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DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH GENERALIZED CONVEXITY

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ABSTRACT

The purpose of this paper is to consider the Mond-Weir type dual model for a class of non-smooth multiobjective semi-infinite programming problem. In this work, we use generalization of convexity namely $G-(F,\theta)$ convexity and Kuhn-Tucker constraint qualification, to prove new duality results for such semi-infinite programming problem. Weak, strong and converse duality theorems are derived. Some previous duality results for differentiable multiobjective programming problems turn out to be special cases for the results described in the paper.

Keywords: $G-(F,\theta)$ *Convex Functions, Semi-Infinite Programming, Efficient Solution*

1. INTRODUCTION

In recent years, there has been considerable interest in so-called semi-infinite programming problem--the optimization of an objective function in finitely many variables over a feasible region defined by an infinite number of constraints, since this model arises in a large number of applications in different fields of mathematics, economics and engineering, i.e., control of robots, mechanical stress of materials, and air pollution abatement etc. We can see in [1, 2]. To date, many authors investigated the optimality conditions and duality results for semi-infinite programming problems. In particular, Kanzi and Nobakhtian[3] established some alternative theorems and several necessary optimality conditions of Fritz-John and Karush-Kuhn-Tucher type for nonsmooth semi-infinite programming problem. In [4], they also established necessary and sufficient optimality conditions under various constraints qualifications for nonsmooth semi-infinite programming problem using Clarke subdifferential. We also refer [5, 6] to understand different aspects of semi-infinite programming.

On the other hand, the concept of convexity and generalized convexity plays a central role in mathematical economics, management science, and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. To relax convexity assumptions imposed on theorems on optimality conditions for generalized mathematical programming problems, various generalized convexity notations have been introduced. In particular, the concept of generalized $(F,\rho)-$convexity, introduced by Preda [7] was in turn an extension of the convexity and was used by several authors to obtain relevant results. In [8, 9], the concept of $V-\rho-$invexity and $(F,\alpha,\rho,d)-$ convexity were introduced, respectively. Furthermore, duality in mathematical programming has not only used in many theoretical and computational developments in mathematical programming itself but also used in economics, control theory, business problems and other diverse fields. A large literature was developed around generalized convexity and its applications in multiobjective programming. Many authors investigated the optimality conditions and duality results for multiobjective programming problems under the conditions of generalized convexity. In [10], the sufficient optimality conditions and duality results were obtained under the generalized convex functions.

In this paper, motivated by the above work, several duality results are established for a class of multiobjective semi-infinite programming problem involving the new generalized convexity

2. DEFINITIONS AND PRELIMINARIES

In the section, we define a kind of generalized convex functions about the Clarke subgradient.

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Let $f: X \to R$ be locally Lipschitz, where $X \subseteq R^n$ is an open set. Then the Clarke directional derivative of f at $x \in X$ is defined by

$$f^0(x;d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+td) - f(y)}{t}.$$

The Clarke subgradient is given by

$$\partial f(x) = \{\xi \in R^n \,|\, \langle \xi, d \rangle \le f^0(x;d), \forall d \in R^n\}.$$

Definition2.1. A functional $F: X \times X \times R^n \to R$ $(X \subseteq R^n)$ is said to be sublinear about the third variable, if for all $(x_1, x_2) \in X \times X$. It satisfies

$$F(x_1, x_2; \alpha_1 + \alpha_2) \le F(x_1, x_2; \alpha_1) + F(x_1, x_2; \alpha_2),$$
$$\forall \alpha_1, \alpha_2 \in R^n.$$

$$F(x_1, x_2; r\alpha) = rF(x_1, x_2; \alpha_1), \forall r \in R_+, \alpha \in R^n.$$

By the above inequality, it is clear that $F(x_1, x_2; 0) = 0$.

We suppose that X is nonempty open subset of R^n, $f: X \to R$ is local Lipschitz function at $x^0 \in X$, $F: X \times X \times R^n \to R$ is sublinear about the third variable, $b: X \times X \times [0,1] \to R_+, \phi: R \to R$, $\lim_{\lambda \to 0^+} b(x, x^0; \lambda) = b(x, x^0), \alpha: X \times X \to R_+ \setminus \{0\}, \rho \in R$, $\theta: X \times X \to R^n$, where θ is vectorial application.

