CALCULATION OF THE WIENER INDEX FOR SOME PARTICULAR TREES

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ABSTRACT

The wiener index W (G) of a connected graph G is the sum of the distances between all pairs of vertices of G. In this paper, we give theoretical results for calculating the wiener index for some composites trees (chain-trees, path-trees, etc.). In the end we give an application of its results on particular families of trees. These formulas are the part of a future attempt to demonstrate the wiener index conjecture (see introduction).

1. INTRODUCTION

A graph $G = (V, E)$ consists of a finite non-empty set V= $v_0, v_1, \dots, v_{m-1}$ (the vertex set of *G*), and a set *E* (of two elements subsets of V, the edge set of *G*). We also write V (G), E (G) for the vertex (resp. edge) set of *G*. A tree is connected graph without cyclic [5, 6]. The distance d(u, v) between the vertices *u* and *v* of the graph *G* is equal to the length of the shortest path that connects *u* and *v,* [1,7] The Wiener index W(G) of a connected graph G is the sum of all the distances between pairs of vertices of G.

$$W(G) = \sum_{u \in V(G)} \sum_{v \in V(G)} d(u,v)$$

This index was introduced by the chemist Wiener [2] in the study of relations between the structure of organic compounds and their properties. It has since been studied extensively by both chemists and mathematicians, especially for trees; see the survey [3, 6, 7, 8, 9] for many results and references, the wiener index of a vertex v in G is defined as:

$$W(v,G) = \sum_{u \in V(G)} d(u,v)$$

The Wiener index is, apart from a constant factor, the geometric mean of the extremely values, which are given for the star E_m and the path P_m respectively [1]:

$$(m-1)^2 = W(E_m) \le W(T_m) \le W(P_m) = \binom{m+1}{3}$$

It has same result for a map C_m

$$W(E_m) \le W(C_m) \le W(P_m)$$

Where E_m is a map defined in article [2].

Conjecture: [Wiener Index Conjecture [4, 10]]

Except for some finite set, every positive integer is the Wiener index of a tree.

2. THE MAIN RESULT

Let E_m be star tree with ***m*** vertices

$v_0, v_1, \dots, v_{m-1}$ $(m \ge 3)$.(See Fig. 1)

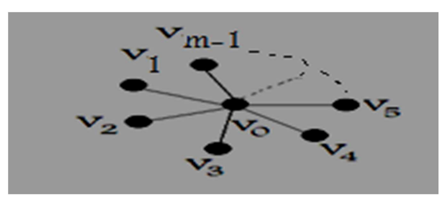

Fig. 1 Star tree E_m

Lemma 1 :

We have: $W(v_0, E_m) = m-1$,

$W(v_i, E_m) = 2m-3 \quad for\ i = 1, 2, \cdots, m-1$. □

Let P_m Chain Tree with m vertices

$v_0, v_1, \dots, v_{m-1}$ $(m \ge 2)$ $(see\ Fig. 2)$.

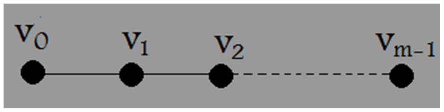

Fig. 2 Path tree P_m

Lemma 2:

We have

$$W(v_i, P_m) = \frac{m^2 - m(2i-1) + 2i^2}{2}$$
$$for\ i = 0,1,\cdots,m-1$$

Proof:

$$W(v_i\,, P_m) = (1 + 2 + \cdots + i) +$$

$$\big(1 + 2 + \cdots + (m-i)\big)$$

$$= \frac{i(i+1)}{2} + \frac{(m-i)(m-i+1)}{2}$$

$$= \frac{1}{2}\left(m^2 - m(2i-1) + 2i^2\right).$$ ▢

Let T_m a tree with m vertices for a vertex, s $\notin$ T (see Fig. 3)

