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# NONLINEAR PDE APPROACH FOR OPTION PRICING WITH STOCHASTIC VOLATILITY BY USING FUZZY SETS THEORY

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**ABSTRACT**

This paper is to price European options for assets with stochastic volatility in using fuzzy set theory. The main idea is to transform the probability distribution of stochastic volatility to its possibility distribution (from `volatility smile to volatility frown') and reduce the problem to a fuzzy stochastic process for underlying asset with volatility as a fuzzy number associated with initial stochastic volatility. We\ then price the corresponding European options by introducing the non-linear fuzzy PDE approach.

**Keywords**: *European Option, Stochastic Volatility, Fuzzy Sets, Fuzzy Stochastic Process, Nonlinear PDE*

## 1 INTRODUCTION

Local volatility can be viewed as the market's view of the future value of volatility when asset price is $S$ at time $t$. When we retrieve the local volatility surface from the prices of market traded instruments we are under the assumption that we have a distribution of option prices of all strikes and maturities. This is not realistic.

The Black-Scholes model assumes that volatility of an asset return is constant. However, actual volatilities cannot be expected to be constants. The implied volatility smile is clear evidence of this. Volatilities do vary from time to time and appear to exhibit stochastic properties. In 1993, Heston [1] proposed a stochastic volatility model. The underlying asset $S_t$ in the risk-neutral world and the variance follow

$$\begin{cases} \dfrac{dS_t}{S_t} & = r_t dt + \sigma_t dW_t^1 \\ d\sigma_t^2 & = k(\theta - \sigma_t^2)dt + \gamma\sigma_t dW_t^2 \end{cases} \quad (1)$$

where $r_t$ is the risk-free interest rate at time $t$, $\sigma_0$ and $\theta$ are short-term and long-term volatilities, $k > 0$ is the mean reversion speed, $\gamma > 0$ is the volatility of the variance $\sigma_t^2$, $W_t^1$ and $W_t^2$ are two Wiener processes with correlation $\rho$. For simplicity, we consider $\rho = 0$.

The advantage of this two-factor model is that it fits markets better than the Black-Scholes one-factor geometrical Brownian motion model, and it can account for the implied volatility smile. However, the model also has disadvantages:

high complexity of both stochastic methods and PDE methods for pricing contingent claims;

the presence of information incompleteness for the process of $\sigma_t^2$: there may not be enough data available to develop a probabilistic distribution for $\sigma_t^2$, more specifically, to determine $\theta$ and $\gamma$.

Uncertainty can arise from information incompleteness as well as from randomness. Probability theory provides a quantitative tool for randomness while possibility theory provides a qualitative tool for incompleteness. Probability and possibility capture different facets of uncertainty. As stated by Dubois et al. [2], it is useful to transform a probability measure to a possibility measure when computing with possibilities is simpler than with probabilities, or when dealing with information incompleteness. Therefore, the objective of this paper is to transform $\sigma_t^2$ from its probabilistic representation to its possibility distribution. In other words, we will derive the possibility distribution of $\sigma_t^2$ from its probability density function. The result will be

$$\frac{dS_t}{S_t} = r_t dt + \tilde{\sigma}_t dW_t^1 \quad (2)$$

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where $\tilde{\sigma}$ is a fuzzy number associated with the second equation in (1).

In other words, we write the dynamics in a standard lognormal form, according to which the role of the volatility parameter $\sigma$ is clear, and, indeed, is given by the quadratic variation of the log price. However, in practice, estimating the volatility from log returns may not be as straightforward as the question assumes. We have in mind the situation where there are insufficient data to obtain a reliable estimate of the volatility. In such a setting, it will be natural to be able to assign confidence levels, or beliefs, to a range to possible values for the volatility - i.e. to model the volatility as a fuzzy number.

We then apply the above fuzzy stochastic model to option pricing problems. The advantages of the new model (2) are

• it captures the information contained by the Heston model;

• it reduces the complexity to approximately a one factor model;

• the PDE approach for derivatives pricing can be easily applied.