Definition2.2. f is said to be $G-(F,\theta)$ convex at $x^0 \in X$, if for all $x \in X$, there exists b, φ, α, ρ and θ, such that

$$b(x, x^0)\phi[f(x) - f(x^0)] \ge$$
$$F(x, x^0; \alpha(x, x^0)\xi) + \rho\|\theta(x, x^0)\|^2, \forall \xi \in \partial f(x^0)$$

Definition2.3. f is said to be strict $G-(F,\theta)$ convex at $x^0 \in X$, if for all $x \in X$, there exists b, φ, α, ρ *and* θ, such that

$$b(x, x^0)\phi[f(x) - f(x^0)] >$$
$$F(x, x^0; \alpha(x, x^0)\xi) + \rho\|\theta(x, x^0)\|^2, \forall \xi \in \partial f(x^0)$$

Definition2.4. f is said to be $G-(F,\theta)$ pseudo-convex at $x^0 \in X$, if for all $x \in X$, there exists b, φ, α, ρ *and* θ, such that

$$b(x, x^0)\phi[f(x) - f(x^0)] < 0 \Rightarrow F(x, x^0; \alpha(x, x^0)\xi)$$
$$+\rho\|\theta(x, x^0)\|^2 < 0, \forall \xi \in \partial f(x^0)$$

Definition2.5. f is said to be $G-(F,\theta)$ strict pseudo-convex at $x^0 \in X$, if for all $x \in X, x \ne x^0$, there exists b, φ, α, ρ *and* θ, such that

$$b(x, x^0)\phi[f(x) - f(x^0)] \le 0 \Rightarrow F(x, x^0; \alpha(x, x^0)\xi)$$
$$+\rho\|\theta(x, x^0)\|^2 < 0, \forall \xi \in \partial f(x^0)$$

Definition2.6. f is said to be $G-(F,\theta)$ quasi-convex at $x^0 \in X$, if for all $x \in X$, there exists b, φ, α, ρ *and* θ, such that

$$b(x, x^0)\phi[f(x) - f(x^0)] \le 0 \Rightarrow F(x, x^0; \alpha(x, x^0)\xi)$$
$$+\rho\|\theta(x, x^0)\|^2 \le 0, \forall \xi \in \partial f(x^0)$$

Definition2.7. f is said to be $G-(F,\theta)$ weak quasi-convex at $x^0 \in X$, if for all $x \in X$, there exists b, φ, α, ρ *and* θ, such that

$$b(x, x^0)\phi[f(x) - f(x^0)] < 0 \Rightarrow F(x, x^0; \alpha(x, x^0)\xi)$$
$$+\rho\|\theta(x, x^0)\|^2 \le 0, \forall \xi \in \partial f(x^0)$$

3. DAULITY THEOREMS

Now we consider the following multiobjective semi-infinite programming problem

$$\begin{aligned} & \min imize f(x) = (f(x_1), f(x_2), \cdots, f(x_p)) \\ \text{(SIVP)} \quad & \text{subject to } g(x,u) \leqq 0, u \in U, \\ & x \in X. \end{aligned}$$

where $X \subseteq R^n$ is a nonempty open set, f_i : $X \to R (i = 1, 2, \cdots, p)$, $g: X \times U \to R^m$ and $U \subset R^m$ is an infinite index set. We suppose that f_i and g are locally Lipschitz and Clarke subdifferentiable at x. We put $X^0 = \{x \,|\, g(x,u) \leqq 0, x \in X, u \in U\}$ for the feasible set of problem (SIVP).

Now we define

$$\Delta = \{i \,|\, g(x, u^i) \leqq 0, x \in X, u^i \in U\};$$
$$I(x^0) = \{i \,|\, g(x^0, u^i) = 0, x^0 \in X, u^i \in U\};$$

$U^* = \{u^i \in U \,|\, g(x, u^i) \leqq 0, x \in X, i \in \Delta\}$, which is countable subset of U;

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$\Lambda=\{\mu_i \mid \mu_i \geqq 0, i\in\Delta\}$, which means that $\mu_i \geqq 0$ for all $i\in\Delta$, and only finitely many are strictly positive.