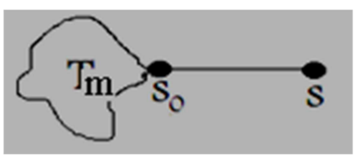

Fig. 3

Lemma 3:

$$W(T_m \cdot \{s\}) = w(T_m) + W(s_0, T_m) + m$$

Proof:

$$W(T_m \cdot \{s\}) = \sum_{u \in V(T_m)} \sum_{v \in V(T_m)} d(u,v)$$

$$+ \sum_{u \in V(T_m)} d(u,s)$$

$$= W(T_m) + \sum_{u \in V(T_m)} d(u,s)$$

Width $d(u\,,s) = d(u\,,s_0) + d(s_0\,, s)$ then:

$$W(T_m \cdot \{s\}) = w(T_m) + W(s_0, T_m) + m\,.$$ ▢

Let T_{m_1} and T_{m_2} be two trees that possess respectivelym_1, m_2 vertices connected by a vertex s (see Fig. 4)

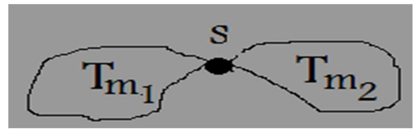

Fig. 4 The tree $T_{m_1} \cdot T_{m_2}$

Lemma 4:

The wiener index of $T_{m_1} \cdot T_{m_2}$ *is:*

$$W(T_{m_1}.T_{m_2}) = W(T_{m_1}) + W(T_{m_2}) +$$

$$(m_1 - 1)w(s\,, T_{m_2}) + (m_2 - 1)w(s\,, T_{m_1})$$

Proof:

$$V^*(T_{m_i}) = V(T_{m_i}) \backslash \{s\}, \quad i = 1,2$$

$$W(T_{m_1} \cdot T_{m_2}) = \sum_{u \in V^*(T_{m_1} \cdot T_{m_2})} \sum_{v \in V^*(T_{m_1} \cdot T_{m_2})} d(u\,,v)$$

$$= \sum_{u \in V^*(T_{m_1})} \sum_{v \in V^*(T_{m_1})} d(u\,,v) +$$

$$\sum_{u \in V^*(T_{m_2})} \sum_{v \in V^*(T_{m_2})} d(u\,,v) + \sum_{\substack{u \in V^*(T_{m_1}) \\ v \in V^*(T_{m_2})}} d(u,v)$$

$$= W(T_{m_1}) + W(T_{m_2}) + \sum_{\substack{u \in V^*(T_{m_1}) \\ v \in V^*(T_{m_2})}} \big(d(u\,,s) + d(s,v)\big)$$

$$= W(T_{m_1}) + W(T_{m_2}) + \sum_{\substack{u \in V^*(T_{m_1}) \\ v \in V^*(T_{m_2})}} d(u,s) + \sum_{\substack{u \in V^*(T_{m_1}) \\ v \in V^*(T_{m_2})}} d(s,v)$$

$$= W(T_{m_1}) + W(T_{m_2}) + (m_1 - 1)w(s\,, T_{m_2}) +$$

$$(m_2 - 1)w(s\,, T_{m_1})\,.$$ ▢

We generalize the lemma 4:

Let T_N be the tree formed by the trees $T_{m_1}, T_{m_2}, \ldots, T_{m_n}$ that possess respectively $m_1, m_2, \ldots, m_n$ vertices connected by a vertex s (see Fig . 5).

$$V^*(T_{m_i}) = V(T_{m_i}) \setminus \{s\}$$

$$N = \sum_{i=1}^{n} m_i - n + 1$$

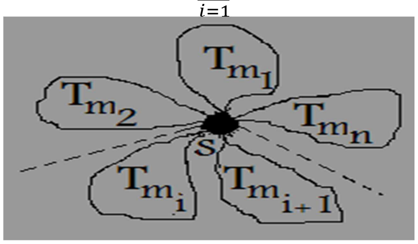

Fig. 5: Star- trees

Lemma 5:

The wiener index of star - tree T_N is:

$$W(T_N) = \sum_{i=1}^{n} w(T_{m_i}) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[(m_j - 1) w(s, T_{m_i}) + (m_i - 1) w\left(s, T_{m_j}\right) \right]$$

