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A NEW OPTIMAL QUADRATIC PREDICTOR OF A RESIDUAL LINEAR MODEL IN A FINITE POPULATION

[1]MOHAMMED EL HAJ TIRARI, [2]ABDELLATIF EL AFIA, [2]RDOUAN FAIZI

[1]Insea, Rabat-Instituts, BP 6217, Rabat, Maroc

[1]Crest-Ensai, Rue Blaise Pascal, BP 37203, 35172 Bruz cedex, France

[2]Ensias, Rabat - Instituts, BP 713, Rabat, Maroc

E-mail: [1]tirari@insea.ac.ma, [2]elafia@ensias.ma, [2]faizi@ensias.ma

ABSTRACT

In this paper, we deal with the problem related to the prediction of a residual quadratic form of a super-population model in the case of a finite population. We propose a new optimal quadratic predictor of this quadratic form for a linear Gaussian model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$. The main result of this paper is used to derive an optimal quadratic predictor of the residual sum of squares of a linear Gaussian model.

Keywords: *Survey Sampling, Finite Population, Super-Population Model, Quadratic Prediction*

1. INTRODUCTION

Linear models play a key role in improving the precision of parameter estimators of a finite population since they allow to take into account the auxiliary information available on population units. Indeed, using the predictive approach in the sampling theory has enabled to construct the Best Linear Unbiased Predictor (BLUP) of the total of an interest variable (Royall, 1971; Cassel *et al.*, 1993; Valliant *et al.*, 2000). However, the optimality of this predictor cannot be guaranteed unless the linear model adopted is valid. Therefore, analysis of the quality of the linear model plays a role in the construction of the predictor BLUP. This analysis is carried out through the residues of the linear model, especially the residual sum of squares. This is why it is very important to predict the residual sum of squares of a linear model or any other quadratic form of these residues.

Under the predictive approach by assuming the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ with $\mathbf{e} = \mathrm{N}\left(\mathbf{0}, \sigma^2\mathbf{V}\right)$, Liu and Rong (2007) were mainly interested in predicting a positive quadratic form $\mathbf{Y}'\mathbf{Q}\mathbf{Y}$ where $\mathbf{Q}$ is a symmetric and nonnegative definite matrix satisfying $\mathbf{Q}\mathbf{X} = \mathbf{0}$. Thus, considering the class of invariant predictors (see definition below), Liu and Rong propose an optimal quadratic predictor following the minimization criterion of the Mean Squared Error under the model.

In this paper, we focus on cases in which the function to predict is a quadratic residue, of a multiple linear model, $\mathbf{e}'\mathbf{Q}\mathbf{e}$ where $\mathbf{Q}$ is a symmetric matrix. We propose a new optimal predictor of this quadratic form whose expression is much simpler than that of the predictor of Liu and Rong. Therefore, Section 2 is devoted to notations and definitions used in this work. In Section 3, we put forward an optimal quadratic predictor of $\mathbf{e}'\mathbf{Q}\mathbf{e}$ following the minimization of the Mean Squared Error under the model. Finally, in Section 4, the main results of this paper are applied to the problem of predicting the residual sum of squares of a linear Gaussian model.

2. NOTATIONS AND DEFINITIONS

Given a finite population $U = \{1, ..., k, ..., N\}$ composed of N units, focus will laid on a variable of interest $\mathbf{y} = \left(y_1, ..., y_N\right)'$. For this reason, a sample s of n units is selected from the population U and the values of $\mathbf{y}$ are observed for these units. We note the set of units U unselected by r. We have values of p auxiliary variables $X_1, ..., X_p$ which can be represented by matrix $\mathbf{X} = \left(\mathbf{x}_1, ..., \mathbf{x}_k, ..., \mathbf{x}_N\right)'$ where $\mathbf{x}_k$ is the vector of values of p variables $X_1, ..., X_p$ for a unit $k \in U$.

To simplify the notations in this paper, for any matrix $\mathbf{A}$, we note $\mathbf{A} \in \mathbb{R}_{m,n}$ if $\mathbf{A}$ is a real matrix;

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$\mathbf{A} \in \square_n^{sy}$ if $\mathbf{A} \in \square_{n,n}$ and it is symmetric; $\mathbf{A} \in \square_n^{\geq}$ if $\mathbf{A} \in \square_n^{sy}$ and is nonnegative definite.

