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RECURSIVE RELATION FOR COUNTING THE COMPLEXITY OF BUTTERFLY MAP

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ABSTRACT

Graph theory is used to represent a communication network by expressing its linkage structure, the vertices represent objects and the pairs called edges or represent the interconnections between objects. The exact geometric positions of vertices or the lengths of the edges are not important. The purpose of this paper is to find a recursive relation counting the number of spanning tree in Butterfly map that illustrate a connection problem suggested by Erdös and Rényi in 1962: Let there be given a country with n cities, n large, so that a direct air connection between two cities would require two busy airports. Considering the capacity limits of the airport, what is the number of flights that would get a passenger from any one city to another so that he needs change planes not more than once? Many results of this problem were given in the past [1], [4].

Keywords: *Graph, map, spanning trees, complexity.*

1. INTRODUCTION

When talking about the applications of graph theory to non-mathematical problems, Erdös and Rényi are invariably mentioned, it cannot be denied that a number of papers that a started a new areas of research concerning the theory of graphs claimed to have been motivated by the possibility of applications. The following "Real life problem" is was considered by Erdös and Rényi and classified as a problem of the pseudo application type: Suppose there are n cities such that the airport of each city can handle at most k flights, it is desirable to schedule the flights in such a way that from each city it is possible to fly to another city with at most t stops along the way. What is the number of flights which must be set up to satisfy the stated requirement? [1], [2], [4]. The solution of this problem requires basic definitions of graph.

A graph is a pair $G = (V, E)$ of sets satisfying $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. The elements of V are the vertices (or nodes) of the graph G, the elements of E are its edges (or lines). A simple graph is a graph having no loops or multiple edges, it is called connected if any two of its vertices are linked by a path in G, and otherwise, G is disconnected. Note that we mainly deal with connected graph [3].

If G has a u,v-path (a path whose vertices of degree 1 (its endpoints) are u and v), then the distance from u to v, written $d_G(u,v)$ or simply it is the least length of a u,v-path. The diameter $D_G = \max_{u,v \in V(G)} d(u,v)$ [1], [3].

A map M is a graph Γ embedded into surface X (that is, considered as a subset $\Gamma \subseteq X$). A plannar drawing of a map is a rendition of the map on a plane with the vertices at distinct locations and no edge intersections. The complexity of a map M is the number of spanning trees in this map which are composed by all vertices and some (or perhaps all) of the edge of M, it is denoted by $\tau(M)$ and it can be calculated using the Matrix Tree Theorem [7]:

If $L^*(M)$ is a matrix obtained by deleting row i and column j of the Laplacian matrix $L(M)$, then

$$\tau(M) = (-1)^{i+j} \det L^*(M)$$

The advantage of the Matrix Tree Theorem is that the determinant is easy to compute, however, it can only give the number of spanning trees in families of graph G and it cannot a priori find out a recursion which produces the sequence of spanning trees. Many recurrences are found for some planar graphs and the main source of these results was based on the Cayley's Formula [7].

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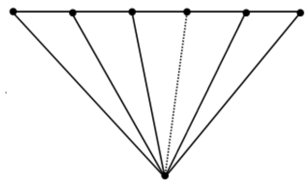
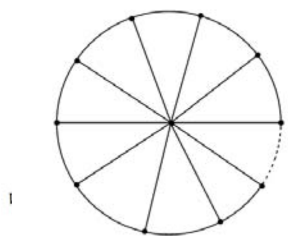
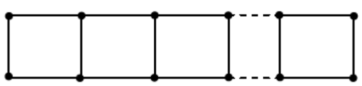
The following table gives the complexity of some maps.

Map	Description & Complexity
	*The complexity of m-Fan chains with* $m+2$ *vertices is given by* $$\tau(F_m)=\frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{m+1}-\left(\frac{3-\sqrt{5}}{2}\right)^{m+1}\right),\ m\geq 1$$ [5], [7]
	*The complexity of wheel with* $m+1$ *vertices is given by* $$\tau(W_{m+1})=(\frac{3+\sqrt{5}}{2})^m+(\frac{3-\sqrt{5}}{2})^m-2,\quad m\geq 3$$ [5]
	*The complexity of m-Grid chains with* $2m+2$ *vertices is given by* $$\tau(G_m)=\frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{m+1}-\left(2-\sqrt{3}\right)^{m+1}\right),\ m\geq 1$$ [6], [7]

Now we can resume the airport problem by: What is the least number $E(n,k,D)$ of edges in a graph G with n vertices having degree at most k and diameter at most D where $D=t+1$ and t is the number of stops along the way? Some partial results were given in the past by Erdös and Rényi [4]. The exact value for $E(n,k,D)$ was given for the extremal graph with n vertices having diameter 2, $degree \leq k$, $\frac{(2n-2)}{3}\leq k\leq n-5$, $E(n,k,D)=2n-4$ edges [2].