This fuzzy price model is related to the concept of uncertain volatility model or model uncertainty in general, that has been extensively explored in both economics and finance. It is widely accepted that the assumption of constant volatility in financial models is incompatible with derivatives prices observed in the market. This problem has several principal solutions: 1) volatility can be made a deterministic function of time and the underlying asset price; 2) it can be made stochastic, introducing one additional source of randomness. To find this deterministic function (or `volatility surface') or random process, however, poses difficulties in the framework of arbitrage pricing theory. Strongly related to finding the right way to model volatility is the problem to measure the exposure of options portfolios to volatility risk. The main question here is: how does the model value of the portfolio change if the volatility assumptions turn out to be false?

This problem can be approached by using the uncertain volatility model developed by Avellaneda et al. [3] as a starting point. Uncertain volatility models select a concrete volatility surface among a candidate set of volatility surfaces, and answer the sensitivity question by computing an upper bound for the value of the portfolio under any candidate volatility. A lower bound can be computed as well by inverting the position. This is achived by choosing the local volatility $\sigma(S_t, t)$ among two extremal values $\sigma_{\min}$ and $\sigma_{\max}$ such that the value of the portfolio is maximized locally.

Uncertain volatility scenarios may generalize this approach: given a model that exhibits uncertainty in some of its coefficients (the volatility, in particular), instantiate those uncertain coefficients such that some objective is fulfilled. This objective is called a scenario (see [4]).

The original uncertain volatility model in Avellaneda et al. [3] is a worst-case scenario for the sell-side. By maximizing the portfolio value and changing accordingly, sellers are guaranteed coverage against adverse market behavior if the realized volatility belongs to the candidate set. Worst-case prices are nonlinear, due to diversification of volatility risk. Worst-case evaluation is based on a nonlinear HJB equation that generalizes Black-Scholes by adjusting the local volatility based on the local gamma.

This paper is organized as follows. Section 2 introduces the basics of fuzzy sets theory and the transformation from a probability distribution to its possibility distribution. With this transformation, we transform a stochastic volatility to a volatility described by a fuzzy number. Meanwhile, we transform the Heston model to a fuzzy stochastic model. Section 3 introduces the development of a PDE approach for option pricing based on the fuzzy stochastic process obtained in the last section. Section 4 presents case studies. Conclusions are drawn in Section 5.

## 2 DEVELOPING POSSIBILITY REPRESENTATION FROM PROBABILITY REPRESENTATION

In this paper, we are going to use the possibility measure [5] , [6] based on the fuzzy set theory [7].

### 2.1 Fuzzy Sets

Let $X$ be a universal set and $A$ be a subset of $X$. The fuzzy set $\tilde{A}$ is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$. The value $\mu_{\tilde{A}}(a)$ can be interpreted as the membership degree of the point $a$ in the set $\tilde{A}$. We denote the $\alpha$ -cut

$$\tilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\},$$

i.e., the level set which consists of points whose membership value no less than $\alpha$. The $\alpha$ -cut of a

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fuzzy set is a classical (crisp) set and it satisfies the property that

$$\alpha_n < \alpha_{n+1} \Leftrightarrow \tilde{A}_{\alpha_{n+1}} \subset \tilde{A}_{\alpha_n}.$$

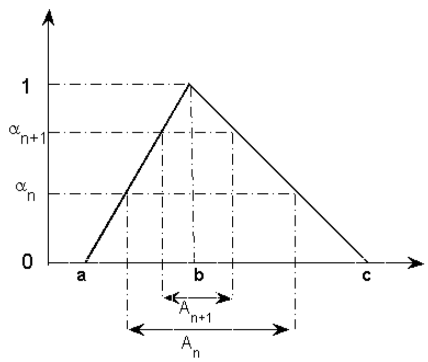


*Figure 1: A Triangular Fuzzy Number And Its $\alpha$ -Cuts.*

A fuzzy number is defined to be a normal and convex fuzzy set with closed and bounded $\alpha$ -cut for any $\alpha \in [0,1]$, where normal means that there exists $x$ such that $\mu_{\tilde{A}}(x) = 1$; and convex means that

$$\mu_{\tilde{A}}\left(\lambda x + (1-\lambda)y\right) \geq \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\right), \quad \forall \lambda \in [0,1].$$

The simplest type of fuzzy number is a triangular fuzzy number which is often denoted by $[a,b,c]$, where $[a,c]$ is its support or 0-cut and $b$ is the point with full membership degree 1. A triangular fuzzy number and its $\alpha$ -cuts are illustrated in Figure 1. Other generic types of fuzzy numbers are trapezoidal and bell-shaped fuzzy numbers (see [8], [9], [10] etc). Adaptive fuzzy numbers have been introduced in [11].