For any $U^*=\{u^i\in U \mid g(x,u^i)\leqq 0, x\in X, i\in\Delta\}$, the Mond-Weir type dual model for (SIVP) is given by

$$\text{(SIVD)}\quad \begin{aligned} &\max f(v)\\ &s.t.\ 0\in\sum_{j=1}^{p}\lambda_j\partial f_j(v)+\sum_{i\in\Delta}\mu_i\partial g(v,u^i);\\ &\sum_{i\in\Delta}\mu_i\partial g(v,u^i)\geqq 0;\\ &\lambda_j\geqq 0,\ j=1,2,\cdots,p,\sum_{j=1}^{p}\lambda_j=1,\mu_i\in\Lambda,i\in\Delta \end{aligned}$$

Let

$W^0=\{(v,u^i,\lambda,\mu) \mid 0\in\sum_{j=1}^{p}\lambda_j\partial f_j(v)+\sum_{i\in\Delta}\mu_i\partial g(v,u^i);$

$\sum_{i\in\Delta}\mu_i\partial g(v,u^i)\geqq 0;\lambda_j\geqq 0,\ j=1,2,\cdots,p,\sum_{j=1}^{p}\lambda_j=1,$

$\mu_i\in\Lambda, i\in\Delta, u^i\in U^*\subset U\}$ denote the set of all feasible solutions of (SIVD).

The following notation conventions are used in this paper:

For $x,y\in R^n, x=(x_1,\cdots,x_n)^T, y=(y_1,\cdots,y_n)^T$, where the superscript T denotes the transpose of a vector,

(i) $x<y \Leftrightarrow x_i<y_i, i=1,\cdots,n$;

(ii) $x\leq y \Leftrightarrow x_i\leqq y_i, i=1,\cdots,n$, and at least one $x_{i_0}<y_{i_0}$ holds for some i_0;

(iii) $x\leqq y \Leftrightarrow x_i\leqq y_i, i=1,\cdots,n$;

(iv) $x\geqq y \Leftrightarrow x_i\geqq y_i, i=1,\cdots,n$.

Definition3.1. $x^*\in X^0$ is said to be a weak efficient solution of (SIVP), if there is no other $x\in X^0$, such that

$$f(x)<f(x^*)$$

Definition3.2. $x^*\in X^0$ is said to be an efficient solution of (SIVP), if there is no other $x\in X^0$, such that

$$f(x)\leq f(x^*)$$

It's meaning that there is no other $x\in X^0$, such that $f_j(x)\leqq f_j(x^*),\forall j\in\{1,2,\cdots,p\}$ and $f_k(x)<f_k(x^*),k\neq j$.

Definition3.3. $x^*\in X^0$ is said to be a properly efficient solution of (SIVP), if x^* is an efficient solution, and there exists a real number $M>0$, such that for all $k\in\{1,2,\cdots,p\}$, only one of the following systems holds:

$$\begin{cases} f_j(x)-f_j(x^*)<0,\forall x\in X^0\\ M(f_k(x)-f_k(x^*))+(f_j(x)-f_j(x^*))\leqq 0,\\ \forall j\in\{1,2,\cdots,p\},\ j\neq k \end{cases}$$

Theorem3.1. (Weak duality)

Let $x\in X^0,(v,u^j,\lambda,\mu)\in W^0$, for $v\in X^0,\lambda_j\geqq 0,$ $j=1,2,\cdots,p$ with $\sum_{j=1}^{p}\lambda_j=1,\mu_i\in\Lambda,i\in\Delta$, assume there exists $F,b_0,\phi_0,b_1,\phi_1,\alpha,\rho_j(j=1,2,\cdots,p),\tau_i$ $(i\in\Delta),\theta$, such that

(i) $b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)]$

$\geqq F(x,v,\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j)+\sum_{j=1}^{p}\lambda_j\rho_j\|\theta(x,v)\|^2,$