Let's take the same number of cities near each airport denoted by i, one stop along the way $t=1$, then $D=2$, In this paper, we calculate the complexity of the map B_i that illustrate the airport problem $E(n,k,D)$ with $n=3i+3$, $k=2i+2$ and $D=2$.

2. MAIN RESULTS

Let M be a map of type $M=M_1\ddagger M_2$ where $\ddagger$ is a simple path that contains $k+1$ vertices and k edges (see fig 1).

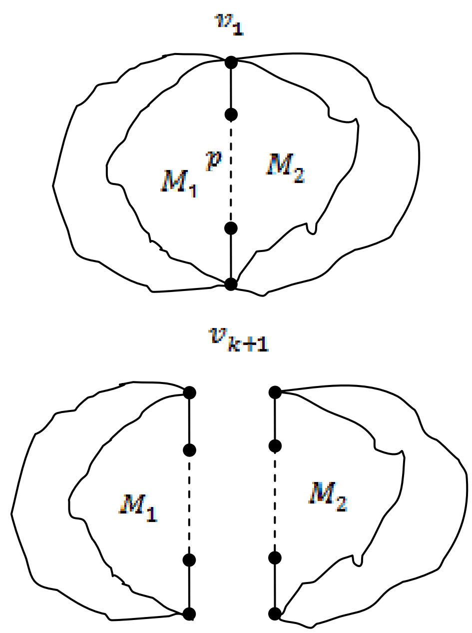


Figure1: Map $M=M_1\ddagger M_2$

Theorem 2.1 The complexity of the map M such that v_1 and v_{k+1} two vertices of the map M connected by a simple path

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$p = v_1, v_2, \dots, v_k, v_{k-1}$ that contains k edges (see fig 1) is given by

$$\tau(M) = \tau(M_1) \times \tau(M_2) - k^2\tau(M_1 - p) \times \tau(M_2 - p)$$

The maps $M_1 - p$ and $M_2 - p$ are obtained by deleting the path p from M_1 and M_2.

Let M_i be a map with $i+2$ vertices (cities) and $2i+1$ edges (see fig 2).

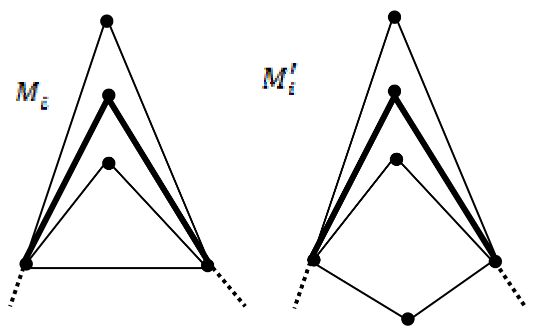

Figure2: Maps M_i and M'_i

Lemma 2.2 The complexity of the map M_i (see fig 2) is given by

$$\tau(M_i) = 2^i\left(1 + \frac{i}{2}\right), \qquad i \geq 1$$

Proof. Let be $\tau(M_i)$ the complexity of the map M_i, we cut along the bold line and we use the Theorem2.1, then we obtain the following system:

$$\begin{cases} \tau(M_i) = 4\tau(M_{i-1}) \quad 4\tau(M_{i-2}), & i \geq 3 \\ \tau(M_1) = 3 \\ \tau(M_2) = 8 \end{cases}$$

The characteristic equation is $r^2 - 4r + 4 = 0$, therefore the solution of this equation is $r = 2$ hence $\tau(M_i) = (\lambda i + \mu)2^i,\ \lambda, \mu \in \mathbb{R},\ i \geq 1$.
Using the initial conditions we obtain $\lambda = \frac{1}{2}$ and $\mu = 1$, hence the result.□

Let M'_i be a map with $i+3$ vertices (cities) and $2i+2$ edges (see fig 2).

Lemma 2.3 The complexity of the map M'_i (see fig 2) is given by

$$\tau(M'_i) = 2^i(i+1), \qquad i \geq 1$$

Proof. Let be $\tau(M'_i)$ the complexity of the map M'_i, we cut along the bold line and we use the Theorem2.1, then we obtain the following system:

$$\begin{cases} \tau(M'_i) = 4\tau(M'_{i-1}) - 4\tau(M'_{i-2}), & i \geq 4 \\ \tau(M_1) = 4 \\ \tau(M_2) = 12 \end{cases}$$

The characteristic equation is $r^2 - 4r + 4 = 0$, therefore the solution of this equation is $r = 2$ hence $\tau(M_i) = (\lambda i + \mu)2^i,\ \lambda, \mu \in \mathbb{R},\ i \geq 1$.
Using the initial conditions we obtain the complexity of the map M'_i. □

Let B_i be the Butterfly map (see fig 3) that illustrates the airport problem $E(3i+3, 2i+2, 2)$, this map contains $6i+2$ edges which are the solution of the problem and $3i+1$ faces found by using the Euler Formula [7]. The goal of this paper is to calculate the number of spanning trees of this map.

