, \forall x \in X$$

Hence, (II) has a solution. Thus the lemma is proved.

Lemma3.2. Let $\bar{x} \in X^0$ be an optimal solution for $(SIP)_\lambda$. Further, we assume that $\lambda^T f(x)$ and $g_t(x), t \in T(\bar{x})$ are K_0- subdifferentiable and K_t- subdifferentiable at $\bar{x}$ with respect to local cone approximation K_0 *and* $K_t, t \in T(\bar{x})$, respectively. The assumption A_0 holds. Then, the following system has no solution.

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(III) $\begin{cases}(\lambda^T f)^{K_0}(\bar{x};y)<0\\ g_t^{K_t}(\bar{x};y)\square 0, t\in T(\bar{x})\end{cases}$

Proof: Suppose there exists $\bar{y}$, such that

$$\begin{cases}(\lambda^T f)^{K_0}(\bar{x};\bar{y})<0\\ g_t^{K_t}(\bar{x};\bar{y})\square 0, t\in T(\bar{x})\end{cases}$$

According to the assumption A_0 and the definition of $K-$ directional derivative, there exists $\bar{u}>0$, for any $u\in(0,\bar{u})$, such that

$$\begin{cases}\dfrac{(\lambda^T f)(\bar{x}+u\bar{y})-(\lambda^T f)(\bar{x})}{u}<0\\ \dfrac{g_t(\bar{x}+u\bar{y})-g_t(\bar{x})}{u}\square 0, t\in T(\bar{x})\end{cases}$$

Then if $u\to 0+$, we obtain

$$\begin{cases}(\lambda^T f)(\bar{x}+u\bar{y})-(\lambda^T f)(\bar{x})<0\\ g_t(\bar{x}+u\bar{y})-g_t(\bar{x})\square 0, t\in T(\bar{x})\end{cases}$$

Finally, we have a contradiction. Thus $\bar{x}$ is an optimal solution of $(SIP)_\lambda$. Hence, the system (III) is incompatible.

Lemma3.3. If for a given $\lambda\in\Lambda^{++}(or\ \Lambda^{+})$, $\bar{x}\in X^0$ is an optimal solution for $(SIP)_\lambda$, then $\bar{x}$ is a properly efficient solution for (SIMP).

Theorem3.1. Let us suppose that $f_i(i=1,2,\cdots,p)$ is generalized uniform $K_i-(F,\alpha,\rho_i,d)-$convex function $(\rho_i\geq 0)$ on X^0. If $x^*\in X^0$ is a weak efficient solution of (SIMP), $0\in\partial^{K_i}(f_i-f_i(x^*))(x^*), a<0\Rightarrow\phi(a)<0, b(x,x^*)>0$. Then there exists $\lambda^*\in\Lambda^+$, such that x^* is an optimal solution of $(SIP)_{\lambda^*}$.

Proof: If $x^*\in X^0$ is weak efficient solution of (SIMP), then there exists no $x\in X^0$, such that

$$f_i<f_i(x^*), i=1,2,\cdots,p$$

That is, the following system has no solution in X^0: $f_i-f_i(x^*)<0, i=1,2,\cdots,p$.

Since $f_i(i=1,2,\cdots,p)$ is generalized uniform $K_i-(F,\alpha,\rho_i,d)$ -convex $(\rho_i\geq 0)$ on X^0, now we get

$$\begin{aligned}&b_0(x,x^*)\phi[(f_i(x)-f_i(x^*))-(f_i(x^*)-f_i(x^*))]\\&=b_0(x,x^*)\phi[(f_i(x)-f_i(x^*))]\\&\square\,\mathrm{F}(x,x^*;\alpha(x,x^*)\xi_i)+\rho_i d^2(x,x^*),\\&\forall\xi_i\in\partial^{K_i}(f_i-f_i(x^*))(x^*)\end{aligned}$$

So, $(f_i(x)-f_i(x^*))$ is generalized uniform $K_i-(F,\alpha,\rho_i,d)-$ convex at $X^*, i=1,2,\cdots,p$. Again from lemma3.1, there exists $\lambda^*\in\Lambda^+$, such that for all $x\in X^0$, we get

$$\sum_{i=1}^{p}\lambda^*[(f_i(x)-f_i(x^*)]\square 0$$

That is

$$\lambda^{*T}f(x)\square\lambda^{*T}f(x^*)$$

So, x^* is an optimal solution of $(SIP)_{\lambda^*}$.