$\forall\xi_j\in\partial f_j(v);$

(ii) $-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)]\geqq$

$F(x,v,\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i)+\sum_{i\in I(v)}\mu_i\tau_i\|\theta(x,v)\|^2,$

$\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*;$

(iii) The complementary condition holds, that is, as $g(v,u^j)>0$, we always have $\mu_i=0,i\in\Delta \mid I(v)$;

(iv) $a\leqq 0\Rightarrow\phi_0(a)\leqq 0,\ \phi_1(a)\leqq 0\Rightarrow a<0,$ $b_0(x,v)>0,b_1(x,v)>0$;

(v) $\sum_{j=1}^{p}\lambda_j\rho_j+\sum_{t\in T}\mu_i\tau_i\geqq 0$.

Then we can obtain $f(x)$’ $f(v)$.

Proof: Suppose that the result does not hold, then there exists $x\in X^0$, such that $f(x)\leqq f(v)$. It follows that there exists at least one index k, such that

$$f_k(x)<f_k(v),\ f_j(x)\leqq f_j(v),$$
$$\forall j\in\{1,2,\cdots,p\},\ j\neq k$$

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Since $\lambda_j \geqq 0,\ j=1,2,\cdots,p$ with $\sum_{j=1}^{p}\lambda_j=1$, so we have

$$\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)\leqq 0$$

By (iv), we obtain

$$b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)]\leqq 0$$

Then from (i), we have

$$F(x,v;\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j)+\sum_{j=1}^{p}\lambda_j\rho_j\left\|\theta(x,v)\right\|^2\leqq 0,\forall\xi_j\in\partial f_j(v) \quad (1)$$

According to (iii), it follows that $\mu_i\in\Lambda$, and not all μ_i are zeroes, as $i\in I(v)$.

Thus we have

$$\sum_{i\in I(v)}\mu_i g(v,u^i)\geqq 0$$

By (iv), we get

$$-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)]<0$$

Then (ii) yields

$$F(x,v;\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i)+\sum_{i\in I(v)}\mu_i\tau_i\left\|\theta(x,v)\right\|^2<0,$$
$$\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*,i\in I(v)$$

Using (iii) again, we have

$$F(x,v;\alpha(x,v)\sum_{i\in\Delta}\mu_i\zeta_i)+\sum_{i\in\Delta}\mu_i\tau_i\left\|\theta(x,v)\right\|^2<0,\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*,i\in\Delta \quad (2)$$

Adding (1) and (2), then by the sublinearity of F and (v), we get

$$F(x,v;\alpha(x,v)(\sum_{j=1}^{p}\lambda_j\xi_j+\sum_{i\in\Delta}\mu_i\zeta_i))$$
$$<-(\sum_{j=1}^{p}\lambda_j\rho_j+\sum_{i\in\Delta}\mu_i\tau_i)\left\|\theta(x,v)\right\|^2\leqq 0$$

where $\forall\xi_j\in\partial f_j(v),\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*$.

Hence, we known for all $\xi_j\in\partial f_j(v)$ and $\zeta_i\in\partial g(v,u^i),u^i\in U^*,i\in\Delta$, we have

$$\sum_{j=1}^{p}\lambda_j\xi_j+\sum_{i\in\Delta}\mu_i\zeta_i\neq 0$$

So we have a contradiction. Hence the result follows.

Theorem3.2. (Weak duality)

Let $x\in X^0,(v,u^j,\lambda,\mu)\in W^0$, for $v\in X^0,\lambda_j>0$, $j=1,2,\cdots,p$ with $\mu_i\in\Lambda,i\in\Delta$, assume there exists $F,b_0,\phi_0,b_1,\phi_1,\alpha,\rho_j(j=1,2,\cdots,p),\tau_i(i\in\Delta),\theta$, such that

(i) $b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)]<0$

$\Rightarrow F(x,v,\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j)+\sum_{j=1}^{p}\lambda_j\rho_j\left\|\theta(x,v)\right\|^2<0,$
$\forall\xi_j\in\partial f_j(v);$

(ii) $-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)]\leqq 0$

$\Rightarrow F(x,v,\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i)+\sum_{i\in I(v)}\mu_i\tau_i\left\|\theta(x,v)\right\|^2$
$\leqq 0,\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*;$

(iii) $a<0\Rightarrow\phi_0(a)<0,\ a\geqq 0\Rightarrow\phi_1(a)\geqq 0,$
$b_0(x,v)>0,b_1(x,v)\geqq 0$;

(iv) $\sum_{j=1}^{p}\lambda_j\rho_j+\sum_{t\in T}\mu_i\tau_i\geqq 0$.