Note also $\mathbf{A}'$, $\mathbf{A}^-$, $\mathbf{P_A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^- \mathbf{A}'$ and $rk(\mathbf{A})$ respectively the transpose, the generalized inverse, the orthogonal projector, and the rank of $\mathbf{A}$, $C(\mathbf{A})$ the subspace spanned by the columns of $\mathbf{A}$ and $tr(\mathbf{A})$ the trace of $\mathbf{A}$.

Under the approach based on a model, we assume that the values $y_1,\dots,y_N$ of the interest variable $\mathbf{y}$ are the achievements of a random vector $\mathbf{Y} = (Y_1,\dots,Y_N)'$ whose joint probability distribution ξ is given by a super-population model. In this work, we consider the following linear model:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \quad \text{with } \mathbf{e} = N(\mathbf{0}, \sigma^2\mathbf{V}) \qquad (2.1)$$

where $E_\xi(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$ and $Cov_\xi(\mathbf{Y}) = \sigma^2\mathbf{V}$. The covariance matrix $\mathbf{V} \in \square^N$ is assumed to be known and $\boldsymbol{\beta}$, σ^2 are unknown parameters. To estimate $\boldsymbol{\beta}$, we can use the best linear unbiased estimator $\hat{\boldsymbol{\beta}}$ given by

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s\right)^- \mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{Y}_s \qquad (2.2)$$

where for a given sample *s* of *n* units selected from *U*, vector $\mathbf{Y}_s$ and matrices $\mathbf{X}_s$ and $\mathbf{V}_s$ are defined through the following decomposition of $\mathbf{Y}$, $\mathbf{X}$ and $\mathbf{V}$:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_s \\ \mathbf{Y}_r \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_s \\ \mathbf{X}_r \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} \mathbf{e}_s \\ \mathbf{e}_r \end{bmatrix}$$

and

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_s & \mathbf{V}_{sr} \\ \mathbf{V}_{rs} & \mathbf{V}_r \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \end{bmatrix}$$

where $\mathbf{Y}_s = (Y_1,\dots,Y_n)'$, $\mathbf{X}_s = (\mathbf{x}_1,\dots,\mathbf{x}_n)'$, $\mathbf{V}_s \in \square_n^{\geq}$ and $\mathbf{V}_1 = (\mathbf{V}_s, \mathbf{V}_{sr})'$.

It is also noted by $\mathbf{T}$ the matrix defined by

$$\mathbf{T} = \mathbf{V}_s + \mathbf{X}_s\mathbf{U}\mathbf{X}_s'$$

with $\mathbf{U} \in \square_p^{\geq}$ defined so that $C(\mathbf{T}) = C(\mathbf{X}_s, \mathbf{V}_s)$.

3. THE OPTIMAL QUADRATIC PREDICTOR

Given $\mathbf{e}'\mathbf{Q}\mathbf{e}$ where $\mathbf{Q} \in \square_N^{sy}$ a residual quadratic form of the multiple linear model (2.1). To predict this quadratic form, we can use the quadratic predictor $\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ where $\mathbf{A}_s \in \square_n^{sy}$ and $\hat{\mathbf{e}}_s = \mathbf{Y}_s - \mathbf{X}_s\hat{\boldsymbol{\beta}}$ with $\hat{\boldsymbol{\beta}}$ is given by (2.2). In general, the matrix $\mathbf{A}_s$ is chosen in such a way that the predictor $\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ satisfies an optimality criterion. A choice criterion of the matrix $\mathbf{A}_s$ may minimize the Mean Squared Error (MSE) under the model (2.1). This criterion was used by Liu and Rong (2007) to determine an invariant optimal quadratic predictor of a quadratic form $\mathbf{Y}'\mathbf{Q}\mathbf{Y}$ where matrix $\mathbf{Q}$ satisfies $\mathbf{Q}\mathbf{X} = \mathbf{0}$. Note that a quadratic predictor $\mathbf{Y}_s'\mathbf{A}_s\mathbf{Y}_s$ is said to be invariant if matrix $\mathbf{A}_s$ satisfies $\mathbf{A}_s\mathbf{X}_s = \mathbf{0}$. Thus, to predict a quadratic form $\mathbf{Y}'\mathbf{Q}\mathbf{Y}$, Liu and Rong (2007) demonstrated that the optimal quadratic predictor in the class of invariants and unbiased quadratic predictors is given by $\mathbf{Y}_s'\mathbf{A}_s^*\mathbf{Y}_s$ where

