Journal of Theoretical and Applied Information Technology
15th October 2011. Vol. 32 No.1

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

GRAY CODE AND HAMMING DISTANCE FOR GRAPH OF $S_n(123,132)$

[1]A. JUARNA, [2]A.B. MUTIARA

Faculty of Computer Science and Information Technology, Gunadarma University
Jl. Margonda Raya No.100, Depok 16424, Indonesia
E-mail : [1,2]{ajuarna,amutiara}@staff.gunadarma.ac.id

ABSTRACT

As we noted that an isomorphism between two combinatorial classes is a closeness preserving bijection between those classes, that is, two objects in a class are closed if and only if their images by this bijection are also closed. Often, as in this paper, closeness is expressed in terms of Hamming distance. Isomorphism allows us to find out some properties of a combinatorial class X (or for the graph induced by the class X) if those properties are found in the pre image of the combinatorial class X; some mentioned properties are hamiltonian path, graph diameter, exhaustive and random generation, and ranking and unranking algorithms. Simion and Schmidt showed in 1985 that the cardinality of the set $S_n(123,132)$ length n permutations avoiding the patterns 123 and 132, is 2^{n-1}, but in the other side 2^{n-1} is the cardinality of the set $B_{n-1} = \{0,1\}^{n-1}$ of length (n-1) binary strings. Theoretically, it must exist a bijection between $S_n(123,132)$ and B_{n-1}. In this paper we give a constructive bijection between B_{n-1} and $S_n(123,132)$; we show that it is actually an isomorphism and illustrate this by constructing a Gray code for $S_n(123,132)$ from a known similar result for B_{n-1}.

Keywords: *Pattern-Avoiding Permutations; Binary Strings, Constructive Bijection; Hamming Distance; Combinatorial Isomorphism*.

1. INTRODUCTION

In this paper an *element* denotes a member of a list or set, and a *term* denotes a term in a string or sequence. Let $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ be two strings of same length. We say x and y are *piecewise comparison* if $x_i \le x_j$ whenever $y_i \le y_j$. Let $[n]$ be the set of all non-negative integers less than or equal to n. We denote by S_n the set of all permutations of $[n]$ and its cardinality is obviously $n!$. Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations, $k \le n$. We say π *contains* τ if there exists k integers $1 \le i_1 < i_2 \ldots i_k \le n$ such that *subsequence* $\pi_{i_1} \ldots \pi_{i_k}$ is piecewise comparison to τ; in such context τ is usually called a *pattern*. We say that π *avoids* τ, or π is τ-*avoiding*, if such subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$ and $s_n(\tau)$ is its cardinality. For an arbitrary finite collection of patterns T, we say π avoids T if π avoids any $\tau \in S_k$; the corresponding subset of S_n is denoted by $S_n(T)$ while $s_n(T)$ is its cardinality. For examples, let $T = \{123,231,1324\}$ is a set of patterns. Clearly permutation 1234567 $\notin S_7(T)$ since it contains 123, permutation 652341 $\notin S_6(T)$ since it contain 234 which is piecewise comparison to 123 (and also 231 and 341 which are piecewise comparison to 231), while permutation 4321 $\in S_4(T)$ since it not contain any subsequence which is piecewise comparison to any pattern of T. Also $s_3(123) = 5$ because $S_3(123) = \{132, 213, 231, 312, 321\}$.

Fundamental questions about pattern-avoiding permutations problems are:

1. to determine $s_n(T)$ viewed as a function of n for given T,
2. to find an explicit bijection (a one-to-one and onto correspondence) between $S_n(T)$ and $S_n(T')$ if $s_n(T) = s_n(T')$, and
3. to find relations between $S_n(T)$ and other combinatorial structures.

By determining $s_n(T)$ we mean finding explicit formula, or ordinary or exponential generating functions. From these researches, a number of enumerative results have been proved, new bijections found, and connections to other fields established.

