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AN DIGITAL FILTER DESIGN SCHEME BASED ON MATRIX TRANSFORMATION

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ABSTRACT

This work discussed the design steps of digital filter and methods of transformation from analog domain to digital domain. In order to use computer to solve the problem of recursive function whose order is N, this work derivates a transformed algorithm. The proposed scheme shifts system transformation to a coefficient matrix and a variable matrix. This work explains how to fix coefficient matrix and a variable matrix in detail. And at last, it takes advantage of feasibility and validity of simulation algorithm to verify it. The merits are as follows. First, it is very direct and it applies the calculation between matrix and matrix. And second, it is very general, so no matter it is from simulative low pass to digital low pass, or from simulative high pass to digital high pass, or from simulative band pass to digital band pass, this method can be used. Third, it is the fastest since there are fewest numbers of calculations.

Keywords: *Bilinear Transform, Digital Filter, Matrix Algorithms, Z Transform*

1 INTRODUCTION

Linear transformation is widely used in digital simulation, signal processing, automatic control and system identification. The so-called linear transformation [1] is to use:

$$s = (\frac{2}{T}\frac{Z\text{-}1}{Z+1}) \quad \text{or} \quad z = \frac{1+\frac{T}{2}S}{1-\frac{T}{2}S}$$

To realize the transformation of transfer function from S domain to Z domain or the transformation of transfer function from Z domain to S domain. However the bilinear transformation is a kind of fractional transformation, therefore with the increase of system order the amount of computation will be increasing. If that, to calculate by hand will be a big problem, and it is inconvenient to use this transformation. In order to use computer to solve the problem of order N's transfer function transformation, here present the linear transformation algorithm, the following are the features [2, 3]:

(1)Transformation is realized through transformation matrix and coefficient matrix; it is easy to carry out by computer

(2) The coefficient matrix used in transformation can be directly derived; the elements of coefficient matrix can be done by addition.

(3) The digital low pass, high pass, band pass, band stop can be directly transferred.

2 THEORIES

To a given analog low pass filter prototype [4, 5]:

$$H(S) = \frac{A_0 + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + \cdots + A_n s^n}{B_0 + B_1 s + B_2 s^2 + B_2 s^3 + B_4 s^4 + \cdots + B_n s^n}$$

To order coefficient vector as:

$$\begin{aligned} A &= (A_0, A_1, A_2, A_3, A_4, \cdots, A_n) \\ B &= (B_0, B_1, B_2, B_3, B_4, \cdots, B_n) \end{aligned} \tag{1}$$

To normalize the analog filter, then it changed to be:

$$S = c\frac{1-Z^{-1}}{1+z^{-1}} \qquad c = \cot\frac{\pi f_1}{f_s} \tag{2}$$

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f_1 is the cut-off frequency, f_s is the sampling frequency [6].

Put $S = c\dfrac{1-Z^{-1}}{1+z^{-1}}$ in $H(s)$, and there come out $H(z)$

$$H(z) = H(s)/_{s=c\frac{1-z^{-1}}{1+z^{-1}}} = H(s = \tfrac{1-z^{-1}}{1+z^{-1}})$$

If $H(s)$ is a low pass filter, transfer from the above, the $H(z)$ is also a digital low pass filter.

The prototype of digital filter is $H(z)$

Next to order

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}$$

And the coefficient vector is:

$$\begin{aligned} a &= (a_0, a_1, a_2, a_3, a_4, a_5, \cdots, a_n) \\ b &= (b_0, b_1, b_2, b_3, b_4, b_5, \cdots, b_n) \end{aligned} \tag{3}$$

When $n = 1$,

$$H(z) = \frac{a_0 + a_1 z^{-1}}{b_0 + b_1 z^{-1}} = \frac{A_0 + A_1 c + z^{-1}(A_0 - A_1 c)}{B_0 + B_1 c + z^{-1}(B_0 - B_1 c)}$$

Here to let a matrix to present the computing process of $H(s) \Rightarrow H(z)$

That is $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 c \end{bmatrix}$

When $n = 2$,

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{b_0 + b_1 z^{-1} + b_2 z^{-2}}$$
$$= \frac{A_0 + A_1 c + A_2 c^2 + z^{-1}(2A_0 - 2A_2 c^2) + z^{-2}(A_0 - A_1 c + A_2 c^2)}{B_0 + B_1 c + B_2 c^2 + z^{-1}(2B_0 - 2B_2 c^2) + z^{-2}(B_0 - B_1 c + B_2 c^2)}$$

Here to let a matrix to present the computing process of $H(s) \Rightarrow H(z)$

That is $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 c \\ A_2 c^2 \end{bmatrix}$

And so, when $n = 3, 4, 5, 6, \cdots, n$, we can get the transfer matrix, to express with formula as:

$$a = P_n \cdot A^1 \tag{4}$$

Here $A^1 = \begin{bmatrix} A_0 & A_1 c & A_2 c^2 & \ldots & A_n c^n \end{bmatrix}^T$

Now let's see the P_n matrix, when $n = 1$, $P_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

When $n = 2$, $P_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

When $n = 3$, $P_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

Therefore, to any n, P_n The elements of the first column in P_n comply with a law that is Yang Hui triangle arrangement. Next to introduce Yang Hui triangle arrangement:

n=1 1

n=2 1 1

n=3 1 2 1

n=4 1 3 3 1

n=5 1 4 6 4 1

From the features [7] of Yang Hui triangle arrangement:

① The figures in each line are symmetrical, and turn large from 1, and then turn small, and go back to 1.

