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SPANNING TREES RECURSIONS FOR CROSSES MAPS

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ABSTRACT

It is known that Kirchhoff Matrix Theorem computes the number of spanning trees in any graph G by taking a determinant; so far, many works derived a recursive function to calculate the complexity of certain families of maps specially Grid map. In this paper we give the major recursive formula that counts the number of spanning trees in the general case of grid graph, then we propose some spanning trees recursions for families of planar graphs called crosses maps such as the cross grid and the cross ocotogonal map by using the spanning tree theorem and we give a new algorithm to count the complexity of some particular maps such as the kite map.

Keywords: *Planar graph, map, complexity, spanning tree, grid graph.*

1. INTRODUCTION

Enumeration of trees is an important field of research in graph theory; many works studied the spanning trees of graphs to solve several problems in computer science area. Our research is to find a new method to enumerate spanning trees in any graph (network) G. The first method was proposed by Kirchhoff who defines the complexity of a graph G by the determinant of its laplacian matrix which is easy to compute but it cannot produce the recursion that counts the number of spanning trees, later, many works derive recursive formulas to calculate the complexity of some particular graphs. The purpose of this paper is to give the major recursive formula that counts the number of spanning trees in the map which is formed by m cycles called the general case of grid map then explicit recurrences are found for many cases of crosses maps. As a bulk result, we also give an algorithm to count the number of spanning trees in kite map.

A graph G is an ordered pairs of disjoint sets (V, E) such that E is the subset of the set V^2 of unordered pairs of V with V is the set of vertices and E is the set of edges. We consider only finite graphs. By definition a simple graph does not contain a loop or multiple edges. It is called connected if for every pair of its vertices there is a path joining them; otherwise the graph is disconnected [1].

A map M is a graph r embedded into surface X (that is, considered as a subset $r \subseteq X$. A planar drawing of a map is a rendition of the map on a plane with vertices at distinct locations and no edge intersections. Euler's gave formula that relates the number of vertices, edges and faces of a planar graph: $|V_M| + |F_M| - |E_M| = 2$. The complexity of a map M is the number of spanning trees of this map which are composed from all vertices and some (or perhaps all) of the edge of M and it is denoted by $\tau(M)$. Note that we mainly deal with a connected planar map [6].

Examples:

Let F_m be the Fan map with m+2 vertices, the complexity of F_m is given by:

$$\tau(\mathrm{F_m}) = \frac{1}{\sqrt{5}}((\frac{3+\sqrt{5}}{2})^{m+1} - (\frac{3-\sqrt{5}}{2})^{m+1}), m \geq 1$$

[3], [5].

Let W_{m+1} be the Wheel map with m+1 vertices, the complexity of W_{m+1} is given by:

$$\tau(W_{m+1}) = (\frac{3+\sqrt{5}}{2})^{m+1} + (\frac{3-\sqrt{5}}{2})^{m+1} - 2, m \geq 3$$

[5], [8].

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Let G_m be the m-Grid chain map with 2m+2 vertices, the complexity of G_m is given by:

$$\tau(G_m) = \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{m+1} - (2-\sqrt{3})^{m+1}), m \geq 1$$

[2], [5], [7].

2. MAIN RESULTS

Let C be a map of type $C = C_1 \ddagger C_2$ where ‡ is a simple path $p = v_1, v_2, ..., v_{k+1}$ that contains $k+1$ vertices such as $\deg(v_i) = 2$ for $i = 2,3,...,k$ and k edges (See Figure1).