Proof:

$$W(T_N) = \sum_{u \in V(T_N)} \sum_{v \in V(T_N)} d(u, v)$$

$$= \sum_{u \in V^*(T_{m_{n-1}})} \sum_{v \in V^*(T_{m_n})} d(u, v) + \sum_{u \in V^*(T_{m_{n-2}})} \sum_{v \in V^*(T_{m_{n-1}})} d(u, v) + \sum_{u \in V^*(T_{m_{n-2}})} \sum_{v \in V^*(T_{m_n})} d(u, v) + \sum_{u \in V^*(T_{m_i})} \sum_{v \in V^*(T_{m_{i+1}})} d(u, v) + \cdots + \sum_{u \in V^*(T_{m_i})} \sum_{v \in V^*(T_{m_n})} d(u, v) + \cdots + \sum_{u \in V^*(T_{m_1})} \sum_{v \in V^*(T_{m_2})} d(u, v) + \cdots + \sum_{u \in V^*(T_{m_1})} \sum_{v \in V^*(T_{m_n})} d(u, v)$$

$$W(T_N) = \sum_{i+1}^{n} w(T_{m_i}) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\sum_{u \in V^*(T_{m_i})} \sum_{v \in V^*(T_{m_j})} d(u, v) \right)$$

Or $d(u, v) = d(u, s) + d(s, v)$ then

$$\sum_{u \in V^*(T_{m_i})} \sum_{v \in V^*(T_{m_j})} d(u, v) = (m_j - 1) \sum_{u \in V^*(T_{m_i})} d(u, s) + (m_i - 1) \sum_{v \in V^*(T_{m_j})} d(s, v)$$

$$= (m_j - 1) w(s, T_{m_i}) + (m_i - 1) w\left(s, T_{m_j}\right)$$

hence the result. ▢

Particular case:

If the trees T_{m_i} have the same number of vertices m $(m_i = m \;\; for \;\; i = 1, \ldots, n)$, we have:

$$(1) \begin{cases} w(s, T_{m_i}) = w\left(s, T_{m_j}\right) \\ T_{m_i} = T_{m_j} = T_m \\ N = nm - n + 1 \end{cases} \quad for \; i, j \in \{1, 2, \ldots, n\}$$

We obtain :

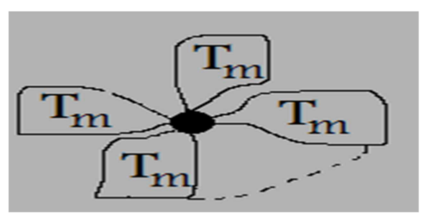

Fig. 6: Star trees T_N: T_m . T_m . …. . T_m

Corollary 1:

The wiener index of T_N is:

$$W(T_N) = n\, w(T_m) + n(n-1)(m-1) w(s, T_m)$$

Proof :

We use (1) in the lemma 5, we have:

$$W(T_N) = n\, w(T_m) + 2(m-1)w(s,T_m)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 1$$

$$= n\, w(T_m) + 2(m-1)w(s,T_m)\frac{n(n-1)}{2}\ . \quad \square$$

Let T_N be the tree formed by trees T_{m_1} , T_{m_2} , T_{m_3} that possess respectively m_1, m_2, m_3 vertices connected by two vertices s_1 , s_2 (see Fig . 7) . We denote by $T_{m_1} - T_{m_2} - T_{m_3}$ the tree T_N $(with\ N = m_1 + m_2 + m_3 - 2)$.

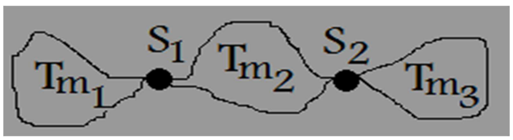

Fig .7

Lemma 6:

The wiener index of T_N is:

$$W(T_N) = W(T_{m_1}) + W(T_{m_2}) + W(T_{m_3}) +$$
$$(m_1-1)[\,W(s_1,T_{m_2}) + W(s_2,T_{m_3})\,] +$$
$$(m_2-1)[\,W(s_1,T_{m_1}) + W(s_2,T_{m_3})\,] +$$
$$(m_3-1)[\,W(s_1,T_{m_1}) + W(s_2,T_{m_2})\,] +$$
$$(m_1-1)(m_3-1)\, d(s_1,s_2)\ .$$