The mathematical apparatus of the theory of fuzzy sets provides a natural basis for the theory of possibility. Zadeh [5], [6] introduced the possibility theory with fuzzy set as a basis by assigning the membership function the role of a possibility distribution function. A fuzzy variable is associated with a possibility distribution in much the same manner as a random variable is associated with a probability distribution.

### 2.2 From Probability Density Function To Membership Function

In general, a variable may be associated both with a possibility distribution and a probability distribution, with the connection between the two representations being the possibility/probability consistency principle [2] . Also, it is shown by Dubois, et al [2] that a possibility measure can be viewed as an upper probability function. However, some information is lost when transforming a probability representation to a possibility distribution simply because that we go from point-valued probabilities to interval-valued ones. A reasonable transformation from probability to possibility should keep as much information as possible and respect the preference preservation principle, i.e.,

$$\pi(x) \geq \pi(x') \Leftrightarrow p(x) \geq p(x').$$

where $p$ is the associated probability density function and $\pi$ is the possibility function. In this context, we follow Zadeh [5], [6] and take the membership function $\mu$ as the possibility function $\pi$ .

In this paper, we will use the transformation introduced by Dubois, Prade and Sandri [12] in 1993. Although we refer the readers to [12] for details, we outline the method here. Suppose we have a unimodal continuous probability density function (pdf) $p$ with bounded support $[a,c]$, such that $p$ is increasing on $[a,b]$ and decreasing on $[b,c]$, where $b$ is the modal value of $p$. Define a function $f:[a,b] \to [b,c]$ by

$$f(x) = max\{y \mid p(y) \geq p(x)\}.$$

Then the possibility distribution $\mu$ can be defined by

$$\mu(x) = \mu(f(x)) = \int_{-\infty}^{x} p(y)dy + \int_{f(x)}^{\infty} p(y)dy. \quad (3)$$

If we denote the cumulative distribution function by

$$P(x) := P(X \leq x) = \int_{-\infty}^{x} p(y)dy$$

then by (3)

$$\mu(x) = P(x) + 1 - P(f(x)).$$

The above idea is illustrated in Figure 2. For a random variable $X$ , with the probability density function shown as in Figure 2 (left), for any $x \in [a,b]$, if $s_1 = P(x)$ and $s_2 = 1 - P(f(x))$ , then the possibility function $\mu(x) = \mu(f(x)) = \alpha = s_1 + s_2$ , as shown in Figure 2 (right). As can be seen, the support of the membership function and the pdf are the same, and

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points with higher probability (likelihood) have the higher possibility.

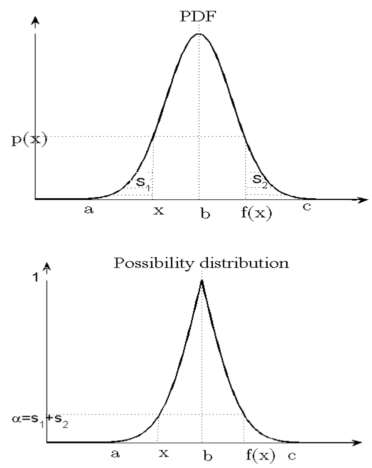


*Figure 2: Transformation From Probability (Up) To Possibility (Bottom).*

The above transformation is for continuous pdfs. For discrete pdfs, see [12], [13].