Problems of pattern avoiding permutations appeared for the first time when Knuth [5], in his text book, posed a sorting problem using single stack. This problem actually is the 312-patterns avoiding permutations. In the other section of his

Journal of Theoretical and Applied Information Technology
15th October 2011. Vol. 32 No.1

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

book, he showed that the cardinality of all three-length-patterns-avoiding permutations is the Catalan numbers. Investigations on problems of pattern avoiding permutations then become wider to some set of patterns of length three, four, five, and so on, some combinations of these patterns, generalized patterns, and permutations avoiding some patterns while in the same time containing exactly a numbers of other patterns.

Pattern avoiding permutations have been proved as useful language in a variety of seemingly unrelated problems, from theory of Kazhdan-Lusztig polynomials, to singularities of Schubert varieties, to Chebyshev polynomials, to rook polynomials for a rectangular board, to various sorting algorithms, sorting stacks and sortable permutations [4], statistic permutation [6], also in practical application such as on cryptanalysis (see [7] for example).

The first systematic study of patterns avoiding permutations undertaken in 1985 when Simion and Schmidt [9] solved the problem with patterns come from every subset of S_3. The idea of this paper is the following propositions,

Proposition 1 (see [9]) The number of (123,132)-avoiding permutations in S_n, $n \geq 1$ is $s_n(123,132) = 2^{n-1}$.

Proof. Let $\pi \in S_n(123,132)$. If $\pi_n = n$ then $\pi = (n-1)(n-2)...1n$. If $\pi_k = n$ then $\pi_1 > \pi_2 > ... > \pi_{k-1}$ in order to avoid 123; on the other hand, in order to avoid 132, $\pi_i > (n-k)$ if $i < k$. Hence, $\pi_i = n-i$ for $1 \leq i \leq k-1$, while $\pi_{k+1}\pi_{k+1}...\pi_n$, must be a (123,132)-avoiding permutation in S_{n-k}. Thus, $s_1(123,132) = 1$, and for $n > 1$, $s_n(123,132) = 1 + \sum_{k=1}^{n-1} s_k(123,132)$. The solution for this recurrence relation is: $s_n(123,132) = 2^{n-1}$. □

The cardinality of set $S_n(123,132)$, as stated by Simion-Schmidt, is the number of elements of B_{n-1}, the set of all binary strings having length $(n-1)$ without any restriction. This paper gives (in the next section) constructive bijection between B_{n-1} and $S_n(123,132)$. Then, in section 3 we show that this bijection is actually isomorphism. Remark that is not always the case: a bijection between combinatorial classes may magnify the distance between two consecutive objects. This result allows us to construct in section 4 a Gray code for $S_n(123,132)$. In the final part some concluding remarks are given.

2. CONSTRUCTIVE BIJECTION BETWEEN B_{n-1} AND $S_n(123,132)$

Simion and Schmidt proved that cardinality of set $S_n(123,132)$ is 2^{n-1}, but the 2^{n-1} is also cardinality of B_{n-1}, set of all binary strings of length n-1. Theoretically it must be exists a bijection between $S_n(123,132)$ and B_{n-1}; here we construct such a bijection.

The general pattern of $\pi \in S_n(123,132)$, as is mentioned in Proposition 1, can be described as three parts as,

$$\pi = \underbrace{\pi_1\pi_2\cdots\pi_{k-1}}_{(1)}\underbrace{\pi_k}_{(2)}\underbrace{\pi_{k+1}\cdots\pi_{n-1}\pi_n}_{(3)} \qquad (1)$$

where

1. $\pi_1 = n$, $\pi_2 = n-1$, ..., $\pi_{k-1} = \pi_{k-2} = 1$, (eventually empty)
2. $\pi_k = n$,
3. $\pi_{k+1}...\pi_n \in S_{n-k}(123,132)$ (also, eventually empty)

For example, Figure 1 is the matrix representation of permutation 6573421 ∈ $S_7(123,132)$.