Then we can obtain $f(x)$ ’ $f(v)$.

Proof: Suppose that the result does not hold, then there exists $x\in X^0$, such that $f(x)\leqq f(v)$. It follows that there exists at least one index k, such that

$$f_k(x)<f_k(v),\ f_j(x)\leqq f_j(v),$$
$$\forall j\in\{1,2,\cdots,p\},\ j\neq k$$

Since $\lambda_j>0,\ j=1,2,\cdots,p$, it follows that

$$\sum_{j=1}^{p}\lambda_j f_j(x)<\sum_{j=1}^{p}\lambda_j f_j(v)$$

According to (iii), we have

$$b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)]<0$$

Then (i) yields

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$$F(x,v;\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j)+\sum_{j=1}^{p}\lambda_j\rho_j\|\theta(x,v)\|^2<0,\forall\xi_j\in\partial f_j(v) \quad (3)$$

Since for all $u^i\in U^*, g(v,u^i)=0$, as $\mu_i\in\Lambda$, $i\in I(v)$, it follows that

$$\sum_{i\in I(v)}\mu_i g(v,u^i)\geqq 0$$

By (iii), we have

$$-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)]\leqq 0$$

Then from (ii), we get

$$F(x,v;\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i)+\sum_{i\in I(v)}\mu_i\tau_i\|\theta(x,v)\|^2\leqq 0,$$
$$\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*,i\in I(v)$$

Let $\mu_i=0$, for all $i\in\Delta\setminus I(v)$, it follows that

$$F(x,v;\alpha(x,v)\sum_{i\in\Delta}\mu_i\zeta_i)+\sum_{i\in\Delta}\mu_i\tau_i\|\theta(x,v)\|^2\leqq 0,\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*,i\in\Delta \quad (4)$$

Adding (3) and (4), then by the sublinearity of F and (iv), we have

$$F(x,v;\alpha(x,v)(\sum_{j=1}^{p}\lambda_j\xi_j+\sum_{i\in\Delta}\mu_i\zeta_i))$$
$$<-(\sum_{j=1}^{p}\lambda_j\rho_j+\sum_{i\in\Delta}\mu_i\tau_i)\|\theta(x,v)\|^2\leqq 0$$

Where $\forall\xi_j\in\partial f_j(v),\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*$.

Hence, we known for all $\xi_j\in\partial f_j(v)$ and $\zeta_i\in\partial g(v,u^i),u^i\in U^*,i\in\Delta$, we have

$$\sum_{j=1}^{p}\lambda_j\xi_j+\sum_{i\in\Delta}\mu_i\zeta_i\neq 0$$

But which contradicts the constraint condition of (SIVD). Hence the result follows.

Theorem3.3. (Weak duality)

Let $x\in X^0,(v,u^j,\lambda,\mu)\in W^0$, for $v\in X^0,\lambda_j\geqq 0$, $j=1,2,\cdots,p$ with $\mu_i\in\Lambda,i\in\Delta$, assume there exists $F,b_0,\phi_0,b_1,\phi_1,\alpha,\rho_j(j=1,2,\cdots,p),\tau_i(i\in\Delta),\theta$, such that

(i) $b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)]\leqq 0$

$\Rightarrow F(x,v,\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j)+\sum_{j=1}^{p}\lambda_j\rho_j\|\theta(x,v)\|^2$
$<0,\forall\xi_j\in\partial f_j(v)$;

(ii) $-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)]\leqq 0$

$\Rightarrow F(x,v,\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i)+\sum_{i\in I(v)}\mu_i\tau_i\|\theta(x,v)\|^2$
$\leqq 0,\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*$;

(iii) $a<0\Rightarrow\phi_0(a)\leqq 0,\ a\geqq 0\Rightarrow\phi_1(a)\geqq 0,$
$b_0(x,v)>0,b_1(x,v)\geqq 0$;

(iv) $\sum_{j=1}^{p}\lambda_j\rho_j+\sum_{t\in T}\mu_i\tau_i\geqq 0$;

Then we can obtain $f(x)$ ’ $f(v)$.

