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THE CONCEPT OF HEIGHT IN A MATRIX AND RELATED PROPERTIES

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ABSTRACT

Matrix has no value whereas a determinant has a value. An attempt is made to associate a value to a matrix, whose elements are integers. For that, a concept of "Height" is introduced in the matrices over Z. As soon as an integral value is associated with every matrix, an attempt is made to introduce odd, even and prime matrices etc. Then related number theoretical properties are proved.

Keywords: *Integral matrices, Odd and Even Matrices - Prime and Composite Matrices, Basic Prime Matrix, Relatively Prime Matrices, GCD of two Matrices, Congruences of two Matrices*

1. INTRODUCTION

Much of the number theory is concerned with the properties of Primes [1,7]. Every integer is either a prime or it can be uniquely expressed as a product of powers of Prime. In the set of all integers the following are some properties of Odd, Even integers and a first attempt is made to introduce the concepts in matrix theory.

(i) Addition of two even integers is an even integer.

(ii) Addition of two odd integers is also an even integer.

(iii) Addition of an odd integer and even integer is odd.

(iv) The set of all even integers is a normal subgroup of (Z, +) [10].

2. HEIGHT OF A MATRIX

Here Z, the set of all integers is an abelian group under usual addition. M_{mxn}, the set of all mxn matrices whose entries are integers, is an abelian group under matrix addition[2].

$f: M_{mxn} \rightarrow Z$ is defined as

$$f(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij},$$

$A=(a_{ij})_{mxn} \in M_{mxn}$.

f is a homomorphism[5] from M_{mxn} to Z, because if $A,B \in M_{mxn}$, where $A=(a_{ij})_{mxn}$ and $B=(b_{ij})_{mxn}$,

$A+B=(c_{ij})_{mxn}$ where $c_{ij}=a_{ij}+b_{ij}$

$$\text{Now } f(A+B) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} (a_{ij}+b_{ij})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}$$

$f(A+B) = f(A) + f(B)$

Therefore f is a homomorphism from M_{mxn} to Z.

Definition:-1

Let $A,B \in M_{mxn}$. A and B are said to be related if f(A) = f(B). That is, $\sum_i\sum_j a_{ij} = \sum_i\sum_j b_{ij}$ where $A=(a_{ij})$ and $B=(b_{ij})$. In symbols, it is written as A~B.

Example:-1

Let $A=\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$ and

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B = $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

A, B$\in$ M_{2x2} .f(A) = f(B) = 6

∴ A and B are related.

Theorem: 1

The above relation is an equivalence relation.

Proof:

(i) Clearly A ~A

∴ This relation is reflexive.

(ii) Suppose A~B, then f(A)=f(B)

∴ Clearly B~A

∴ This relation is symmetric.

(iii) Suppose A~B and B~C

This implies that f(A) = f(B)

and f(B) = f(C).

∴ We get f(A)=f(C)

∴ A~C

∴ This relation is transitive.

From (i), (ii) and (iii), the above relation is an equivalence relation.

Definition:-2

Let A$\in$ M_{mxn}. The height of the matrix, denoted by h_A is defined as

h_A= $\sum_i \sum_j a_{ij}$

Height of a matrix may be negative, zero or positive.

∴

Note: 1

Let A=$(a_{ij})_{mxn}$ and B =$(b_{ij})_{pxq}$ be any two matrices.

If $\sum_i \sum_j a_{ij} = \sum_i \sum_j b_{ij}$ then also we say A and B are related that is if h_A=h_B, we say A~B. In general case also it can be proved that '~' is an equivalence relation.

Example:-2

Let A = $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ $\in$ M_{2x2} .

Then h_A = 6

Definition:-3

Let A=(a_{ij}) $\in$ M_{mxn}. If h_A = $\Sigma\Sigma a_{ij}$ is an even integer, A is said to be an even matrix and if h_A is an odd integer , A is said to be an odd matrix.

Example:-3

Let A = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\in$ M_{2x2} . Then h_A = 4

∴ A is an even matrix .

Let B= $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\in$ M_{2x2}. Then h_B = 3

∴ B is an odd matrix .

Theorem:-2

The set of all even matrices E_{mxn} in M_{mxn} is a normal subgroup of M_{mxn}.

Proof:

Let E_{mxn} denote the set of all even matrices.

Let A = (a_{ij}) and B = (b_{ij}) $\in$ E_{mxn}

f(A) = $\Sigma\Sigma a_{ij}$ and

f(B) = $\Sigma\Sigma b_{ij}$

are even integers.