$$\mathbf{A}_s^* = \lambda^*\mathbf{N}_{\mathbf{X}_s} + \mathbf{N}_{\mathbf{X}_s}\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\mathbf{N}_{\mathbf{X}_s}$$

with

$$\lambda^* = \frac{tr\left(\mathbf{Q}\mathbf{V} - \mathbf{Q}\mathbf{V}_1\mathbf{N}_{\mathbf{X}_s}\mathbf{V}_1'\right)}{rk(\mathbf{T}) - rk(\mathbf{X}_s)}$$

and $\mathbf{N}_{\mathbf{X}_s} = \mathbf{T}^+ - \mathbf{T}^+\mathbf{X}_s\left(\mathbf{X}_s'\mathbf{T}^+\mathbf{X}_s\right)\mathbf{X}_s'\mathbf{T}^+$.

The predictor proposed by Liu and Rong can be used to predict a quadratic form of residues $\mathbf{e}'\mathbf{Q}\mathbf{e}$ of the multiple linear model (2.1) as the invariance condition implies that $\mathbf{Y}_s'\mathbf{A}_s\mathbf{Y}_s = \hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ but not the best to use in the case of predicting $\mathbf{e}'\mathbf{Q}\mathbf{e}$. Indeed, by considering the class of quadratic predictors $\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ (without imposing the invariance condition), we propose a quadratic predictor of $\mathbf{e}'\mathbf{Q}\mathbf{e}$ whose expression is much simpler than that of the predictor of Liu and Rong while also being optimal in the class of quadratic predictors. Note that the prediction Mean Squared Error (MSE) of $\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ cannot be calculated exactly but can be approximated by that of the quadratic form $\mathbf{e}_s'\mathbf{A}_s\mathbf{e}_s$ with $\mathbf{e}_s = \mathbf{Y}_s - \mathbf{X}_s\boldsymbol{\beta}$ where $\boldsymbol{\beta}$ is the coefficients vector of the multiple linear model (2.1). In fact, an approximation of the prediction mean squared error under the model (2.1) is given by

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$$\begin{aligned} MSE_\xi\left(\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s\right) &\approx MSE_\xi\left(\mathbf{e}_s'\mathbf{A}_s\mathbf{e}_s\right) \\ &\approx \mathrm{E}_\xi\left(\mathbf{e}_s'\mathbf{A}_s\mathbf{e}_s - \mathbf{e}'\mathbf{Q}\mathbf{e}\right)^2 \\ &\approx 2\sigma^4 tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{A}_s\mathbf{V}_s\right) \\ &\quad -4\sigma^4 tr\left(\mathbf{A}_s\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\right) \\ &\quad +2\sigma^4 tr\left(\mathbf{QVQV}\right) \\ &\quad +\sigma^4\left[tr\left(\mathbf{A}_s\mathbf{V}_s\right) - tr\left(\mathbf{QV}\right)\right]^2 \end{aligned}$$

This expression of $MSE_\xi\left(\mathbf{e}_s'\mathbf{A}_s\mathbf{e}_s\right)$ is obtained by using the fact that for any variable $\mathbf{Z} \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega})$, we have

$$Cov_\xi\left(\mathbf{Z}'\mathbf{AZ}, \mathbf{Z}'\mathbf{BZ}\right) = 2tr\left(\mathbf{A\Omega B\Omega}\right) + 4\boldsymbol{\mu}'\mathbf{A\Omega B}\boldsymbol{\mu}$$

for all symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ (see for example, Schott, 2005, p.418). In addition, we have

$$\mathrm{E}_\xi\left(\mathbf{e}_s'\mathbf{A}_s\mathbf{e}_s - \mathbf{e}'\mathbf{Q}\mathbf{e}\right) = tr\left(\mathbf{A}_s\mathbf{V}_s\right) - tr\left(\mathbf{QV}\right)$$