### 2.3 Application To The Stochastic Volatility Model

Consider the Cox-Ingroll-Ross model, for $v_t := \sigma_t^2$,

$$dv_t = k(\theta - v_t)dt + \gamma\sqrt{v_t}dW_t. \qquad (4)$$

The probability density function of $v_t$ at time $t$, conditional on its value at time 0, is

$$p(v_t) = ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2\sqrt{uv}\right) \qquad (5)$$

where

$$\begin{aligned} c &= \frac{2k}{\gamma^2(1-e^{(-kt)})}, \\ u &= cv_0e^{(-kt)}, \\ v &= cv_t, \\ q &= 2k\frac{\theta}{\gamma^2}-1, \end{aligned} \qquad (6)$$

$I_q$ is the modified Bessel function of the first kind of order $q$.

*Table 1: Model Parameters*

| $\gamma$ | $\theta$ | $k$ | $v_0$ |
|---|---|---|---|
| 0.3877 | 0.0354 | 4.3253 | 0.0174 |

For the parameters given in Table 1, the probability density functions for $v_t$ at $t = 0.5$ and $t = 1$ are shown in Figure 3.

The probability density function of $v_t$ can also be described in terms of the noncentral chi-square distribution (see [14], [15], [16] for example).

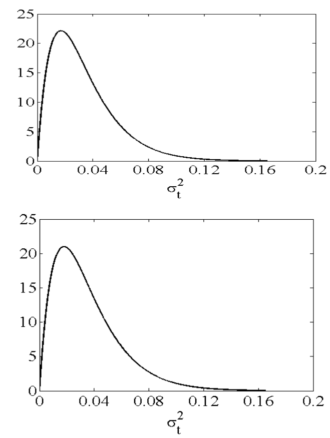


*Figure 3: Probability Density Of $\sigma_t^2$ At $t = 0.5$ Year (Up) And $t = 1$ Year (Bottom).*

We now apply the method (3) to develop the fuzzy distribution $\mu(v_t)$. Let $\overline{v}_t$ be the modal value or maximum value of $v_t$ then

$$\begin{aligned} \mu(v_t) &= P(v_t) + 1 - P(f(v_t)), \quad v_t \le \overline{v}_t \\ \mu(f(v_t)) &= \mu(v_t), \end{aligned} \qquad (7)$$

where

$$P(v) = \int_0^v p(u_t)du_t, \qquad (8)$$

and $p(\cdot)$ is defined in (5). The fuzzy distributions for $v_t$ with parameters given in TABLE 1 are shown in Figure 4.

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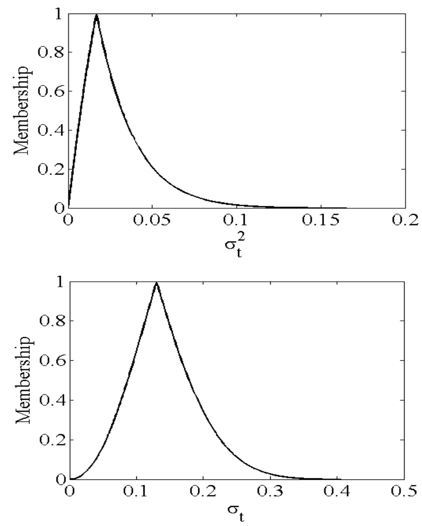


*Figure 4: Fuzzy Distributions For $v_t = \sigma_t^2$ (Up) And $\sigma_t$ (Bottom) At $t = 1$ Year.*

## 3 FUZZY PDE APPROACH FOR PRICING EUROPEAN OPTIONS

Once the fuzzy parameter $\tilde{\sigma}_t$ of the underlying asset is obtained, the option price at any time $t$ is expected to be a fuzzy number. However, how can the option price be determined? This work was addressed in [17] where we developed a non-linear PDE with fuzzy parameters for determining the fuzzy option prices. Compared to [17], the major contribution of this paper is that we develop the fuzzy volatility by transforming the stochastic volatility, while the main contribution of [17] is that we built a framework for option pricing with prescribed fuzzy parameters. For clarity, we briefly review the non-linear fuzzy PDE model below (for more details, see Appendix or [17]).