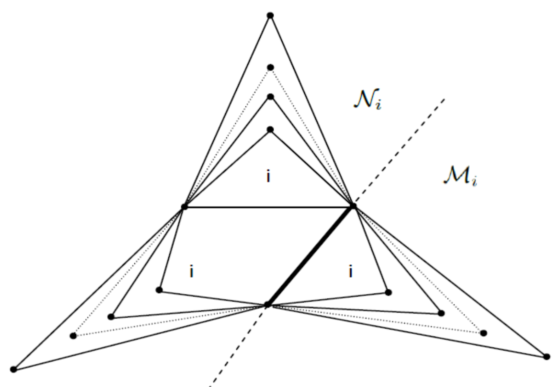

Figure3: Butterfly Map

Let N_i be a map of type $N_i = M_i \ddagger M'_i$ with $2i+3$ vertices and $4i+2$ edges (see fig 4).

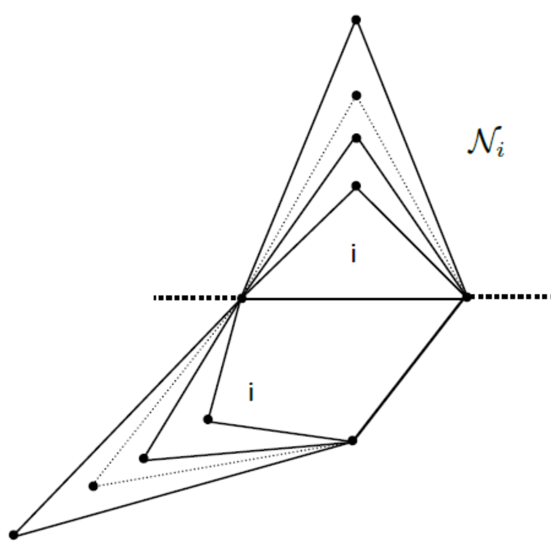

Figure4: Map N_i

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Lemma 2.4 The complexity of the map N_i (see fig 4) is given by

$$\tau(N_i) = 2^{i-1}(\frac{1}{2}i^2 + 3i + 2), \qquad i \geq 1$$

Proof. We cut along the bold line (see fig 4) and we use the Theorem 2.1 then we obtain

$$\tau(N_i) = \tau(M_i) \times \tau(M_i') - (\tau(M_i'))^2$$

We obtain the complexity of the map N_i by replacing $\tau(N_i)$ and $\tau(M_i')$ by their expressions.

Theorem2.5

Let $\tau(B_i)$ be the complexity of the map that illustrates the airport problem $E(3i+3, 2i+2, 2)$ (see fig 3) then

$$\tau(B_i) = 2^{3i-2}(2+i)(2+3i), \qquad i \geq 1$$

Proof. Let B_i be the Butterfly map (see fig 3). From the Theorem 2.1 we have $B_i = N_i \ddagger M_i$ then

$$\tau(B_i) = \tau(N_i) \times \tau(M_i) - \tau(M_{i-1}') \times (\tau(M_i) \times \tau(M_{i-1}'))$$

, we obtain the co□plexity of the map B_i by replacing $\tau(N_i)$, $\tau(M_i)$ and $\tau(M_i')$ by their expressions.

Now we can generalize the previous Theorem, let's i, j, l the cities next each airport, the butterfly map that represents the Erdös and Rényi problem contains $i+j+l+3$ vertices and $2(i+j+l)+2$ edges. The following Theorem gives the complexity of this map.

Theorem2.6 (Generalization of the proposed Theorem (2.5))

The complexity of the map $B_{i,j,l}$ with i, j, l cities respectively near each airport is given by

$$\tau(B_{i,j,l}) = 2^{i+j+l-1} \times \left[\left(2(1+j) + i\left(1+\frac{i}{2}\right)\right)\left(1+\frac{l}{2}\right) - \frac{lj}{2}\left(1+\frac{i}{2}\right)\right],$$

$i, j, l \geq 1$

3. CONCLUSION

The spanning Trees are used in different domain to solve some problems such that loop switching in computer network, then the solution was to use the spanning tree protocol to eliminate the loop link which creates redundant information or the airport problem in transportation network, etc... The goal of this paper was to find a recursive function for counting the number of spanning tree of the map that illustrates the airport problem with the same number of cities near each airport. Finally, we found explicit formula that generalizes this result. In this work we use the Theorem of spanning tree which gives a recursive relation to count the complexity of the map C such that its decomposition is by a simple path; the future research direction is to find another decomposition which can be a tree or a map.

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