For example, Figure 1 is the matrix representation of permutation 6573421 ∈ $S_7(123,132)$.

If we trace the terms of π in (1) from the left to the right, at first we will find π_1 as the second largest term in π (after n). If we remove π_1, then π_2 again will be the second largest, and so until π_{k-1}. Next, $\pi_k = n$ is the largest term of π. This tracing and interpretation is similar for the third part of π until one place before the largest term.

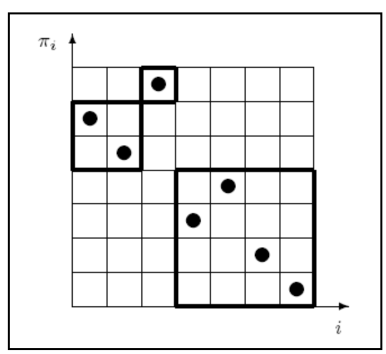

Figure 1. π = 6573421 ∈ $S_7(123,132)$ consist of three part as is mentioned by (1). Notice that the third part is an element of $S_4(123,132)$, the first stage in the verification of π = 6573421 as element of $S_7(123,132)$ recursively using (1).

Now, we associate $\pi \in S_n(123,132)$ to s, a binary string of length $(n-1)$, and assign the largest of π whenever we find 1 in s and assign the second

Journal of Theoretical and Applied Information Technology
15th October 2011. Vol. 32 No.1

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

largest of π whenever we find 0 in s. It is easy to see that this construction is a bijection, so we get the following proposition:

Proposition 2 For each $n \geq 1$, there exists a constructive bijection between B_{n-1} and $S_n(123,132)$.
Proof. Let $s = s_1s_2... s_n \in B_{n-1}$. We construct its corresponding $\pi \in S_n(123,132)$ by determining π_i, $1 \leq i < n$, as follows: *if* $X_i = \{1, 2, ..., n\} - \{\pi_1, \pi_2, ..., \pi_{i-1}\}$, then set:

$$\pi_i = \begin{cases} \textit{largest element in } X_i \textit{ if } s_i = 1 \\ \textit{second largest element in } X_i \textit{ if } s_i = 0 \end{cases} \quad (2)$$

and π_n is the single element in X_n. For examples, $0000 \in B_4$ produces $43215 \in S_5(123,132)$, $10110 \in B_5$ will produce $645312 \in S_6(123,132)$, and $010110 \in B_6$ will produce $6745312 \in S_7(123,132)$. □

Table I shows the set B_4 together with its image, the set $S_5(123,132)$.

TABLE I. THE LIST B_4 AND ITS IMAGE, $S_5(123,132)$, BY BIJECTION (2).

rank	B_4	S_5(123,132)
1	0000	43215
2	0001	43251
3	0011	43521
4	0010	43512
5	0110	45312
6	0111	45321
7	0101	45231
8	0100	45213
9	1100	54213
10	1101	54231
11	1111	54321
12	1110	54312
13	1010	53412
14	1011	53421
15	1001	53241
16	1000	53214

3. ISOMORPHISM BETWEEN B_{n-1} AND $S_n(123,132)$

A graph associated with a combinatorial class is a graph where objects of the class act as vertices of the related graph. Two vertices of this graph are connected (or adjacent) if the associated two combinatorial objects *are closed*, that is fulfill a predetermined condition(s), usually in the term of Hamming distances. Two graphs G and H are said to be isomorphic if there is a bijection φ such that (u,v) is an edge in G if and only if $(\varphi(u), \varphi(v))$ is an edge in H.

Before exploring the graph associated with the combinatorial classes B_{n-1} and $S_n(123,132)$ and showing the isomorphism between the two graph, we define the closeness properties of two elements of B_{n-1} and $S_n(123,132)$ and then give a theorem concerning the isomorphism.