Proof: The proof is similar to the theorem 3.2.

Theorem3.4. (Weak duality)

Let $x\in X^0,(v,u^j,\lambda,\mu)\in W^0$, for $v\in X^0,\lambda_j\geqq 0$, $j=1,2,\cdots,p$ with $\sum_{j=1}^{p}\lambda_j=1,\mu_i\in\Lambda,i\in\Delta$, assume there exists $F,b_0,\phi_0,b_1,\phi_1,\alpha,\rho_j(j=1,2,\cdots,p),\tau_i$ $(i\in\Delta),\theta$, such that

(i) $b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x)-\sum_{j=1}^{p}\lambda_j f_j(v)]\leqq 0$

$\Rightarrow F(x,v,\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j)+\sum_{j=1}^{p}\lambda_j\rho_j\|\theta(x,v)\|^2\leqq 0,$
$\forall\xi_j\in\partial f_j(v)$;

(ii) $-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)]\leqq 0$

$\Rightarrow F(x,v,\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i)+\sum_{i\in I(v)}\mu_i\tau_i\|\theta(x,v)\|^2<0,$
$\forall\zeta_i\in\partial g(v,u^i),u^i\in U^*$;

(iii)
$a\leqq 0\Rightarrow\phi_0(a)\leqq 0,\ a\geqq 0\Rightarrow\phi_1(a)\geqq 0,b_0(x,v)>0,$
$b_1(x,v)\geqq 0$;

(iv) $\sum_{j=1}^{p}\lambda_j\rho_j+\sum_{t\in T}\mu_i\tau_i\geqq 0$;

Then we can obtain $f(x)$ ’ $f(v)$.

Proof: Suppose that the result does not hold, then there exists $x\in X^0$, such that $f(x)\leqq f(v)$. It follows that there exists at least one index k, such that

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$$f_k(x) < f_k(v),\ f_j(x) \leqq f_j(v),$$
$$\forall j \in \{1,2,\cdots,p\},\ j \neq k$$

Since $\lambda_j \geqq 0,\ j=1,2,\cdots,p$ with $\sum_{j=1}^{p}\lambda_j = 1$, it follows that

$$\sum_{j=1}^{p}\lambda_j f_j(x) \leqq \sum_{j=1}^{p}\lambda_j f_j(v)$$

According to (iii), we have

$$b_0(x,v)\phi_0[\sum_{j=1}^{p}\lambda_j f_j(x) - \sum_{j=1}^{p}\lambda_j f_j(v)] \leqq 0$$

Then (i) yields

$$F(x,v;\alpha(x,v)\sum_{j=1}^{p}\lambda_j\xi_j) + \sum_{j=1}^{p}\lambda_j\rho_j\|\theta(x,v)\|^2 \leqq 0, \forall \xi_j \in \partial f_j(v) \quad (5)$$

Since for all $u^i \in U^*, g(v,u^i) = 0$, as $\mu_i \in \Lambda$, $i \in I(v)$, it follows that

$$\sum_{i\in I(v)}\mu_i g(v,u^i) \geqq 0$$

By (iii), we have

$$-b_1(x,v)\phi_1[\sum_{i\in I(v)}\mu_i g(v,u^i)] \leqq 0$$

Then from (ii), we get

$$F(x,v;\alpha(x,v)\sum_{i\in I(v)}\mu_i\zeta_i) + \sum_{i\in I(v)}\mu_i\tau_i\|\theta(x,v)\|^2 < 0, \forall \zeta_i \in \partial g(v,u^i), u^i \in U^*, i \in I(v)$$