Hence, a quadratic form $\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ is an optimal predictor of $\mathbf{e}'\mathbf{Q}\mathbf{e}$ in the class of quadratic predictors if the matrix $\mathbf{A}_s$ minimizes

$$tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{A}_s\mathbf{V}_s\right) - 2tr\left(\mathbf{A}_s\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\right) + \frac{1}{2}\left[tr\left(\mathbf{A}_s\mathbf{V}_s\right) - tr\left(\mathbf{QV}\right)\right]^2$$

In what follows, we will adopt the approach used by Liu and Rong (2007) so as to get an optimal quadratic predictor of the quadratic form $\mathbf{e}'\mathbf{Q}\mathbf{e}$. However, to look for matrix $\mathbf{A}_s$ minimizing $MSE_\xi\left(\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s\right)$ in the class of quadratic predictors, we will be restricted to matrices $\mathbf{A}_s$ satisfying the condition $C\left(\mathbf{A}_s\right) \subseteq C\left(\mathbf{V}_s\right)$. This restriction is reasonable and necessary as $\hat{\mathbf{e}}_s'\mathbf{A}_s\hat{\mathbf{e}}_s$ is equal to $\hat{\mathbf{e}}_s'\mathbf{P}_{\mathbf{V}_s}\mathbf{A}_s\mathbf{P}_{\mathbf{V}_s}\hat{\mathbf{e}}_s$ with a probability equal to 1, using the fact that $\hat{\mathbf{e}}_s \in C\left(\mathbf{V}_s\right)$ is satisfied almost surely.

Thus, the problem of finding the optimal quadratic predictor $\hat{\mathbf{e}}_s'\mathbf{A}_{s,opt}\hat{\mathbf{e}}_s$ of $\mathbf{e}'\mathbf{Q}\mathbf{e}$ is reduced to finding matrix $\mathbf{A}_{s,opt}$, which is the solution to the following minimization problem:

$$\min_{\mathbf{A}_s \in \mathbb{R}_n^{sy} | C(\mathbf{A}_s) \subseteq C(\mathbf{V}_s)} \left\{ tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{A}_s\mathbf{V}_s\right) - 2tr\left(\mathbf{A}_s\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\right) + \frac{1}{2}\left[tr\left(\mathbf{A}_s\mathbf{V}_s\right) - tr\left(\mathbf{QV}\right)\right]^2 \right\} \quad (3.3)$$

We note that as $C\left(\mathbf{V}_1'\right) = C\left(\mathbf{V}_s\right)$, we have $\mathbf{V}_1' = \mathbf{V}_s\mathbf{V}_s^{-1}\mathbf{V}_1'$ and

$$\begin{aligned} tr\left(\mathbf{A}_s\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\right) &= tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\mathbf{V}_s^{-1}\mathbf{V}_s\right) \\ &= tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{G}\mathbf{V}_s\right) \end{aligned}$$

where $\mathbf{G} = \mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{Q}\mathbf{V}_1\mathbf{V}_s^{-1}$.

Moreover, by considering $\tilde{\mathbf{A}}_s = \mathbf{V}_s^{\frac{1}{2}}\mathbf{A}_s\mathbf{V}_s^{\frac{1}{2}}$ and $\tilde{\mathbf{G}} = \mathbf{V}_s^{\frac{1}{2}}\mathbf{G}\mathbf{V}_s^{\frac{1}{2}}$, the minimization problem (3.3) can be rewritten as follows

$$\min_{\tilde{\mathbf{A}}_s \in \mathbb{R}_n^{sy} | C(\tilde{\mathbf{A}}_s) \subseteq C(\mathbf{V}_s)} \left\{ tr\left(\tilde{\mathbf{A}}_s - \tilde{\mathbf{G}}\right)^2 + \frac{1}{2}\left[tr\left(\mathbf{A}_s\mathbf{V}_s\right) - tr\left(\mathbf{QV}\right)\right]^2 \right\} \quad (3.4)$$