Theorem3.2. (Necessary optimality condition)

Let us suppose that $\lambda^T f(x)$ and $g_t(x)$ $(t\in T)$ are K_0- subdifferentiable and K_t- subdifferentiable at $\bar{x}\in X^0$, respectively. The assumption A_0 holds. If $\bar{x}$ is an optimal solution of $(SIP)_\lambda$, and the constraint qualification C_0 holds. Then, there exists $(\mu_t)_{t\in T}\in R^{(T)}$, such that

$$0\in\partial^{K_0}(\lambda^T f)(\bar{x})+\sum_{t\in T}\mu_t\partial^{K_t}g_t(\bar{x}), \qquad (1)$$

$$\mu_t g_t(\bar{x})=0, t\in T \qquad (2)$$

$$(\mu_t)_{t\in T}\square 0 \qquad (3)$$

Proof: since $(SIP)_\lambda$ satisfies the constraint qualification C_0 at $\bar{x}$, then we obtain $\Omega_-^0\neq\Phi$.

Suppose $T(\bar{x})=\Phi$, and $\bar{x}$ is an optimal solution of $(SIP)_\lambda$, from the theorem1.3, $0\in\partial^{K_0}(\lambda^T f)(\bar{x})$, and as $\mu\in R_+^{(T)}$, we also obtain $\mu_t=0$, for all $t\in T$.

So, (1), (2), (3) hold.

Suppose $T(\bar{x})\neq\Phi$, and $\bar{x}$ is an optimal solution of $(SIP)_\lambda$. According to the definition of $K-$ subdifferentiable and lemma3.2, we get the following system has no solution.

$$\begin{cases}\xi^T y<0, & \forall\xi\in\partial^{K_0}(\lambda^T f)(\bar{x})\\ \zeta_t^T y\square 0, & \forall\zeta_t\in\partial^{K_t}g_t(\bar{x}), t\in T(\bar{x})\end{cases}$$

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That is, the following system has solution.

$$\begin{cases} \xi^T y \square 0, & \forall \xi \in \partial^{K_0}(\lambda^T f)(\bar{x}) \\ -\zeta_t^T y \square 0, & \forall \zeta_t \in \partial^{K_t} g_t(\bar{x}), t \in T(\bar{x}) \end{cases}$$

Then using the Farkas lemma, there exists $\mu_t \square 0, t \in T(\bar{x})$, such that

$$\xi = -\sum_{t \in T(\bar{x})} \mu_t \zeta_t$$

Let $\mu_t = 0$, if $t \in T \setminus T(\bar{x})$, then we get

$$\xi + \sum_{t \in T} \mu_t \zeta_t = 0$$

So, we obtain

$$0 \in \partial^{K_0}(\lambda^T f)(\bar{x}) + \sum_{t \in T} \mu_t \partial^{K_t} g_t(\bar{x}),$$
$$\mu_t g_t(\bar{x}) = 0, t \in T$$
$$(\mu_t)_{t \in T} \square 0$$

Using the theoren3.1 and 3.2, we can easily obtain the following result.

Theorem3.3. (Necessary optimality condition)

Let us suppose that $f_i (i = 1, 2, \cdots, p)$ is generalized uniform $K_i - (F, \alpha, \rho_i, d) -$ conve $(\rho_i \square 0)$ on X^0, $g_t(x)(t \in T)$ are $K_t -$ subdifferentiable at $\bar{x}$. If $\bar{x} \in X^0$ is a weak efficient solution of (SIMP), $0 \in \partial^{K_i}(f_i - f_i(\bar{x}))(\bar{x})$, $a < 0 \Rightarrow \phi(a) < 0, b(x, \bar{x}) > 0$, and there exists $\lambda \in \Lambda^+$, such that $(SIP)_\lambda$ satisfies the constraint qualification C_0 holds. Then, there exists $(\mu_t)_{t \in T} \in R^{(T)}$, such that

$$0 \in \partial^{K_0}(\lambda^T f)(\bar{x}) + \sum_{t \in T} \mu_t \partial^{K_t} g_t(\bar{x}),$$
$$\mu_t g_t(\bar{x}) = 0, t \in T$$
$$(\mu_t)_{t \in T} \square 0$$