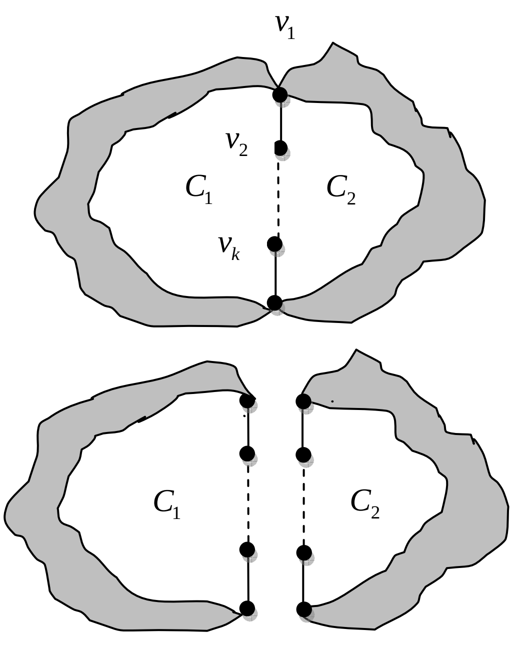

Figure1: Map C, C_1 and C_2

Theorem 1: (Spanning Trees Theorem) the complexity of the map C such that v_1 and v_{k+1} two vertices of C connected by a simple $p = v_1, v_2, ..., v_{k+1}$ that contains k edges (see Figure 1) is given by:

$$\tau(C) = \tau(C_1) \times \tau(C_2) - k^2 \tau(C_1 - p)\tau(C_2 - p)$$

such that $C_1 - p$ and $C_2 - p$ are the maps obtained by deleting the path p [6].

Let S_m be a map formed by m cycles whose lengths are equal, we denote h the length of each cycle. The i-th and the (i+1)-th cycles have a common path of length k, $i = 1,2,....,m-1$ as illustrated in Figure 2.

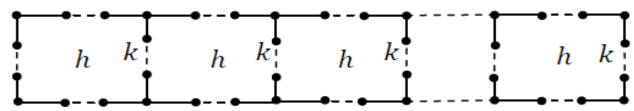

Figure 2: Map S_m

Theorem 2: The number of spanning trees in the map S_m is given by the following recursion:

$$\tau(S_m) = \frac{1}{\sqrt{h^2-4k^2}}((\frac{h+\sqrt{h^2-4k^2}}{2})^{m+1} - (\frac{h-\sqrt{h^2-4k^2}}{2})^{m+1}), m \geq 1, h \geq 2k+1$$

With $\tau(S_0) = 1$, S_0 is an open cycle.

Proof: Let S_m be the map illustrated in Figure 2, we cut along the last cycle and we apply the Theorem 1 then:

$\tau(S_m) = h\tau(S_{m-1}) - \tau(S_{m-2})k^2$, hence we obtain the following system:

$$\begin{cases} \tau(S_m) = h\tau(S_{m-1}) - k^2\tau(S_{m-2}) & \\ \tau(S_1) = h, & m \geq 3 \\ \tau(S_2) = h^2 - k^2 & \end{cases}$$

The characteristic equation is

$r^2 - hr + k^2 = 0, \Delta = h^2 - 4k^2$, however $h \geq 2k \Rightarrow \Delta > 0$, therefore, the solutions of this equation are: $r_1 = \frac{h-\sqrt{h^2-4k^2}}{2}$ and $r_2 = \frac{h+\sqrt{h^2-4k^2}}{2}$ hence

$$\tau(S_m) = \alpha(\frac{h-\sqrt{h^2-4k^2}}{2})^m + \beta(\frac{h-\sqrt{h^2-4k^2}}{2})^m,$$

$\alpha, \beta \in \Re, m \geq 1$

Using the initial conditions

$\tau(S_1) = h, \tau(S_2) = h^2 - k^2$, we obtain:

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$$\alpha = \frac{-h^2 + h\sqrt{h^2 - 4k^2} + 2k^2}{\sqrt{h^2 - 4k^2}\,(h - \sqrt{h^2 - 4k^2})},$$

$$\beta = \frac{h^2 + h\sqrt{h^2 - 4k^2} - 2k^2}{\sqrt{h^2 - 4k^2}\,(h + \sqrt{h^2 - 4k^2})}.$$

3. APPLICATIONS

3.1 Cycle Case

3.1.1 Cross Grid Map

The cross grid map is a planar drawing of graph in which vertices are located at grid points of an integer grid and it forms a cross. The cross grid map C_m is formed by connecting 4 m-Grid chains maps and it contains 8m+4 vertices, 12m+4 edges and 4m+2 faces as illustrated in Figure 3.