② The figure numbers of the number n line are n.

③ The sum of figures in number n line is 2^{n-1}

④ each figure is the sum of its left and right figure in the previous line, which is $(C^i_{n+1} = C_n^{\ i} + C^{i-1}_{\ n})$

⑤ The first figure of number n line is 1,

The second is $1 \times (n-1)$,

The third is $\dfrac{1 \times (n-1) \times (n-2)}{2}$,

The fourth is $\dfrac{1 \times (n-1) \times (n-2) \times (n-3)}{1 \times 2 \times 3}$,

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And so,

The number n figure is $\frac{(n-1)!}{(i-1)!(n-i)!}$

What's more, to extract all the elements of first column in transfer matrix P_n , and get the follows:

n=0 1

n=1 1 1

n=2 1 2 1

n=3 1 3 3 1

n=4 1 4 6 4 1

Here actually the n is the previous given n-1, thus, use n+1 to instead the $\frac{(n-1)!}{(i-1)!(n-i)!}$, then it becomes $\frac{n!}{(i-1)!(n-i+1)!}$ which is also

$$P_n = \frac{n!}{(i-1)!(n-i+1)!} \tag{5}$$

As a result, to any P_n, it is only need to know its n, and then the elements in the first column can be written down.

3 FEATURES

From the above discussion, we can get 3 features of transfer matrix [8]:

First, the first line elements of P_n only can be and all are 1,

Second, the first column elements, if i and n are known, then $P_{i,1}$ can be known. That is

$$P_{i,1} = \frac{n!}{(n-i+1)!(i-1)!} \quad i = n+1, n, \cdots, 2, 1 \tag{6}$$

Third, there is a special relation in the matrix, That is

$$P_{i,j} = P_{i-1,j} + P_{i-1,j+1} + P_{i,j+1} \tag{7}$$

and $\begin{matrix} i = 2, 3, \cdots, n+1 \\ j = 1, 2, \cdots, n \end{matrix}$

From these 3 features, if the order n is known, the any transfer matrix P_n can be calculated out.

4 EXAMPLES

Example 1, to use bilinear transformation to design a Butterworth digital low pass filter[9] of order 3, sampling frequency is $f_s = 4kHZ$, (sampling period is T=250us), and the 3dB cut off frequency $f_c = 1kHZ$, and analog Butterworth filter of 3 order is

$$H(s) = \frac{1}{1 + 2(\frac{s}{\Omega_c}) + 2(\frac{s}{\Omega_c})^2 + (\frac{s}{\Omega_c})^3}$$

Solve: to order $\Omega_c = 1$, $c = \cot\frac{\pi f_1}{f_s} = 1$, and

$H(s) = \frac{1}{1 + 2s + 2s^2 + s^3}$, from formula (2) ,

$$A_0 = 1, A_1 = 0, A_2 = 0, A_3 = 0$$
$$B_0 = 1, B_1 = 2, B_2 = 2, B_3 = 1$$

Apply formula (6) and (7) , it is easy to get:

$$P_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

To use the previous formula (4) , and put $a = P_n \cdot A^1$ in,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 c \\ A_2 c^2 \\ A_3 c^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 c \\ B_2 c^2 \\ B_3 c^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

That is $\begin{matrix} a_0 = 1, \ a_1 = 3, \ a_2 = 3, \ a_3 = 1 \\ b_0 = 6, \ b_1 = 0, \ b_2 = 2, \ b_3 = 0 \end{matrix}$

Thus get the system function of digital filter:

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{6 + 2z^{-2}}$$

Example 2: sampling frequency[10] is $f_s = 100kHZ$, T=10us, it is required to design a Butterworth digital band pass filter of order 3, and its up and down 3dB cut off frequency is $f_2 = 37.5kHZ$, $f_1 = 12.5KHZ$.