Proof :

$$W(T_{m_1} - T_{m_2} - T_{m_3}) =$$

$$\sum_{u\in(T_{m_1},T_{m_2},T_{m_3})}\ \sum_{v\in(T_{m_1},T_{m_2},T_{m_3})} d(u,v)$$

$$= W(T_{m_1}) + W(T_{m_2}) + W(T_{m_3}) +$$

$$\sum_{\substack{u\in V(T_{m_1})\\ v\in V(T_{m_2})}} d(u,s_2) + d(s_1,v) +$$

$$\sum_{\substack{u\in V(T_{m_1})\\ v\in V(T_{m_3})}} \big(d(u,s_1) + d(s_1,s_2) + d(s_2,v)\big) +$$

$$\sum_{\substack{u\in V(T_{m_2})\\ v\in V(T_{m_3})}} \big(d(u,s_2) + d(s_2,v)\big)$$

$$= W(T_{m_1}) + W(T_{m_2}) +$$

$$W(T_{m_3})(m_2-1)\, W(s_1,T_{m_1}) +$$

$$(m_1-1)\, W(s_1,T_{m_2}) + (m_3-1)\, W(s_1,T_{m_1}) +$$
$$(m_1-1)(m_3-1)d(s_1,s_2) +$$
$$(m_1-1)\, W(s_2,T_{m_3}) + (m_3-1)\, W(s_2,T_{m_2}) +$$
$$(m_2-1)\, W(s_2,T_{m_3})$$

hence the result . $\square$

We generalize the lemma 6:

Let T_N be tree formed by the trees $T_{m_1}, T_{m_2}, \dots, T_{m_n}$ that possess respectively $m_1, m_2, \dots, m_n$ vertices, those trees are connected by the vertices $s_1, s_2, \cdots, s_{n-1}$ (see Fig .8). We denote the tree $T_{m_1} - T_{m_2} - \cdots - T_{m_n}$ by T_N

$$where\ N = \sum_{i=1}^{n} m_i - (n-1)$$

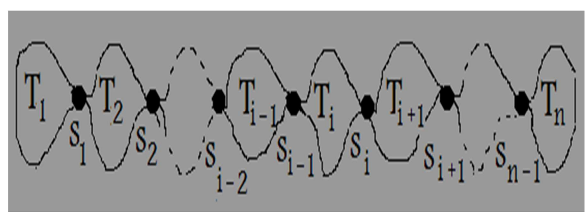

Fig. 8 The path-trees

Lemma 7:

The wiener index of T_N is:

$$W(T_N) = \sum_{i=1}^{n} w(T_i) +$$

$$\sum_{i=1}^{n-1}[(m_{i+1}-1)w(s_i,T_i) + (m_i-1)w(s_i,T_{i+1})] +$$

$$\sum_{i=1}^{n-2}\sum_{j=i+2}^{n}[\,(m_j-1)w(s_i,T_i) +$$

$$(m_i-1)w(s_{j-1},T_j) + (m_i-1)(m_j-1)d(s_i,s_{j-1})\,]$$

Proof:

$V(T_i)$ is the set of the vertices of T_i and

$V^*(T_i) = V(T_i)\backslash\{s_i\}$

$$W(T_N) = \sum_{i=1}^{n} w(T_i) + \sum_{u\in V^*(T_{n-1})} \sum_{v\in V^*(T_n)} d(u,v) +$$

$$\sum_{u\in V^*(T_{n-2})} \sum_{v\in V^*(T_{n-1})} d(u,v) + \sum_{u\in V^*(T_{n-2})} \sum_{v\in V^*(T_n)} d(u,v) + \cdots +$$

$$\sum_{u\in V^*(T_i)} \sum_{v\in V^*(T_{i+1})} d(u,v) + \sum_{u\in V^*(T_i)} \sum_{v\in V^*(T_{i+2})} d(u,v)$$

$$+ \cdots + \sum_{u\in V^*(T_i)} \sum_{v\in V^*(T_j)} d(u,v) +$$

$$\sum_{u\in V^*(T_i)} \sum_{v\in V^*(T_n)} d(u,v) + \cdots + \sum_{u\in V^*(T_1)} \sum_{v\in V^*(T_2)} d(u,v) +$$

$$\sum_{u\in V^*(T_1)} \sum_{v\in V^*(T_3)} d(u,v) + \cdots + \sum_{u\in V^*(T_i)} \sum_{v\in V^*(T_n)} d(u,v)$$