Let $\mu_i = 0$, foe all $i \in \Delta \setminus I(v)$, it follows that

$$F(x,v;\alpha(x,v)\sum_{i\in\Delta}\mu_i\zeta_i) + \sum_{i\in\Delta}\mu_i\tau_i\|\theta(x,v)\|^2 < 0, \forall \zeta_i \in \partial g(v,u^i), u^i \in U^*, i \in \Delta \quad (6)$$

Adding (5) and (6), then by the sublinearity of F and (iv), we have

$$F(x,v;\alpha(x,v)(\sum_{j=1}^{p}\lambda_j\xi_j + \sum_{i\in\Delta}\mu_i\zeta_i)) < -(\sum_{j=1}^{p}\lambda_j\rho_j + \sum_{i\in\Delta}\mu_i\tau_i)\|\theta(x,v)\|^2 \leqq 0$$

where $\forall \xi_j \in \partial f_j(v), \forall \zeta_i \in \partial g(v,u^i), u^i \in U^*$.

Hence, we known for all $\xi_j \in \partial f_j(v)$ and $\zeta_i \in \partial g(v,u^i), u^i \in U^*, i \in \Delta$, we have

$$\sum_{j=1}^{p}\lambda_j\xi_j + \sum_{i\in\Delta}\mu_i\zeta_i \neq 0$$

But which contradicts the constraint condition of (SIVD). Hence the result follows.

Theorem3.5. (Strong duality)

Suppose that x^* is a properly efficient solution of (SIVP), and the Kuhn-Tucker constraint qualification is satisfied at x^*. Then there exists $(\lambda^*,\mu^*) \geq 0$, such that $(x^*,u^i,\lambda^*,\mu^*)$ is feasible solution of (SIVD). Furthermore, the two problems (SIVP) and (SIVD) have the same objective value. Furthermore, if the hypothesis of theorem 3.2 is also satisfied for all $x \in X^0$ and $(v,u^i,\lambda,\mu) \in W^0$, then $(x^*,u^i,\lambda^*,\mu^*)$ is a properly efficient solution for (SIVD).

Proof: Since x^* is a properly efficient solution of (SIVP), and the Kuhn-Tucker constraint qualification is satisfied at x^*, then there exists $\lambda_j^* > 0, j=1,2,\cdots,p$ and $\mu_i^* \in \Lambda, i \in I(x^*)$ (not all μ_i are zeroes), such that for any $u^i \in U^*$, we have

$$0 \in \sum_{j=1}^{p}\lambda_j^*\partial f_j(x^*) + \sum_{i\in I(x^*)}\mu_i^*\partial g(x^*,u^i),$$
$$\sum_{i\in I(x^*)}\mu_i^*\partial g(x^*,u^i) = 0, \forall u^i \in U^*, i \in I(x^*),$$

Let $\mu_i = 0$, as $i \in \Delta \setminus I(x^*)$, it follows that

$$0 \in \sum_{j=1}^{p}\lambda_j^*\partial f_j(x^*) + \sum_{i\in\Delta}\mu_i^*\partial g(x^*,u^i),$$
$$\sum_{i\in\Delta}\mu_i^*\partial g(x^*,u^i) = 0, \forall u^i \in U^*, i \in \Delta,$$

Hence, $(x^*,u^i,\lambda^*,\mu^*)$ is feasible solution of (SIVD).

It is clear that the two problems have the same objective value at x^* and $(x^*,u^i,\lambda^*,\mu^*)$.

Since x^* is a properly efficient solution of (SIVP), let $M = (p-1)\max_{1\leqq j,k\leqq p}\frac{\lambda_k^*}{\lambda_j^*}$.