Indeed, we have this equivalence between (3.3) and (3.4) as

$$\begin{aligned} tr\left(\tilde{\mathbf{A}}_s - \tilde{\mathbf{G}}\right)^2 &= tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{A}_s\mathbf{V}_s\right) + tr\left(\mathbf{G}\mathbf{V}_s\mathbf{G}\mathbf{V}_s\right) \\ &\quad - 2tr\left(\mathbf{A}_s\mathbf{V}_s\mathbf{G}\mathbf{V}_s\right) \end{aligned}$$

and $\mathbf{G}\mathbf{V}_s\mathbf{G}\mathbf{V}_s$ is independent of $\mathbf{A}_s$. The expression of matrix $\tilde{\mathbf{A}}_s$ minimizing (3.4) is given in the following lemma:

LEMMA 1 : *The matrix* $\tilde{\mathbf{A}}_s$ *which is the unique solution of* (3.4) *is given by*

$$\tilde{\mathbf{A}}_{s,opt} = \mathbf{P}_{\mathbf{V}_s}\left(\lambda\mathbf{P}_{\mathbf{V}_s} + \tilde{\mathbf{G}}\right)\mathbf{P}_{\mathbf{V}_s} \quad (3.5)$$

where

$$\lambda = \frac{tr\left(\mathbf{QV}\right) - tr\left(\tilde{\mathbf{G}}\mathbf{P}_{\mathbf{V}_s}\right)}{tr\left(\mathbf{P}_{\mathbf{V}_s}\right) + 2} \quad (3.6)$$

Proof: the matrix $\tilde{\mathbf{A}}_{s,opt}$ given by (3.5) satisfies that $C\left(\tilde{\mathbf{A}}_{s,opt}\right) = C\left(\mathbf{V}_s\right)$. Moreover, for any matrix $\tilde{\mathbf{A}}_s \in \mathbb{R}_n^{\geq}$ satisfying $C\left(\tilde{\mathbf{A}}_s\right) \subseteq C\left(\mathbf{V}_s\right)$, we have

$$\mathbf{P}_{\mathbf{V}_s}\mathbf{B}_s = \mathbf{B}_s \quad \text{where} \quad \mathbf{B}_s = \tilde{\mathbf{A}}_s - \tilde{\mathbf{A}}_{s,opt}$$

and

$$tr\left(\tilde{\mathbf{A}}_s - \tilde{\mathbf{G}}\right)^2 + \frac{1}{2}\left[tr\left(\tilde{\mathbf{A}}_s\right) - tr\left(\mathbf{QV}\right)\right]^2$$

is equal to

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$$tr\left(\tilde{\mathbf{A}}_{s,opt}-\tilde{\mathbf{G}}\right)^2+\frac{1}{2}\left[tr\left(\tilde{\mathbf{A}}_{s,opt}\right)-tr\left(\mathbf{QV}\right)\right]^2$$
$$+tr\left(\mathbf{B}_s^2\right)+\Delta+\frac{1}{2}\left[tr\left(\mathbf{B}_s\right)\right]^2+$$

with

$$\begin{aligned}\boldsymbol{\Delta} &= 2\lambda tr\left(\mathbf{B}_s\right)+tr\left(\mathbf{B}_s\right)\left[tr\left(\tilde{\mathbf{A}}_{s,opt}\right)-tr\left(\mathbf{QV}\right)\right]\\ &= 2\lambda tr\left(\mathbf{B}_s\right)+tr\left(\mathbf{B}_s\right)\left[\lambda tr\left(\mathbf{P}_{\mathbf{V}_s}\right)+tr\left(\tilde{\mathbf{G}}\mathbf{P}_{\mathbf{V}_s}\right)-tr\left(\mathbf{QV}\right)\right]\\ &= tr\left(\mathbf{B}_s\right)\left[\lambda\left(2+tr\left(\mathbf{P}_{\mathbf{V}_s}\right)\right)-\lambda\left(2+tr\left(\mathbf{P}_{\mathbf{V}_s}\right)\right)\right]=\mathbf{0}\end{aligned}$$

So, $tr\left(\tilde{\mathbf{A}}_s-\tilde{\mathbf{G}}\right)^2+\frac{1}{2}\left[tr\left(\tilde{\mathbf{A}}_s\right)-tr\left(\mathbf{QV}\right)\right]^2$ is greater than $tr\left(\tilde{\mathbf{A}}_{s,opt}-\tilde{\mathbf{G}}\right)^2+\frac{1}{2}\left[tr\left(\tilde{\mathbf{A}}_{s,opt}\right)-tr\left(\mathbf{QV}\right)\right]^2$ and we have equality if and only if $\mathbf{B}_s=\mathbf{0}$, this means that $\tilde{\mathbf{A}}_s=\tilde{\mathbf{A}}_{s,opt}$, thus completing the proof.