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Solve: first to get the each critical frequency of the needed digital filter. And the up and down cut off frequency is

$$\omega_1 = 2\pi f_1 T = 2\pi \times 12.5\times 10^3 \times 10\times 10^{-6} = 0.25\pi$$
$$\omega_2 = 2\pi f_2 T = 2\pi \times 37.5\times 10^3 \times 10\times 10^{-6} = 0.75\pi$$

Analog low pass cut off frequency is

$$\begin{aligned}\Omega_c &= \frac{2}{T}[\tan(\frac{\omega_2}{2}) - \tan(\frac{\omega_1}{2})] \\ &= \frac{2}{T}[\tan(\frac{3\pi}{8}) - \tan(\frac{\pi}{8})] \\ &= 0.25\pi\end{aligned}$$

From the appendix 1, the transfer formula of transformation from analog low pass prototype to digital band pass is (8), to use this to get $H(Z)$, which is the needed digital band pass filter prototype.

$$s = D[\frac{1 - Ez^{-1} + z^{-2}}{1 - z^{-2}}] = \frac{4}{T}\frac{1 + z^{-2}}{1 - z^{-2}} \tag{8}$$

We can get D is

$$D = \Omega_c \cot(\frac{\omega_2 - \omega_1}{2}) = \frac{4}{T}\cot(\frac{\pi}{4}) = \frac{4}{T}$$

We can get E is

$$E = 2\frac{\cos(\frac{\omega_2 + \omega_1}{2})}{\cos(\frac{\omega_2 - \omega_1}{2})} = 2\frac{\cos(\frac{\pi}{2})}{\cos(\frac{\pi}{4})} = 0$$

We use this to substitute, and get $H(z)$, which is the needed digital band pass filter prototype. This relationship formula is also the transformation formula from analog low pass prototype to digital band pass. And from N=3, get the system function of Butterworth filter of 3 order is

$$H(s) = \frac{1}{1 + 2(\frac{s}{\Omega_c}) + 2(\frac{s}{\Omega_c})^2 + 1(\frac{s}{\Omega_c})^3}$$

We put it back in the S,

$$H(z) = H(s)|_{s = \frac{4}{T}\frac{1+z^{-2}}{1-z^{-2}}}$$
$$= \frac{1}{1 + 2(\frac{1+z^{-2}}{1-z^{-2}}) + 2(\frac{1+z^{-2}}{1-z^{-2}})^2 + 1(\frac{1+z^{-2}}{1-z^{-2}})^3}$$
$$H(z) = \frac{1 - 3z^{-2} + 3z^{-4} - z^{-6}}{6 + 2z^{-4}}$$

Now change to calculate in another way, firstly to transfer the analog low pass to analog band pass, and then use the transfer matrix P_n, from the appendix, the formula of transformation from analog low pass prototype to analog band pass is:

$$s \to \Omega_c \frac{s^2 + \Omega_h \Omega_l}{s(\Omega_h - \Omega_l)} \tag{9}$$

Ω_h, Ω_l: they are the actual band pass's up and down cut-off frequency.

From formula (2), the ω_1 and ω_2 is known, the corresponding analog low pass corner frequency is

$$\Omega_1 = \frac{2}{T}\tan(\frac{\omega_1}{2}) \;, \Omega_2 = \frac{2}{T}\tan(\frac{\omega_2}{2})$$

Now let's normalize, to order c=1, that is $\frac{2}{T} = 1$, thus,

$$\Omega_1 \cdot \Omega_2 = 1, \quad \Omega_2 - \Omega_1 = 2$$

And from the appendix, the formula should be:

$$s \to \Omega_c \frac{s^2 + 1}{2s} \tag{10}$$

Put the above formula back to system function of analog low pass filter, then:

$$H(s) = \frac{1}{1 + 2(\frac{s^2+1}{2s}) + 2(\frac{s^2+1}{2s})^2 + (\frac{s^2+1}{2s})^3}$$
$$= \frac{s^3}{\frac{1}{8} + \frac{1}{2}s + \frac{11}{8}s^2 + 2s^3 + \frac{11}{8}s^4 + \frac{1}{2}s^5 + \frac{1}{8}s^6}$$

Here, n=6, from previous transfer matrix P_n, when n=6,

$$P_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & 20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Therefore, the corresponding coefficient of digital filter prototype can be achieved from formula (4), they are:

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$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & 20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & 20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/8 \\ 1/2 \\ 11/8 \\ 2 \\ 11/8 \\ 2 \\ 1/8 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Get

$$a_0 = 1, a_1 = 0, a_2 = -3, a_3 = 0, a_4 = 3, a_5 = 0, a_6 = -1$$
$$b_0 = 6, b_1 = 0,\ b_2 = 0,\quad b_3 = 0,\ b_4 = 2, b_5 = 0,\ b_6 = 0$$

Then write the digital band pass filter's system function:

$$H(z) = \frac{1 - 3z^{-2} + 3z^{-4} - z^{-6}}{6 + 2z^{-4}}$$

5 CONCLUSIONS

From the previous analysis, matrix computing is the key of bilinear transformation, its workload is: $2^{n+2}(n+2)$, and the workload of common computing is : $2(n+1)^2$, therefore, when n is a large number, the matrix computing is much faster than the regular computing. Obviously, it is easy to realize by computer procedure, because matrix computing is easy to carry out by computer. This method also can be used to design digital filter, it is only needed to design the ideal analog low pass, high pass, band pass, and band stop filter, and then to directly bilinear transfer to get digital low pass, high pass, band pass, and band stop filter by matrix. If there can be a little change, then it can output graphs and data, and it can be very clear and easy to understand. This whole process can be done by computer at once.

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