We have for $u \in V^*(T_i)$ and $v \in V^*(T_j)$:

$$d(u,v) = d(u,s_i) + d(s_i,v)$$

$$\sum_{u\in V^*(T_i)} \sum_{v\in V^*(T_{i+1})} d(u,v) =$$

$$(m_{i+1}-1)w(s_i,T_i) + (m_i-1)w(s_i,T_{i+1})$$

And we have

for $u \in V^*(T_i)$ and $v \in V^*(T_j)$, $j \geq i+2$:

$$d(u,v) = \left(d(u,s_i) + d(s_i, s_{j-1}) + d(s_{j-1},v)\right)$$

$$\sum_{u\in V(T_i)^*} \sum_{v\in V(T_j)^*} d(u,v) =$$

$$(m_j-1)w(s_i,T_i) + (m_i-1)w(s_{j-1},T_j) +$$

$$(m_i-1)(m_j-1)d(s_i,s_{j-1})$$

hence the result . ▢

Particular case:

If $m_i = m$ for $i = 1, \ldots, n$ then

$$(2)\begin{cases} W(s_i,T_m) = W(s_1,T_m) & \text{for } i = 1,\ldots,n-1 \\ d(s_1,s_2) = d(s_i,s_{i+1}) & \text{for } i = 1,\ldots,n-2 \\ N = nm - n + 1 \end{cases}$$

T_N is the tree $T_m - T_m - \cdots - T_m$ (see Fig. 9)

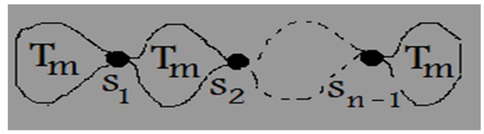

Fig. 9: The chain trees

Corollary 2:

The wiener index of T_N *is:*

$$W(T_N) = n\,w(T_m) + n(m-1)(n-1)w(s_1,T_m)$$

$$+\frac{n(n-2)(n-1)(m-1)^2}{6}\, d(s_1,s_2)$$

Proof:

We use (1) in lemma 7:

$$w(T_N) = n\,w(T_m) + 2(m-1)(n-1)w(s_1,T_m) +$$

$$2(m-1)(s_1,T_m)\sum_{i=1}^{n-2}\sum_{j=i+1}^{n} 1 +$$

$$(m-1)^2 \sum_{i=1}^{n-2}\sum_{j=i+2}^{n} d(s_i,s_{j-1})$$

where

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-2}\left((n-1)-i\right) = \frac{(n-1)(n-2)}{2}$$

We poses

$d = d(s_i,s_{i+1})$,

We have $d(s_i,s_{i+2}) = 2d$, ...

$d(s_i,s_{n-1}) = (n-1-i)d$,

Then:

$$\sum_{j=i+1}^{n} d(s_i,s_{j-1}) = d\left(1+2+\cdots+(n-1-i)\right)$$

$$= \frac{d}{2}[n^2 + n + i(1-2n) + i^2]$$

$$\sum_{i=1}^{n-2}\sum_{j=i+2}^{n} d(s_i, s_{j-1}) = \frac{d}{2}(n^2+n)(n-2) + \frac{d}{2}(1-2n)\sum_{i=i}^{n-2} i + \frac{d}{2}\sum_{i=1}^{n-2} i^2$$

$$= \frac{d}{2}n(n+1)(n-2) + \frac{d}{2}(1-2n)\frac{(n-2)(n-1)}{2} + \frac{(n-2)(n-1)(2n-3)d}{12} = \frac{n(n-1)(n-2)d}{6}.$$