4. WOLF TYPE DUALITY

In this section, we consider the following Wolf type dual model for (SIMP):

$$\text{(SIWD)} \quad \begin{aligned} &\max f(u) + \sum_{t \in T} \mu_t g_t(u) e \\ &s.t.\ 0 \in \partial^{K_0}(\lambda^T f)(u) + \sum_{t \in T} \mu_t \partial^{K_t} g_t(u); \\ &\lambda \in \Lambda^+; \\ &(\mu_t)_{t \in T} \in R_+^{(T)}. \end{aligned}$$

where e is a p-dimensional vector whose all components are all ones.

Theorem4.1. (Weak duality)

Let x and (u, λ, μ) be feasible solutions of (SIMP) and (SIWD) respectively. Assumption A_0 holds, and suppose

(i) $\lambda^T f$ is generalized uniform $K_0 - (F, \alpha, \rho_0, d)$ -convex at u with respect to φ_0 and b ;

(ii) For all $t \in T$, g_t is generalized uniform $K_t - (F, \alpha, \tau_t, d)$ -convex at u with respect to ϕ_1 and b ;

(iii) $\phi_0(a) = a,\ \phi_1(a) = a$;

(iv) $\rho_0 + \sum_{t \in T} \mu_t \tau_t \square 0$;

Then we can obtain

$$f(x) \square f(u) + \sum_{t \in T} \mu_t g_t(u) e$$

Proof: Since x and (u, λ, μ) are feasible solutions of (SIMP) and (SIWD) respectively, it follows that

$$\mu_t g_t(x) \square 0, \forall t \in T \tag{4}$$

And

$$0 \in \partial^{K_0}(\lambda^T f)(u) + \sum_{t \in T} \mu_t \partial^{K_t} g_t(u)$$

That is, $\exists \xi \in \partial^{K_0}(\lambda^T f)(u)$ and $\zeta_t \in \sum_{t \in T} \partial^{K_t} g_t(u)$, $t \in T$, such that

$$0 = \xi + \sum_{t \in T} \mu_t \zeta_t \tag{5}$$

By hypothesis (i), we obtain

$$\begin{aligned} &b(x, u)\phi_0[(\lambda^T f)(x) - (\lambda^T f)(u)] \\ &\square F(x, u; \alpha(x, u)\xi) + \rho_0 d^2(x, u) \end{aligned}$$

Using (iii), we get

$$\begin{aligned} &(\lambda^T f)(x) - (\lambda^T f)(u) \\ &\square b(x, u)^{-1}[\alpha(x, u)F(x, u; \xi) + \rho_0 d^2(x, u)] \end{aligned} \tag{6}$$

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By hypothesis (ii), we have

$$b(x,u)\phi_1[g_t(x)-g_t(u)] \square F(x,u;\alpha(x,u)\zeta_t)+\tau_t d^2(x,u)$$

Using (iii) and $(\mu_t)_{t\in T}\in R_+^{(T)}$, we obtain

$$\sum_{t\in T}\mu_t g_t(x)-\sum_{t\in T}\mu_t g_t(u)\square b(x,u)^{-1}[\alpha(x,u)F(x,u;\sum_{t\in T}\mu_t\zeta_t)+\sum_{t\in T}\mu_t\tau_t d^2(x,u)]$$

Then by inequality (4), it follows that

$$\sum_{t\in T}\mu_t g_t(u)+b(x,u)^{-1}[\alpha(x,u)F(x,u;\sum_{t\in T}\mu_t\zeta_t)+\sum_{t\in T}\mu_t\tau_t d^2(x,u)]\square 0 \qquad (7)$$