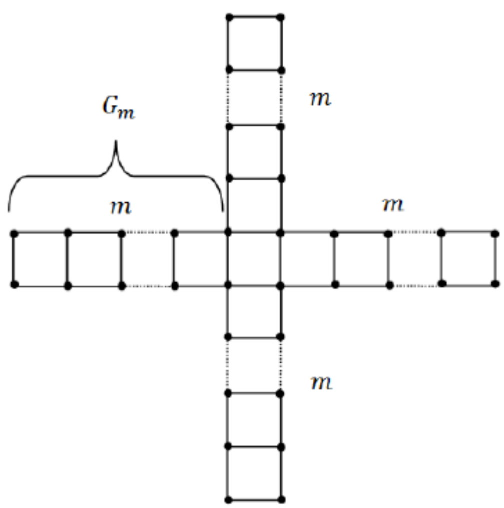

Figure 3: Map C_m

Theorem 3: The number of spanning trees in the map C_m is given by the following recurrence:

$$\tau(C_m) = g_m^2 (g_{2m+1} - 2g_{m-1} g_m - g_{m-1}^2), m \geq 1$$

Such that $\tau(C_0) = \tau(G_1) = 4$ and

$$g_m = \tau(G_m) = \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{m+1} - (2-\sqrt{3})^{m+1})$$

$, m \geq 1.$

3.1.2 Cross Octogonal Map

Let H_m be the m-octogonal chain map, it is a particular case of S_m where h=8 and k=1 (Theorem 2). The map H_m contains 6m+2 vertices, 7m+1 edges and m+1 faces as illustrated in Figure 4.

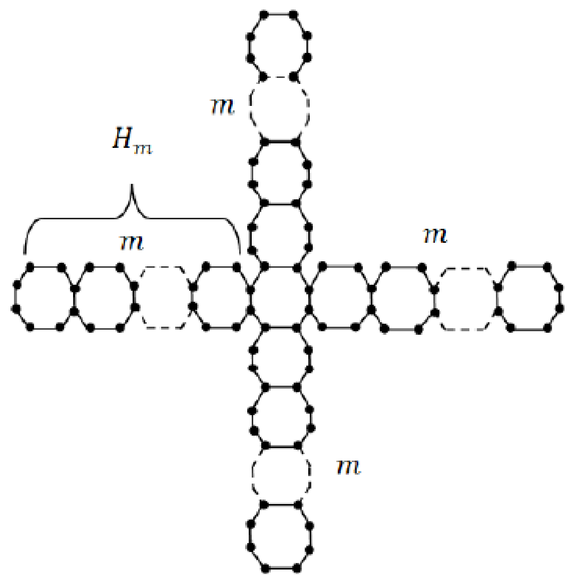

Figure 4: Map N_m

Corollary 1: The number of spanning trees in the map H_m is given by the following recursion:

$$h_m = \tau(H_m) = \frac{1}{2\sqrt{15}}((4+\sqrt{15})^{m+1} - (4-\sqrt{15})^{m+1}),$$

$m \geq 1$

Theorem 4: The complexity of the map N_m is given by

$$\tau(N_m) = h_m^2 (h_{2m+1} - 2h_{m-1} h_m + h_{m-1}^2), m \geq 1$$

with $\tau(N_0) = \tau(H_1) = 8$.