Now f(A-B)=f(A)-f(B)

since f is homomorphism.

$$f(A-B) = f(A) - f(B) = \Sigma\Sigma a_{ij} - \Sigma\Sigma b_{ij}$$

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We know that subtraction of two even integers is even.

$\therefore \Sigma\Sigma a_{ij} - \Sigma\Sigma b_{ij}$ is even.

$\therefore$ f(A-B) is an even integer.

$\therefore$ A-B is an even matrix.

$\therefore$ A-B $\in E_{mxn}$

$\therefore E_{mxn}$ is a subgroup of $M_{mxn.}$

Since $M_{mxn.}$ is an abelian , E_{mxn} is a normal subgroup of $M_{mxn.}$

Theorem:3

Matrix addition of two even matrices is an even matrix.

Proof:

Let $A=(a_{ij})_{mxn}$ and $B=(b_{ij})_{mxn}$ be two even matrices.

f(A) and f(B) are even numbers.

f(A)+f(B) is an even numbers

f(A)+f(B)=f(A+B),since f is a homomorphism

f(A+B) is an even number.

A+B is an even matrix.

Addition of two even matrices is an even matrix.

Theorem: 4

1.

Matrix addition of two odd matrices is an even matrix.

Proof:

Let $A=(a_{ij})_{mxn}$ and $B=(b_{ij})_{mxn}$ be two odd matrices.

f(A) and f(B) are odd numbers.

f(A)+f(B) is an odd number.

f(A)+f(B)=f(A+B),

since f is a homomorphism

f(A+B) is an odd number.

A+B is an odd matrix.

Addition of two odd matrices is an even matrix.

Theorem:5

Matrix addition of an odd matrix and an even matrix is an odd matrix.

Proof:

Let $A=(a_{ij})_{mxn}$ and $B=(b_{ij})_{mxn}$ be odd and even matrices respectively.

f(A) and f(B) are odd and even integers respectively

f(A)+f(B) is an odd integer.

f(A)+f(B)=f(A+B),since f is a homomorphism

f(A+B) is an odd integer.

A+B is an odd matrix.

Addition of an odd matrix and an even matrix is an odd matrix.

Note: 2

Theorems 2 to 5 can also be proved using the definition of height of matrix

Note: 3

The transpose of an even matrix is even.

Note: 4

The transpose of an odd matrix is odd.

3. PRIME AND COMPOSITE MATRICES

Definition:4

A matrix A$\in$ M_{mxn} is said to be prime,if h_A is a prime integer.

Definition:5

A matrix A$\in$ M_{mxn} is said to be composite,if h_A is a composite integer.

Example:4

Let A= $\begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix} \in M_{2x2}$

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$\therefore$ h_A=13.

$\therefore$ A is a prime matrix.

Let B = $\begin{pmatrix} 1 & 3 \\ 6 & 8 \end{pmatrix} \in M_{2x2}$

$\therefore$ h_B=18.

$\therefore$ B is a composite matrix.

Definition:6

If A= (p), where p is a prime integer then A is called Basic prime matrix.

Example: 5

If A=(3), then A is a Basic prime matrix.

Theorem: 6

Every integral matrix is related to either a Basic prime matrix or it is related to a product of powers of Basic prime matrices [3,9].

Proof:

Case: (i)

Let A be a prime matrix and let hA=p, where p is a prime integer.

Then A~ (p) where (p) is a Basic prime matrix.

Case: (ii)

Suppose A is a composite matrix.

If h_A=c, then 'c' is a composite number.

We know that every composite number can be uniquely expressed as a product of powers of prime numbers.

Let c= $p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots\ldots . p_r^{\alpha_r}$,

where each p_i is prime, for i = 1 to r.

Let A_i = (p_i), i= 1 to r.

A_i is a Basic prime matrix, for i=1 to r.

$$(A_i)^{\alpha_i} = (p_i)^{\alpha_i} = (p_i^{\alpha_i})$$

$$\therefore (A_1)^{\alpha_1}\,(A_2)^{\alpha_2} \ldots\ldots (A_r)^{\alpha_r} = (p_1)^{\alpha_1}\,(p_2)^{\alpha_2} \ldots\ldots (p_r)^{\alpha_r}$$

$$= (p_1^{\alpha_1})\,(p_2^{\alpha_2}) \ldots\ldots . (p_r^{\alpha_r}) = (p_{\alpha 1}^{\alpha_1}\, p_{\alpha 2}^{\alpha_2} \ldots\ldots . p_{\alpha r}^{\alpha_r}) = (c)$$

$$\therefore\ (A_1)^{\alpha_1}\,(A_2)^{\alpha_2} \ldots\ldots (A_r)^{\alpha_r} = (c)$$

Since h_A = c , A~ (c)

$\therefore$ A ~ $(A_1)^{\alpha_1}\,(A_2)^{\alpha_2} \ldots\ldots (A_r)^{\alpha_r}$,

$\therefore$ where each (A_i) is a Basic prime matrix.