Adding (6) and (7), then using (5), we have

$$(\lambda^T f)(x)\square(\lambda^T f)(u)+\sum_{t\in T}\mu_t g_t(u)+(\rho_0+\sum_{t\in T}\mu_t\tau_t)d^2(x,u)b(x,u)^{-1}$$

By hypothesis (4), we get

$$(\lambda^T f)(x)\square(\lambda^T f)(u)+\sum_{t\in T}\mu_t g_t(u)$$

That is

$$f(x)\square f(u)+\sum_{t\in T}\mu_t g_t(u)e\,.$$

Theorem4.2. (Strong duality)

Let assumption A_0 hold. Suppose that $\bar{x}$ is a weakly efficient solution of (SIMP) for which the Kuhn-Tucker constraint qualification is satisfied. Then, there exists $\bar{\lambda}\in\Lambda^+,(\bar{\mu}_t)_{t\in T}\in R_+^{(T)}$ such that $(\bar{x},\bar{\lambda},\bar{\mu})$ is feasible for (SIWD) and the objective function values of (SIMP) and (SIMD) are equal.

Furthremore, if the conditions of theorem 4.1 hold for all feasible solutions of (SIMP) and (SIMD), then $(\bar{x},\bar{\lambda},\bar{\mu})$ is a weakly efficient solution of (SIWD).

Proof: Since $\bar{x}$ is a weakly efficient solution of (SIMP) for which the Kuhn-Tucker constraint qualification is satisfied, it follows that there exists $\bar{\lambda}\in\Lambda^+,(\bar{\mu}_t)_{t\in T}\in R_+^{(T)}$ satisfying the following necessary conditions

$$0\in\partial^{K_0}(\lambda^T f)(\bar{x})+\sum_{t\in T}\mu_t\partial^{K_t}g_t(\bar{x})$$

$$\bar{\mu}_t g_t(\bar{x})=0,t\in T$$

$$\lambda\in\Lambda^+,(\bar{\mu}_t)_{t\in T}\in R_+^{(T)}.$$

Hence, $(\bar{x},\bar{\lambda},\bar{\mu})$ is feasible for (SIWD) and the objective function values of (SIMP) and (SIMD) are equal.

By contrary method, suppose that $(\bar{x},\bar{\lambda},\bar{\mu})$ is not a weakly efficient solution of (SIWD). Then, there exists a feasible solution (x^*,λ^*,μ^*) of (SIWD), such that

$$f(\bar{x})+\sum_{t\in T}\bar{\mu}_t g_t(\bar{x})e<f(u^*)+\sum_{t\in T}\mu_t^* g_t(u^*)e$$

Since $\bar{\mu}_t g_t(\bar{x})=0$, it follows that

$$f(\bar{x})<f(u^*)+\sum_{t\in T}\mu_t^* g_t(u^*)e$$

We have a contradiction with the result of theorem4.1. Hence, $(\bar{x},\bar{\lambda},\bar{\mu})$ is a weakly efficient solution of (SIWD).

Remark4.1. It may be noted that, if the constraint condition $0\in\partial^{K_0}(\lambda^T f)(u)+\sum_{t\in T}\mu_t\partial^{K_t}g_t(u)$ is replaced by

$$0\in\sum_{i=1}^{p}\lambda_i\partial^{K_i}f_i(u)+\sum_{t\in T}\mu_t\partial^{K_t^*}g_t(u) \qquad (8)$$

then we can give the following theorem.

Theorem4.3. (Weak duality)

Let x and (u,λ,μ) be feasible solutions of (SIMP) and (SIWD) respectively. Assumption A_0 holds, and suppose

(i) For all $i\in\{1,2,\cdots,p\}$, f_i is generalized uniform $K_i-(F,\alpha_i,\rho_i,d)$-convex at u with respect to φ_0 and b_i;

(ii) For all $t\in T$, g_t is generalized uniform $K_t^*-(F,\alpha_t^*,\tau_t,d)$-convex at u with respect to ϕ_1 and b_t^*;

(iii) $\phi_0(a)=a,\ \phi_1(a)=a$;

(iv) $\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)}{\alpha_i(x,u)}=1,\frac{b_t^*(x,u)}{\alpha_t^*(x,u)}=1,t\in T$;