Proof: **(Theorem 3 and 4)** let $\tau(C_m)$ and $\tau(G_m)$ be the complexities of the cross grid map C_m and the m-grid chain map G_m respectively, we denote $g_m = \tau(G_m)$, from the

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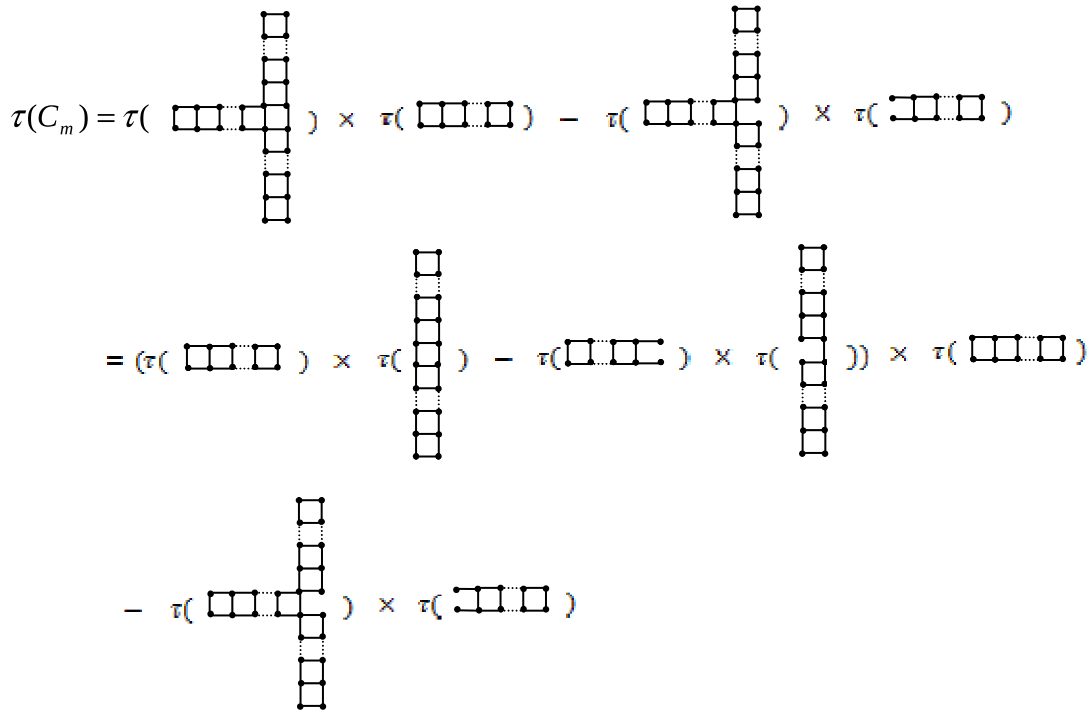
Theorem 1, we decompose the map C_m into two planar maps then we get the following result:



Hence

$\tau(C_m) = g_m^2 \times g_{2m+1} - g_{m-1} \times g_m^3 - g_m^3 \times g_{m-1}$
$- g_m^2 \times g_{m-1}^2$.

Let $\tau(N_m)$ be the complexity of the map N_m and $\tau(H_m) = h_m$, the proof of the Theorem 4 is similar to the previous one.

Numerical result

The following table gives some values for the complexity of the maps C_m and N_m by using the formula given in Theorem 3 and 4.

Table 1: Some values of the complexities $\tau(C_m)$ and $\tau(N_m)$

m	$\tau(N_m)$	$\tau(C_m)$
1	30784	784
2	11828×10^4	152100
3	45422×10^7	29507×10^3
4	17459×10^{11}	57241×10^5
5	67076×10^{14}	11105×10^8
6	25771×10^{18}	21542×10^{10}

3.1.3 General Case

Let R_m be the map that generalizes the two previous maps, it is formed by 4 m-cycles whose lengths are equals as illustrated in Figure 5.