Suppose we know the fuzzy parameter $\tilde{\sigma}$. We denote the $\alpha$-cut of $\tilde{\sigma}_t$ by

$$\bar{\sigma}_\alpha = \left[\sigma_\alpha^-, \sigma_\alpha^+\right]. \tag{9}$$

Following the arguments in [17] (see the Appendix), the price intervals $[V_\alpha^-, V_\alpha^+]$ solve

$$\frac{\partial V_\alpha^\pm}{\partial t} + \frac{1}{2}\left[\sigma^\pm(\Gamma)\right]^2 S^2 \frac{\partial^2 V_\alpha^\pm}{\partial S^2} + rS\frac{\partial V_\alpha^\pm}{\partial S} - rV_\alpha^\pm = 0, \tag{10}$$

where $\Gamma = \dfrac{\partial^2 V_\alpha}{\partial S^2}$ and

$$\sigma^\pm(\Gamma) = \begin{cases} \sigma_\alpha^\mp & if\,\Gamma \le 0 \\ \sigma_\alpha^\pm & if\,\Gamma > 0 \end{cases}. \tag{11}$$

Furthermore, in the paper [17] we have shown that, in the fuzzy environment, for a belief degree $\alpha$, if an investor believes that the real volatility surely remains in between the corresponding $\alpha$-cut for a short time, then for the investor who takes a short position, $\Delta = \dfrac{\partial V_\alpha^+}{\partial S}(S_t, t)$ provides the optimal strategy, in the sense that if the volatility $\sigma$ really remains in the $\alpha$-level set $\bar{\sigma}_\alpha$, then this investor almost surely has no risk of losing money. Similarly, if he takes a long position, his best hedging strategy is $\Delta = \dfrac{\partial V_\alpha^-}{\partial S}(S_t, t)$.

Thus, once the fuzzy parameter $\tilde{\sigma}_t$ is determined with the method (3), the fuzzy option price can be solved via (10). Please note that, the major contribution of this paper lies in that we first transfer the stochastic volatility from the probability space into the possibility space, and then apply the fuzzy option pricing model [17] to obtain option price.

## 4 CASE STUDY

### 4.1 Vanilla Call

The parameters for Heston model are taken from [18] as shown in Table 1. This set of parameters are calibrated to fit the SPX. We first study a European vanilla call option: the risk free interest rate $r = 0.1$, strike price $K = \$24.5$, maturity $T = 1$ year. We notice that for a vanilla call option price surface, it is always concave up which means that its second derivatives with respect to $S$, i.e., $\Gamma$ is nonnegative. Thus, for each $\alpha$, $\sigma^\pm(\Gamma) = \sigma_\alpha^\pm$, and we only need to solve two Black-Scholes equations in order to obtain the prices band $[V_\alpha^-, V_\alpha^+]$. The fuzzy option price for current stock price $S_0 = 30$ is shown below in Figure 5 (left). Also we have the corresponding dominating hedging strategies at time 0 shown in Figure 5 (right).

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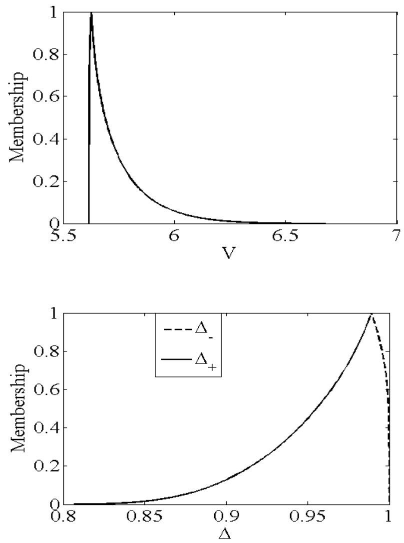

*Figure 5: Fuzzy Option Price (Up) And Delta Hedging Ratio (Bottom).*

The fuzzy option prices with respect to different $S$ at time 0 are shown in Figure 6.

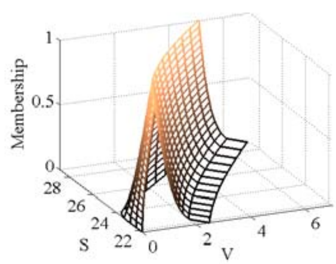

*Figure 6: European Call Option Prices At Time 0.*

## 4.2 Digital Call Option

For exotic options, we have no exact knowledge about the price's concavity. In this case we have to solve these two nonlinear PDEs in (10). The finite difference method is used, at each time time we have to use an iterative method to solve the nonlinear system. The detailed numerical methods are shown in [17]. We study a digital call option with the risk free interest rate $r = 0.1$, strike price $K = \$24.5$, maturity $T = 1$ year. The fuzzy option prices with respect to different $S$ at time 0 are shown in Figure 7.