The result of Lemma 1 can be used to determine the expression of matrix $\mathbf{A}_{s,opt}$, minimizing (3.3).

THEOREM 1 : *The optimal quadratic predictor, according to the minimization of the mean squared error under the model* (2.1), *of* $\mathbf{e}'\mathbf{Qe}$ *where* $\mathbf{Q}\in\mathbb{R}_N^{sy}$ *is given by* $\hat{\mathbf{e}}_s'\mathbf{A}_{s,opt}\hat{\mathbf{e}}_s$ *with*

$$\mathbf{A}_{s,opt}=\lambda\mathbf{V}_s^{-1}+\mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{QV}_1\mathbf{V}_s^{-1} \qquad (3.7)$$

where

$$\lambda=\frac{tr\left(\mathbf{QV}\right)-tr\left(\mathbf{QV}_1\mathbf{V}_s^{-1}\mathbf{V}_1'\right)}{rk\left(\mathbf{V}_s\right)+2} \qquad (3.8)$$

Proof: since $\tilde{\mathbf{A}}_s=\mathbf{V}_s^{\frac{1}{2}}\mathbf{A}_s\mathbf{V}_s^{\frac{1}{2}}$, the matrix $\mathbf{A}_{s,opt}$ minimizing (3.3) is given by

$$\begin{aligned}\mathbf{A}_{s,opt} &= \mathbf{V}_s^{-\frac{1}{2}}\tilde{\mathbf{A}}_{s,opt}\mathbf{V}_s^{-\frac{1}{2}}\\ &= \lambda\mathbf{V}_s^{-\frac{1}{2}}\mathbf{P}_{\mathbf{V}_s}\mathbf{V}_s^{-\frac{1}{2}}+\mathbf{V}_s^{-\frac{1}{2}}\mathbf{P}_{\mathbf{V}_s}\mathbf{V}_s^{\frac{1}{2}}\mathbf{G}\mathbf{V}_s^{\frac{1}{2}}\mathbf{P}_{\mathbf{V}_s}\mathbf{V}_s^{-\frac{1}{2}}\\ &= \lambda\mathbf{V}_s^{-1}+\mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{QV}_1\mathbf{V}_s^{-1}\end{aligned}$$

In addition, the expression (3.8) of λ is deduced from (3.6). This completes the proof of Theorem 1.

Note that the quadratic predictor proposed in Theorem 1 has an expression much simpler than that of Liu's and Rong's predictor by being approximately optimal in the class of quadratic predictors. Moreover, one can easily demonstrate that

$$\begin{aligned}MSE_\xi\left(\hat{\mathbf{e}}_s'\mathbf{A}_{s,opt}\hat{\mathbf{e}}_s\right) &\approx MSE_\xi\left(\mathbf{e}_s'\mathbf{A}_{s,opt}\mathbf{e}_s\right)\\ &\approx 2\sigma^4\left[\lambda^2\left(rk\left(\mathbf{V}_s\right)+2\right)+\alpha\right]\end{aligned}$$

where $\alpha=tr\left(\mathbf{QVQV}\right)-tr\left(\mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{QV}_1\mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{QV}_1\right)$.

Hence, the ability to approximate the MSE of $\hat{\mathbf{e}}_s'\mathbf{A}_{s,opt}\hat{\mathbf{e}}_s$ enables to measure its accuracy unlike that proposed by Liu and Rong whose complexity of its expression makes it impossible to calculate accuracy.