$\therefore$ The integral matrix is related to a product of powers of Basic prime matrices.

Definition: 7

Let A, B be any two integral matrices. Then greatest common divisor of A and B is defined to be G.C.D of h_A and h_B[4,6].

Definition: 8

The two integral matrices [8] are said to be relatively prime, if G.C.D of the two matrices is 1.

Example: 6

Let A= $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ and B= $\begin{pmatrix} 1 & 2 & 8 \\ 6 & 7 & 4 \end{pmatrix}$

h_A = 10 and h_B=28

G.C.D of h_A and h_B is 2.

$\therefore$ We say G.C.D of the matrices A and B is 2.

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Example:

If A= $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and B= $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 7 \end{pmatrix}$,

then h_A=5 and h_B=22.

G.C.D of h_A and h_B is 1.

$\therefore$ The matrices A and B are relatively prime.

Definition: 9

Let A,B,D be any three integral matrices of any order.A is said to be congruent B mod D, if $h_A \equiv h_B(\text{mod } h_D)$, where $h_D \neq 0$.In symbols, it is denoted by

$A \equiv B(\text{mod D})$.

Theorem: 7

Congruence of integral matrices is an equivalence relation.

Proof:

(i) Let $A \in M_{mxn}$, let $D \in M_{mxn}$, where $h_D \neq 0$.

Since $h_A \equiv h_B(\text{mod } h_D)$, $A \equiv A(\text{mod D})$.

$\therefore$ '$\equiv$' is a reflexive relation.

(ii) Let $A \equiv B$ (mod D)

$\therefore h_A \equiv h_B(\text{mod } h_D)$,

$\therefore h_B \equiv h_A(\text{mod } h_D)$,

$\therefore B \equiv A$ (mod D)

$\therefore$ '$\equiv$' is a Symmetric relation.

Let $A \equiv B$ (mod D) and $B \equiv E$ (mod D)

$\therefore$ $h_A \equiv h_B(\text{mod } h_D)$ and

$h_B \equiv h_E(\text{mod } h_D)$,

$\therefore$ $h_A \equiv h_E(\text{mod } h_D)$

$\therefore$ $A \equiv E$ (mod D)

$\therefore$ '$\equiv$' is a transitive relation.

$\therefore$ From (i),(ii) and (iii) , congruences of matrices is an equivalence relation.

Remark:

In the above theorem, all matrices are considered to be m× n matrices. The theorem is true even if A, B, E are of any order, with the condition that $h_D \neq 0$.

Theorem: 8

Let $A, B \in M_{mxn}$,. Let D be any integral matrix with $h_D \neq 0$.

If $A \equiv A'$ (mod D) and $B \equiv B'$ (mod D), then $A \pm B \equiv A' \pm B'$ (mod D)

Proof:

Let $A, B \in M_{mxn}$, and D is an integral matrix with $h_D \neq 0$.

Given that $A \equiv A'$ (mod D) and

$B \equiv B'$ (mod D)

Therefore, $h_A \equiv h_A'(\text{mod } h_D)$ and

$h_B \equiv h_B'(\text{mod } h_D)$

$\therefore$ h_D divides $h_A - h_A'$ and $h_B - h_B'$.

$$\frac{(h_A \pm h_B) - (h_{A'} \pm h_{B'})}{h_D} = \frac{(h_A - h_{A'}) \pm (h_B - h_{B'})}{h_D}$$

$$= \frac{h_A - h_{A'}}{h_D} \pm \frac{h_B - h_{B'}}{h_D}$$

$\therefore$ h_D divides $(h_A \pm h_B) - (h_{A'} \pm h_{B'})$

$\therefore$ $h_A \pm h_B \equiv h_A' \pm h_B'$ (mod h_D)

$\therefore$ $A \pm B \equiv A' \pm B$ (mod h_D).

4. CONCLUSION:

Primes play a fundamental role in the theory of numbers. Some of the most striking results in the number theory, the quadratic reciprocity law, Wilson's theorem and Fermat's theorem, reveal interesting properties of prime. Since prime matrix is defined, a great deal of useful information may be derived. This new idea can be studied with an interest by itself.

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