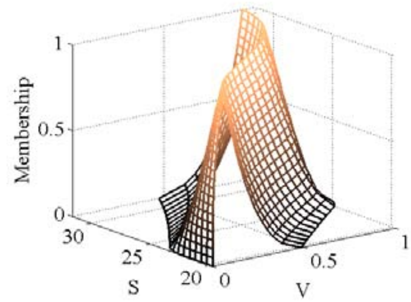

*Figure 7: Digital Call Option Prices At Time 0.*

## 5 CONCLUSION

We applied fuzzy set theory to European option pricing theory for underlying asset with stochastic volatility in Heston model. Our future research is associated with the application of fuzzy set theory to other derivatives pricing for underlying assets with various stochastic volatilities and performing the analysis of obtained results.

## ACKNOWLEDGEMENTS

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## APPENDIX: NONLINEAR PDES FOR OPTION PRICING WITH FUZZY PARAMETERS

We now assume that the parameter $\sigma$ is a fuzzy number. Given $\alpha$, the $\alpha$-cut is

$$\sigma_\alpha = [\sigma_\alpha^-, \sigma_\alpha^+]. \tag{12}$$

For a fixed belief $\alpha$, we have the save assumption as Avellaneda, et al. [3] that the 'true' volatility $\sigma$ is uncertain, but it is going to lie within its certainty band $\sigma_\alpha$. Following Avellaneda [3] and Wilmott [19], the option price will fall into a certainty band which is the $\alpha$-cut of the option price in this context:

$$\tilde{V}_\alpha = [V_\alpha^-, V_\alpha^+].$$

If we allow $\sigma$ at each time $t$ to take any values in $\sigma_\alpha$, and let the stock price follows the geometrical Brownian motion described by

$$\frac{dS}{S} = rdt + \sigma dW \tag{13}$$

and $V(S,t)$ be the corresponding option price, we can set up a hedged portfolio at each time $t$:

$$\Pi = V - \Delta S, \quad \text{with} \quad \Delta = \frac{\partial V}{\partial S}.$$

Actually we do not have knowledge of $\sigma$ beyond that it lies in $\sigma_\alpha$, and so we have to consider extreme prices: highest and lowest prices.

Let us first think from the writer's point of view. The writer will ask the highest $V_\alpha^+$ because he/she wants no risk of losing money. It yields that

$$d\Pi - r\Pi dt \le 0, \quad for \sigma \in \sigma_\alpha.$$

But the writer will have an arbitrage opportunity if

$$\max_{\sigma\in\sigma_\alpha}\{d\Pi - r\Pi dt\} < 0.$$

So, the arbitrage-free highest price $V_\alpha^+$ corresponds to

$$\max_{\sigma\in\sigma_\alpha}\{d\Pi - r\Pi dt\} = 0,$$

$$i.e., \quad \max_{\sigma\in\sigma_\alpha}\{d\Pi\} - r\Pi dt = 0$$

where $\max_{\sigma\in\sigma_\alpha}\{d\Pi\} = \max_{\sigma\in\sigma_\alpha}\left\{\left(\frac{\partial V_\alpha^+}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_\alpha^+}{\partial S^2}\right)dt\right\}.$

If $\Gamma > 0$ then a maximum in $d\Pi$ will require $\sigma = \sigma_\alpha^+$. If $\Gamma \le 0$ then a maximum in $d\Pi$ will require $\sigma = \sigma_\alpha^-$. Similarly, the holder will bid the lowest price $V_\alpha^-$ which involves the same analysis but selecting values of parameters to yield

$$\min_{\sigma\in\sigma_\alpha}\{d\Pi - r\Pi dt\} = 0.$$

Then we have the nonlinear equations (10).