Definition 1

1. Two binary strings B_{n-1} are closed if they differ in a single position.
2. Two permutations in $S_n(123,132)$ are closed if they differ by a transposition of two terms.

Theorem 1 The bijection (2) is a combinatorial isomorphism, that is, two binary strings in B_{n-1} are closed if and only if their images in $S_n(123,132)$ under this bijection are closed.
Proof. Let x and x' be two elements of B_{n-1} which differ at position i, and also, without loss of generality, let $x_i = 1$, and:

$$x = x_1...x_{i-1}10...01x_{j+1}...x_{n-1}$$
$$x = x_1...x_{i-1}00...01x_{j+1}...x_{n-1}$$

With the contiguous sequence of *0*s: $x_{i+1} = x_{i+1} = ... = x_{j-1} = 0$ eventually empty.

- If x_j until x_{n-1} is 0 then $\pi_n = (m\text{-}1)$ for π and m for π'.
- Let m be the largest element in X_i as is mentioned in (2). Let π, $\pi' \in S_n(123,132)$ the images of x and x' by the bijection (2), clearly $\pi_i = m$, $\pi_{i+1} = (m\text{-}2)$, and so on, while $\pi_1' = (m\text{-}1)$, $\pi_{1+1}' = (m\text{-}2)$, and so on. Then the shapes of π and π are:

  $\pi = \pi_1... \pi_{i-1}\ m\ (m\text{-}2)\ ...\ (m\text{-}j\text{+}i\text{+}1)\ (m\text{-}1)\ \pi_{j+1}... \pi_{n-1}\ \pi_n$

  $\pi' = \pi_1... \pi_{i-1}\ (m\text{-}1)\ (m\text{-}2)\ ...\ (m\text{-}j\text{+}i\text{+}1)\ m\ \pi_{j+1}... \pi_{n-1}\ \pi_n$

The case for $x_i = 0$ is similar. □

Since (3) is cyclic, we can draw an $(n\text{-}1)$-cube graph of B_{n-1} and also we can find at least a Hamiltonian cycle in the graph. And since (2) is an isomorphism, we also can draw a congruent graph of $S_n(123,132)$ and also can find the Hamiltonian cycle. Figure 2 shows the two graphs for $n = 4$ together with one of their Hamiltonian path.

4. GRAY CODE FOR $S_n(123,132)$ AND THE HAMMING DISTANCES

A binary string is a string over a binary alphabet, $\{0,1\}$. The set of binary strings of length p codes

Journal of Theoretical and Applied Information Technology
15th October 2011. Vol. 32 No.1

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

the set of non-negative integers over closed interval [0, 2^p-1]. For example, set of all 3 length binary strings is {000, 001, 010, 011, 100, 101, 110, 111} and represents set of all non-negative integers less than or equal to 7, the all non-negative integers over the closed interval [0, 2^3-1].

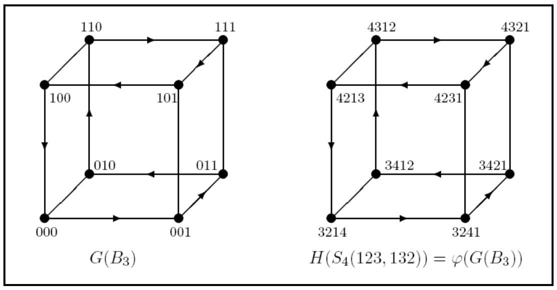

Figure 2. Isomorphism between graph B_3 and graph $S_4(123,132)$. This figure also shows a Hamiltonian cycle in each graph, as is indicated by the arrows. Notice that the Hamiltonian path in $S_4(123,132)$ is the isomorphic image of the path in B_3