Suppose on the contrary that $(x^*,u^i,\lambda^*,\mu^*)$ is not a properly efficient solution of (SIVD). Then there exists

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one $j \in \{1,2,\cdots,p\}$ and $(\bar{v},u^i,\bar{\lambda},\bar{\mu}) \in W^0$, such that $f_j(x^*) < f_j(\bar{v})$ and $f_k(x^*) > f_k(\bar{v})$ for any $j \neq k$. It follows that

$$f_j(\bar{v}) - f_j(x^*) > M[f_k(x^*) - f_k(\bar{v})]$$
$$= (p-1)\max_{1\leqq j,k\leqq p} \frac{\lambda_k^*}{\lambda_j^*}[f_k(x^*) - f_k(\bar{v})]$$
$$\geqq \frac{p-1}{\lambda_j^*}\lambda_k^*[f_k(x^*) - f_k(\bar{v})]$$

So for any $j \neq k$, we get

$$\frac{\lambda_j^*}{p-1}[f_j(\bar{v}) - f_j(x^*)] > \lambda_k^*[f_k(x^*) - f_k(\bar{v})]$$

Summing for any $j \neq k$, then we obtain

$$\lambda_j^*[f_j(\bar{v}) - f_j(x^*)] > \sum_{k\neq j}\lambda_k^*[f_k(x^*) - f_k(\bar{v})]$$

So we get

$$\lambda^{*T} f(\bar{v}) = \sum_{j=1}^{p}\lambda_j^* f_j(\bar{v}) > \sum_{j=1}^{p}\lambda_j^* f_j(x^*) = \lambda^{*T} f(x^*) \quad (7)$$

Then using the result of theorem 3.2, we get

$$f(x^*) ’ \ f(\bar{v})$$

Since $\lambda^* > 0$, it follows that

$$\lambda^* f(x^*) ’ \ \lambda^* f(\bar{v})$$

But which contradicts the inequality (7). Hence, we conclude that $(x^*,u^i,\lambda^*,\mu^*)$ is a properly efficient solution of (SIVD).

Theorem3.6. (Strong duality)

Suppose that x^* is an efficient solution of (SIVP), and the Kuhn-Tucker constraint qualification is satisfied at x^*. Then there exists $(\lambda^*,\mu^*) \geq 0$, such that $(x^*,u^i,\lambda^*,\mu^*)$ is feasible solution of (SIVD). Furthermore, the two problems (SIVP) and (SIVD) have the same objective value. Other hand, if the hypothesis of theorem 3.2 is also satisfied for all $x \in X^0$ and $(v,u^i,\lambda,\mu) \in W^0$. Then $(x^*,u^i,\lambda^*,\mu^*)$ is a properly efficient solution for (SIVD).

Proof: The proof is similar to the theorem 3.5.

Theorem3.7. (Converse duality)

Let $f_j(x^*) = f_j(v^*)(j=1,2,\cdots,p)$ at $x^* \in X^0$ and $(x^*,u^i,\lambda^*,\mu^*) \in W^0$. Suppose that for $\lambda_j^* > 0$ $(j=1,2,\cdots,p)$, $(\sum_{j=1}^{p}\lambda_j^* f_j, \sum_{i\in I(v)}\mu_i^* g(\cdot,u^i)), \forall u^i \in U^*$ satisfies the hypothesis of theorem 3.2 at $v^* \in X^0$, then $(x^*,u^i,\lambda^*,\mu^*)$ is a properly efficient solution of (SIVD).

Proof: Using the result of theorem 3.3, we known v^* is a properly efficient solution of (SIVP). It is clear that x^* is also a properly efficient solution of (SIVP). If it is not true, then there exists one $j \in \{1,2,\cdots,p\}$ and one $\bar{x} \in X^0$, such that $f_j(x^*) > f_j(\bar{x})$, and $f_k(x^*) < f_k(\bar{x})$ for any $k \neq j$. It follows that

$$f_j(x^*) - f_j(\bar{x}) > M[f_k(x^*) - f_k(\bar{x})]$$

Also, we have $f_j(x^*) = f_j(v^*)$ for any $j = 1,2,\cdots,p$.

Now we have a contradiction. Therefore, x^* is also a properly efficient solution of (SIVP).

We can derive that $(x^*,u^i,\lambda^*,\mu^*)$ is a properly efficient solution of (SIVD) like the proof of theorem 3.5.

4. CONCLUSION

Throughout this paper, we have defined a new generalized convex function, extending many well-known classes of generalized convex functions. Furthermore, we have formulated the multi-objective dual problem and proved the results concerning weak and strong duality between the primal (SIVP) and the dual (SIVD), there should be further opportunities for exploiting this structure of the semi-infinite programming problem.

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