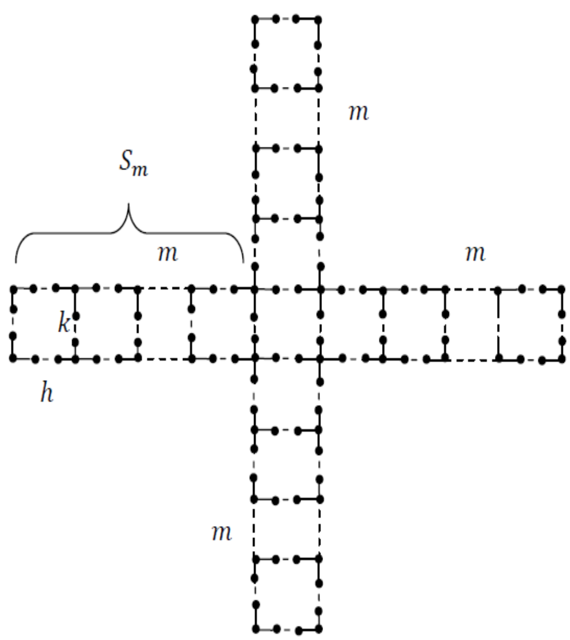


Figure 5: Map R_m

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Theorem 5: The recursive function that calculates the number of spanning trees in the map R_m is given by

$$\tau(R_m)=\tau^2(S_m)(\tau(S_{2m+1})-2k^2\tau(S_{m-1})\tau(S_m)+k^4\tau^2(S_{m-1})), m\geq 1.$$

Proof: Let be $\tau(R_m)$ the complexity of the map R_m and $\tau(S_m)$ the complexity of the map S_m, the proof of this Theorem is similar to the previous one.

Some values of $\tau(R_m)$ was given in Table 1 such that $\tau(R_m)=\tau(C_m)$ if h=4 and k=1 and $\tau(R_m)=\tau(C_m)$ if h=8 and k=1.

3.2 Kite Case

Let's P_k be the k-Kite chains map, if its last edge moves away we obtain the map Q_k as illustrated in Figure 6.

Figure 6: Maps P_k and Q_k

Theorem 6: The complexity of the maps P_k and Q_k such that $p_k=\tau(P_k)$ and $q_k=\tau(Q_k)$ are given by the following system:

$$\begin{cases} p_k=45p_{k-1}-21q_{k-1} \\ q_k=21p_{k-1}-9q_{k-1} \end{cases} \text{With}$$
$$p_1=45, q_1=21$$

Proof: Let denote $p_k=\tau(P_k)$ and $q_k=\tau(Q_k)$, the initials conditions are $p_1=45, q_1=21$, in the sequence of the map P_k, we cut the last kite, and we use Theorem 1 (the same goes for the sequence of the map Q_k).

Numerical Result

The spanning trees sequence of P_k can be found by using the Theorem 6. Therefore we have the following equation

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}=M\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} \text{Where } M=\begin{pmatrix} 45 & -21 \\ 21 & -9 \end{pmatrix}$$

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}=M\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix}=...=M^{k-1}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

We use the square-and-multiply method to compute M^{k-1}, the following table gives some values of $\tau(P_k)$ and $\tau(Q_k)$.

k	$\tau(P_k)$	$\tau(Q_k)$
1	45	21
2	1584	756
3	55404	26460
4	1937520	925344
5	67756176	32359824
6	2369471616	1131641280

Table 2: Some values of the complexities $\tau(P_k)$ and $\tau(Q_k)$

Proposition 1: For a given initials conditions on k blocs, the number of spanning trees in P_k and Q_k can also be found by using the following system:

$$\begin{cases} p_{2k}=p_k^2-q_k^2 \\ q_{2k+1}=p_{k+1}p_k-q_{k+1}q_k \end{cases} \text{With}$$
$$p_1=45, q_1=21$$

Let Z_k and T_k be the maps as illustrated in Figure 7.