A Gray code for binary strings is a listing of all p length p binary strings so that successive strings (including the first and last) differ in exactly one bit position [8]. The simple and best-known example of Gray code for binary strings is binary reflected Gray code which can be described the following recursive definition:

$$B_p = \begin{cases} \varepsilon & p = 0 \\ 0 \cdot B_{p-1} \circ 1 \cdot \overline{B}_{p-1} & p \geq 1 \end{cases} \qquad (3)$$

where ε is empty string, $\alpha \cdot \overline{B}$ is the list obtained by concatenation α to each string of $\overline{B}$, $\circ$ is concatenation operator of two lists, and $\overline{B}$ is the list obtained by reversing B. $Fist(B_p) = 0^p$ since it is constructed by recursively concatenation 0 to ε and so on in p times, while $Last(B_p) = 10^{p-1}$ since it just concatenation 1 to $First(B_{p-1})$ and since $Last(\overline{B}_p) = First(B_p)$. For examples, B_1 = {0, 1}, B_2= {00, 01, 11, 10}, and B_3 = {000, 001, 011, 010, 110, 111, 101, 100}.

Since the first and last elements of B_p also differ in one bit position, the code is in fact a cycle. Generating of (3) can be implemented efficiently as a loop free algorithm [1]. Note that, since a binary Gray code is a cycle, it can be viewed as a Hamilton cycle in the n-cube.

Existence of at least a Hamiltonian cycle in the graph of $S_n(123,132)$, as is showed in the last part of the previous section, is an indication that there is at least a Gray code for $S_n(123,132)$. Since there is a bijection between B_{n-1} and $S_n(123,132)$, here we construct a Gray code for $S_n(123,132)$. By considering bijection (2), Gray code B_p (3) is transformed into following Gray code for $S_n(123,132)$:

$$S_n(123,132) = \begin{cases} \{1\} & n = 1 \\ (n-1) \cdot S^*_{n-1}(123,132) \circ \\ \quad n \cdot \overline{S}_{n-1}(123,132) & n \geq 2 \end{cases} \qquad (4)$$

where $S^*_{n-1}(123,132)$ is $S_{n-1}(123,132)$ after replacing (n-1) with n. This replacement is taken place since 0, which is the prefix to the first part of (3), is associated to (n-1), *the second largest element* as is mentioned in (2). Hence (n-1) must be prefix to the second part of (4). For examples, $S_2(123,132)$ = {12, 21}, $S_3(123,132) = 2\cdot\{13, 31\} \circ 3\cdot\{12, 21\}$ = {213, 231, 321, 312}. Table 1. shows the list of B_4 together with its image, the list of $S_5(123,132)$.

The recursively properties of (4) imply $First(S_n(123,132)) = (n\text{-}1)(n\text{-}2)...1n$. In the other hand, since $Last(\overline{S}_{n-1}(123,132)) = First(S_{n-1}(123,132))$, so $Last(\overline{S}_n(123,132))$ must be $n\cdot(n\text{-}1)\cdot(n\text{-}3)...1(n\text{-}1)$.

Proposition 3. The Hamming distance between two consecutive elements of $S_n(123,132)$ is 2 and, except between the first and the last, the two different terms are adjacent.

Proof. For n = 2 the Hamming distance is between 12 and 21 which is 2. For n > 2, Hamming distance between two consecutive elements of $S_n(123,132)$, except between the first and last elements, is determined recursively by the distance in the smaller list, and so on, and finally by the distance in $S_2(123,132)$ which is 2. Concatenating (n-1) and n, respectively to the two parts of (4), of course will not change the Hamming distance values in each part. Also, replacing (n-1) with n in $S^*_{n-1}(123,132)$ will not change the Hamming distance between each its two consecutive elements. So we only must to check the Hamming distance between $Last((n-1)\cdot S^*_{n-1}(123,132))$ and $First(n \cdot \overline{S}_{n-1}(123,132))$, as follow:

$$\begin{aligned} &Last((n-1)\cdot S^*_{n-1}(123,132)) \\ &\quad = (n-1)\cdot Last(S^*_{n-1}(123,132)) \\ &\quad = (n-1)\cdot n \cdot Last(S_{n-2}(123,132)) \end{aligned}$$

$$First(n \cdot \overline{S}_{n-1}(123,132))$$

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$$= n \cdot First(\overline{S}_{n-1}(123,132))$$
$$= n \cdot (n-1) \cdot Last(S_{n-2}(123,132))$$