4. OPTIMAL QUADRATIC PREDICTOR OF THE RESIDUAL SUMS OF SQUARES

In what follows, we will use the optimal quadratic predictor given by (3.7) to provide a new predictor of the residual sum of squares (RSS) of a regression model given by

$$\left\|\mathbf{e}\right\|^2=\left(\mathbf{Y}-\mathbf{Xb}\right)'\left(\mathbf{Y}-\mathbf{Xb}\right)$$

which corresponds to the quadratic form $\mathbf{e}'\mathbf{Qe}$ with $\mathbf{Q}=\mathbf{I}_N$ where $\mathbf{I}_N$ is the identity matrix. Therefore, Theorem 1 allows us to deduce the following result:

COROLLARY 1 : Under the model (2.1), the optimal quadratic predictor of the residual sum of squares $\left\|\mathbf{e}\right\|^2$ is given by

$$\hat{\mathbf{e}}_s'\mathbf{A}_{s,opt}\hat{\mathbf{e}}_s=\hat{\mathbf{e}}_s'\left[\lambda_0\mathbf{V}_s^{-1}+\mathbf{I}_n+\mathbf{V}_s^{-1}\mathbf{V}_{sr}\mathbf{V}_{rs}\mathbf{V}_s^{-1}\right]\hat{\mathbf{e}}_s$$

where

$$\lambda_0=\frac{tr\left(\mathbf{V}\right)-tr\left(\mathbf{V}_s\right)-tr\left(\mathbf{V}_{sr}\mathbf{V}_s^{-1}\mathbf{V}_{rs}\right)}{rk\left(\mathbf{V}_s\right)+2}$$

Proof: for $\mathbf{Q}=\mathbf{I}_N$, the expression of $\mathbf{A}_{s,opt}$ is reduced to

$$\begin{aligned}\mathbf{A}_{s,opt} &= \lambda_0\mathbf{V}_s^{-1}+\mathbf{V}_s^{-1}\mathbf{V}_1'\mathbf{V}_1\mathbf{V}_s^{-1}\\ &= \lambda_0\mathbf{V}_s^{-1}+\mathbf{V}_s^{-1}\left(\mathbf{V}_s^2+\mathbf{V}_{sr}\mathbf{V}_{rs}\right)\mathbf{V}_s^{-1}\\ &= \lambda_0\mathbf{V}_s^{-1}+\mathbf{I}_n+\mathbf{V}_s^{-1}\mathbf{V}_{sr}\mathbf{V}_{rs}\mathbf{V}_s^{-1}\end{aligned}$$

with

$$\begin{aligned}\lambda_0 &= \frac{tr\left(\mathbf{V}\right)-tr\left(\mathbf{V}_1\mathbf{V}_s^{-1}\mathbf{V}_1'\right)}{rk\left(\mathbf{V}_s\right)+2}\\ &= \frac{tr\left(\mathbf{V}\right)-tr\left(\mathbf{V}_s\right)-tr\left(\mathbf{V}_{sr}\mathbf{V}_s^{-1}\mathbf{V}_{rs}\right)}{rk\left(\mathbf{V}_s\right)+2}\end{aligned}$$

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This completes the proof.

We note that when $\mathbf{V} = diag\left(v_1^2,...,v_N^2\right)$, we have

$$\hat{\mathbf{e}}_s' \mathbf{A}_{s,opt} \hat{\mathbf{e}}_s = \hat{\mathbf{e}}_s' \left[\lambda_0 \mathbf{V}_s^{-1} + \mathbf{I}_n \right] \hat{\mathbf{e}}_s$$

where

$$\lambda_0 = \frac{\sum_{k \in U \setminus s} v_k^2}{n+2}$$

which gets reduced to

$$\hat{\mathbf{e}}_s' \mathbf{A}_{s,opt} \hat{\mathbf{e}}_s = \frac{N+2}{n+2} \left\| \hat{\mathbf{e}}_s \right\|^2$$

when $\mathbf{V} = \mathbf{I}_N$.

5. CONCLUSION

In this paper, we proposed a new quadratic predictor of a residual quadratic form of a linear Gaussian model. This predictor was found out to be optimal according to the minimization of mean squared error under the adopted linear model. The main interest of this result is used to provide an optimal predictor for any quadratic function involving residues of a linear model. This is the case, for example, of the model's residual sum of squares, which is among the criteria used to measure the quality of the model. As such, a new optimal quadratic predictor of the residual sum of squares of a linear Gaussian model is proposed in this pape

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