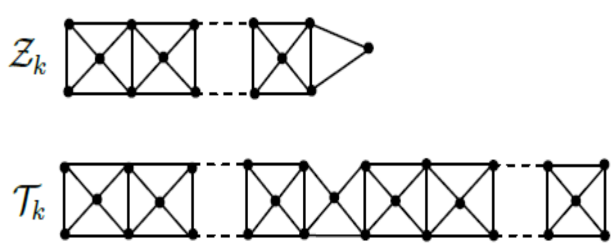

Figure7: Maps Z_k and T_k

Lemma 1: The complexity of the maps Z_k and T_k is given by $z_k=3p_k-q_k$ and

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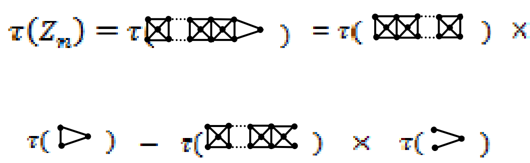
$t_k = 21p_k^2 - 16p_k q_k + 3q_k^2$ with $z_k = \tau(Z_k)$ and $t_k = \tau(T_k)$.

Proof: Let Z_k and T_k be the maps illustrated in Figure 7, from Theorem.1:



This proof is similar for the complexity of T_k.

Let E_k be the cross kite map, it is formed by connecting 4 k-Kite chains map and it contains 12k+9 vertices, 24k+8 edges and 12k+1 faces as illustrated in Figure 8.

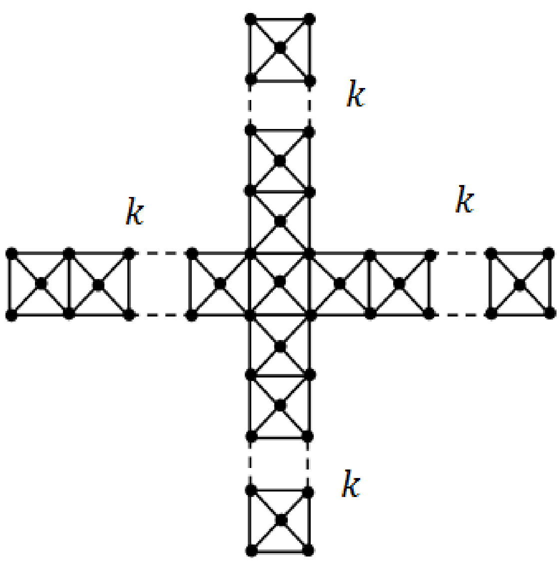


Figure 8: Map E_k

Theorem 7: The complexity of the map E_k is given by the following recurrence:

$$\tau(E_k) = 45p_k^4 - 84p_k^3 q_k + 50p_k^2 q_k^2 - 12p_k q_k^3 + q_k^4$$

$k \geq 1$.

Proof: Let $\tau(E_k)$ be the complexity of the map E_k, we apply the Theorem 1and the Lemma 1 then

$$\tau(E_k) = (p_k p_{2k+1} - p_k t_k)p_k - (p_k t_k - q_k z_k^2)q_k = p_k^2 p_{2k+1} - 2p_k q_k t_k + q_k^2 z_k^2.$$

Numerical result

The following table gives the sequence of complexities $\tau(E_k)$ by using the formula given in Theorem 7.

k	$\tau(E_k)$
1	63628416
2	94718949564672
3	141640142015878594560
4	211833694568250504033140736
5	316814807766899320237496406638592
6	47382279996106378847960373245006315520 0

Table 3: Some values of the complexity $\tau(E_k)$

4. CONCLUSION

The Kirchhoff Theorem give the number of spanning trees in any graph G by calculating the determinant of its Laplacian Matrix. For some planar graphs this can be improved by deriving recursive formula giving this count. In this paper we give a recursive function for counting the number of spanning tree in some crosses maps. We derive such formulas for cross grid and cross octagonal map, and then we give a method for the general case. Since the recursive function is easy to find but for some families of graph this recursion is huge and not easy to be used. Therefore, we propose an algorithm counting the complexity of cross kite map which can be applied to all the planar graphs that satisfy the conditions of spanning trees theorem. Our research perspectives take two directions; the generalization of the spanning tree theorem which is the derivation of a recursive function to count the number of spanning trees in any planar graph, therefore, the decomposition of a map will be not through a simple path but even through a tree or another map. The second direction is to find a general algorithm with logarithmic complexity that counts the number of panning trees in different topologies of network.

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