3. APPLICATION

Let T_N the tree formed by star E_n and a chain P_m: T_N is the tree $E_n \cdot P_m$ where $(N = n + m - 1)$ (see Fig . 10)

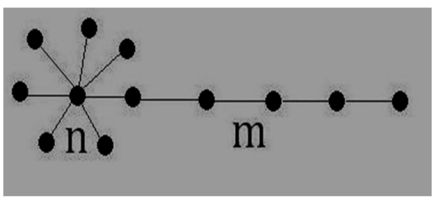

Fig. 10 $E_n \,.\, P_m$

Theorem 1:
The wiener index of T_N is:

$$W(T_N) = (n-1)^2 + \frac{m(m-1)(m+1)}{6} + (m-1)(2n-3) + \frac{(n-1)\,m\,(m-1)}{2}$$

Proof:

It suffices to apply lemmas 1,2 and 4 . □

Let T_N where by the starry tree whose branches are chains $P_m : T_N = P_m \,.\, P_m \,. \dots .\, P_m$ (see Fig . 11). (N=nm-n+1).

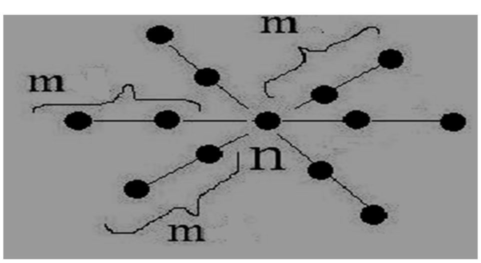

Fig. 11 The tree $P_m \,.\, P_m \,\dots\, P_m$

Theorem 2:
The wiener index of T_N is:

$$W(T_N) = \frac{nm(m-1)(m+1)(3n-1)}{12}$$

Proof :

It suffices to apply corollaire 1. □

Let T_N be the tree formed by a chained stars E_m (See Fig. 12),
T_N *is the tree* $E_m - E_m - \dots - E_m$.
N=nm-n+1

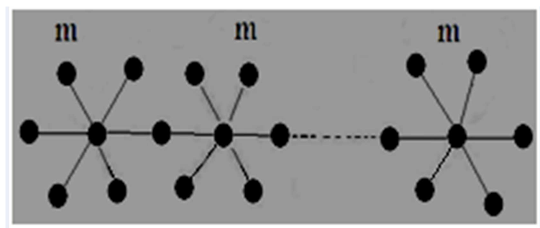

Fig. 12: $E_m - E_m - \dots - E_m$

Theorem 3:

The wiener index of T_N is:

$$W(T_N) = n(m-1)^2 + n(n-1)(m-1)(2m-3) + \frac{n(n-1)(n-2)}{3}(m-1)^2$$

Proof:

It suffices to apply corollary 2. □

Let T_N be the tree formed by n stars E_m connected by a vertex s [4].
T_N is $E_m \,.\, E_m \,. \dots .\, E_m$ (see Fig. 13) .
N=nm-n+1

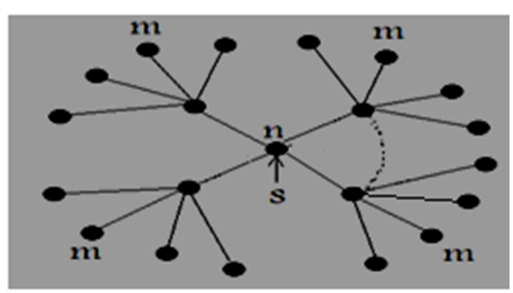

Fig. 13 The tree $E_m \,.\, E_m \,. \dots .\, E_m$

Theorem 4:

The wiener index of T_N *is:*

$$W(T_N) = n(m-1)^2 + n(n-1)(m-1)(2m-3)$$

Proof:

It suffices to apply corollary 1. □

Conclusion

In this article we give the Wiener index of some families of trees. These formulas are useful from the perspective of the proof of the Wiener Index Conjecture given in the introduction to this article, see [4, 10].

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