Clearly the Hamming distance between *Last*((*n*-1)· $S^{*}_{n-1}(123,132)$) and $First(n \cdot \overline{S}_{n-1}(123,132))$ is 2 and adjacent. □

The Hamming distance between the first and the last element of $S_2(123,132)$ is also 2, but the two terms are parted by (*n*-2) other terms since the first element is the image of 0^{n-1}, namely (*n*-1)(*n*-2)...1*n*, while the last is the image of 10^{n-2}, namely *n*(*n*-2)(*n*-3)...1(*n*-3).

5. CONCLUDING REMARKS

Isomorphism between graph of B_{n-1} and graph of $S_n(123,132)$ is more simple than isomorphism between graph of F_{n-1} and graph of $S_n(123,132,213)$, where F_{n-1} is the set of binary strings of length (*n*-1) having no 2 consecutive 1s. The constructive bijection between F_{n-1} and $S_n(123,132,213)$ showed by Simion-Schmidt [9]. There is no Hamiltonian cycle in this case, while Hamming distance between two consecutive elements of $S_n(123,132,213)$, a Gray code for $S_n(123,132,213)$, is also 2, as is showed by Juarna-Vajnovszki [3, 2].

REFERENCES:

[1] J.R. Bitner, G. Ehrlich, and E.M. Reingold. Efficient generation of the binary reflected Gray code. *Communication of the ACM*, 19(9):517-521, 2008.

[2] Juarna and V. Vajnovszki. Combinatorial Isomorphism Between Fibonacci Classes. *Journal of Discrete Mathematical Sciences and Cryptography*, II(2), 2008.

[3] Asep Juarna and Vincent Vajnovszki. Isomorphism between classes counted by Fibonacci numbers. *Words 2005*, pages 51-62, 2005. UQAM - Canada.

[4] Sergey Kitaev and Toufik Mansour. A Survey on Certain Pattern Problems. Technical report, University of Kentucky, 2003.

[5] Donald E. Knuth. *The Art of Programming*, volume I. Addison Wesley, Reading Massachusetts, 1973.

[6] M. Barnabei, F. Bonetti, and M. Silimbani. The Descent Statistic on 123-Avoiding Permutations. *Seminaire Lotharingien de Combinatoire*, (63), 2010.

[7] Nicolas T. Courtis, Gregory V. Brad, Shaun V. Ault. Statistics of Random Permutation and the Cryptanalysis of Periodic Block Ciphers. *J. Math. Crypt.*, (2):1-20, 2008.

[8] Carla Savage. A Survey of Combinatorial Gray Code. *SIAM Review*, :605-629, 1997.

[9] Rodica Simion and Frank W. Schmidt. Restricted Permutations. *Europ. J. Combinatorics*, (6):383-406, 1985.

AUTHOR PROFILES:

A. Juarna is a combinatorlist at Faculty of Computer Science and Information Technology, Gunadarma University, Indonesia. He got his Ph.D *dual degree* in Combinatorics from Universite de Bourgogne-France under supervising of Prof. Vincent Vajnovszki and from Gunadarma University under supervising of Prof. Belawati Widjaja. Some of his papers were presented in some conference such as Words-2005, CANT-2006, GASCom-2006, and some others are published in some journals or research reports such as CDMTCS-242 (2004), CDMTCS-276 (2006), The Computer Journal 60(5)-2007, Taru-DMSC 11(2)-2008.

A.B. Mutiara is a Professor of Computer Science. He is also Dean of Faculty of Computer Science and Information Technology, Gunadarma University, Indonesia.