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(v) $\sum_{i=1}^{p}\frac{\lambda_i\rho_i}{\alpha_i(x,u)}+\sum_{t\in T}\frac{\mu_t\tau_t}{\alpha_t^*(x,u)}\square 0$;

Then, the following inequality cannot hold:

$$f(x)< f(u)+\sum_{t\in T}\mu_t g_t(u)e \qquad (9)$$

Proof: By the contrary method. Suppose that (9) hold. By hypothesis (iii), we get

$$\begin{aligned}&\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(x)}{\alpha_i(x,u)}\\&<\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(u)}{\alpha_i(x,u)}+\sum_{t\in T}\mu_t g_t(u)\end{aligned} \qquad (10)$$

Also, hypothesis (i) yields

$$\begin{aligned}&b_i(x,u)\phi_0[f_i(x)-f_i(u)]\\&\square F(x,u;\alpha_i(x,u)\xi_i)+\rho_i d^2(x,u),\quad \forall\xi_i\in\partial^{K_i}f_i(u)\end{aligned}$$

Using hypothesis (iii) and (iv), we obtain

$$\begin{aligned}&\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(x)}{\alpha_i(x,u)}-\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(u)}{\alpha_i(x,u)}\\&\square F(x,u;\sum_{i=1}^{p}\lambda_i\xi_i)+\sum_{i=1}^{p}\frac{\lambda_i\rho_i}{\alpha_i(x,u)}d^2(x,u)\end{aligned} \qquad (11)$$

By hypothesis (ii), we have

$$\begin{aligned}&b_t^*(x,u)\phi_1[g_t(x)-g_t(u)]\\&\square F(x,u;\alpha_t^*(x,u)\zeta_t)+\tau_t d^2(x,u),\forall\zeta_t\in\partial^{K_t^*}g_t(u)\end{aligned}$$

Using hypothesis (iii) and (iv), we obtain

$$\begin{aligned}&\sum_{t\in T}\mu_t g_t(x)-\sum_{t\in T}\mu_t g_t(u)\\&\square F(x,u;\sum_{t\in T}\mu_t\zeta_t)+\sum_{t\in T}\frac{\mu_t\tau_t}{\alpha_t^*(x,u)}d^2(x,u)\end{aligned} \qquad (12)$$

Since x is a feasible solutions of (SIMP), it follows that

$$\mu_t g_t(x)\square 0,\forall t\in T \qquad (13)$$

By inequality (13), then the inequality (12) becomes

$$\begin{aligned}&\sum_{t\in T}\mu_t g_t(u)+F(x,u;\sum_{t\in T}\mu_t\zeta_t)\\&+\sum_{t\in T}\frac{\mu_t\tau_t}{\alpha_t^*(x,u)}d^2(x,u)\square 0\end{aligned} \qquad (14)$$

Adding (11) and (14), and using the sublinearity of F along with (8), we have

$$\begin{aligned}&\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(x)}{\alpha_i(x,u)}-\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(u)}{\alpha_i(x,u)}\\&\square\sum_{t\in T}\mu_t g_t(u)+(\sum_{i=1}^{p}\frac{\lambda_i\rho_i}{\alpha_i(x,u)}+\sum_{t\in T}\frac{\mu_t\tau_t}{\alpha_t^*(x,u)})d^2(x,u)\end{aligned}$$

By hypothesis (v), we get

$$\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(x)}{\alpha_i(x,u)}\square\sum_{i=1}^{p}\frac{\lambda_i b_i(x,u)f_i(u)}{\alpha_i(x,u)}+\sum_{t\in T}\mu_t g_t(u)$$

This inequality contradicts (10). Hence, the inequality (9) cannot hold.

5. CONCLUSION

In this paper, we have defined a new generalized convex function, extending many well-known classes of generalized convex functions. By utilizing the new convexity, we have achieved some necessary optimality conditions for a class of multiobjective semi-infinite programming problem. Furthermore, we have obtained several duality results between the problem and the Wolf dual problem, there should be further opportunities for exploiting this structure of the semi-infinite programming problem.

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