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# THREE-STEP BLOCK METHODS WITH DIFFERENT OFF-STEP POINTS INTERVAL FOR SOLVING SECOND ORDER INITIAL VALUE PROBLEMS

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**ABSTRACT**

Block methods are numerical methods adopted for the direct solution of higher order ordinary differential equations (ODEs) with no need for reduction to a system of first order ODEs. Hybrid block methods, which combines the use of both on-step and off-step points in the derivation of the block method, are known for performing better in terms of absolute error in comparison with block methods developed using only on-step points. However, when considering multistep methods, the off-step step points can be selected at different intervals, and it is important to know which interval choice for the off-step points gives the best results. This article considers a three-step block method with selection of two off-step points at all three intervals, which leads to developing three different methods. The first method is the three-step block method with off-step points selected within the first interval, while the second and third methods are three-step block methods with off-step points selected in the second and third intervals respectively. These resultant methods are used to solve second order initial value problems and comparison is made among the three block methods, and with existing studies. The three hybrid block methods (HBMs) showed comparable performance among themselves but displayed better accuracy when compared to existing studies in terms of absolute error. The basic properties of the HBMs were also tested and they were found to be zero-stable, consistent, and hence convergent.

**Keywords**: *Hybrid Block Method, Numerical Solution, Second Order, Initial Value Problems*

## 1. INTRODUCTION

Numerical solution of ordinary differential equations (ODEs), either initial value problems (IVPs) or boundary value problems (BVPs), is often sought when the exact solutions of these ODEs fail to exist or are difficult to obtain. Various numerical methods have been developed to obtain numerical solutions to ODEs, such as the Euler [1-5], Runge-Kutta [2, 4-7] and linear multistep methods [4, 8, 9]. The shortcoming identified with Euler and Runge-Kutta methods was the low order of the methods, while the implementation approach of linear multistep methods was computational tasking with low accuracy. The implementation approach of linear multistep methods for solving higher order ODEs was either via reduction to a system of first order ODEs or using predictor-corrector approach. In order to obtain methods with high order, better accuracy and less computational rigor, block methods were developed [10, 11]. However, it was observed that the introduction of off-step points to produce hybrid block methods further improved the accuracy of block methods for solving ODEs [12, 13].

Having established that hybrid block methods have better accuracy, the question of which interval of off-step points would be most accurate comes to mind. A three-step block method is developed in this article for the solution of second order IVPs, and two off-step points are selected at intervals $[x_n, x_{n+1}]$, $[x_{n+1}, x_{n+2}]$, and $[x_{n+2}, x_{n+3}]$. The next section discusses in detail how the three-step block is developed, while subsequent sections discuss about the basic properties of the block method, numerical solution of second order IVPs using the developed block method and a conclusion of the article.

## 2. DEVELOPMENT OF THE THREE-STEP BLOCK METHOD WITH GENERALISED OFF-STEP POINTS PER INTERVAL

Consider the second order IVP of the form

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$$y'' = f(x,y,y'), y(a) = \eta_0, y'(a) = \eta_1, \quad x \in [a,b] \quad (1)$$

where $a, b, \eta_0$, and $\eta_1$ are constants. Also, let the power series expression below

$$y(x) = \sum_{j=0}^{i+c-1} a_j \left(\frac{x - x_n}{h}\right)^j \quad (2)$$

be an approximate solution to Equation (1) whose second derivative is

$$y''(x) = f(x,y,y') = \sum_{j=2}^{i+c-1} a_j \frac{j(j-1)}{h^2}\left(\frac{x - x_n}{h}\right)^{j-2} \quad (3)$$

where $x \in (x_n, x_{n+3}]$ for $n = 0,3,6,\dots,N-3$, $h = x_{\delta+1} - x_\delta$ and $\delta = 0,1,2,\dots,N$ in the interval $[a,b]$, with $i$ and $c$ denoting the number of interpolation and collocation points, respectively.

The derivation of the three-step hybrid block method with two off-step points for three different locations of off-step points through interpolation and collocation is initiated by considering two-off-step points denoted by $x_{n+p}$ and $x_{n+q}$ where $0 < p < q < 3$. Fixing two off-step points in one of the sub-intervals produces three different locations of off-step points and substituting $i = 2$ and $c = 6$ in Equations (2)-(3) yields

$$\sum_{j=0}^{7} a_j \left(\frac{x - x_n}{h}\right)^j = y_{n+\theta} \quad (4)$$

$$\sum_{j=2}^{7} a_j \frac{j(j-1)}{h^2}\left(\frac{x - x_n}{h}\right)^{j-2} = f_{n+\theta}. \quad (5)$$

The unknown coefficients $a_j$'s can be determined by solving Equations (4) and (5) simultaneously depending on the selected interpolation points, where different interpolation produces different values of $a_j$'s . The detailed derivation of the three new generalized three-step hybrid block methods (HBMs) based on the various intervals that the two off-step points are selected is discussed in the following subsections.

### 2.1 Derivation of Three-Step HBM with Two-Off-Step Points Located in $[x_n, x_{n+1}]$

Substituting the obtained values $a_j$'s into Equation (2) gives a continuous implicit hybrid three-step scheme with generalised two off-step points

$$y(x) = \sum_{j=0,p} \alpha_j(x) y_{n+j} + \sum_{j=0}^{3} \beta_j(x) f_{n+j} + \sum_{j=p,q} \beta_j(x) f_{n+j}. \quad (6)$$

with its first derivative

$$y'(x) = \sum_{j=0,p} \frac{\partial}{\partial x}\alpha_j(x) y_{n+j} + \sum_{j=0}^{3} \frac{\partial}{\partial x}\beta_j(x) f_{n+j} + \sum_{j=p,q} \frac{\partial}{\partial x}\beta_j(x) f_{n+j}. \quad (7)$$

Evaluating Equation (6) at the non-interpolating points $x_{n+\gamma}$ $(\gamma = q,1,2,3)$ and Equation (7) at $x_n$ and then combining these equations produces the first three-step block hybrid method (3S2P1) which can be represented in a matrix form as follows

$$A^{3[V1]_2} Y^{3[V1]_2} = B_1^{3[V1]_2} R_1^{3[V1]_2} + B_2^{3[V1]_2} R_2^{3[V1]_2} + h^2\left[D^{3[V1]_2} R_3^{3[V1]_2} + E^{3[V1]_2} R_4^{3[V1]_2}\right]. \quad (8)$$

Multiplying Equation (8) with inverse of $A^{3[3]_2}$ yields

$$I_{5\times5} Y^{3[V1]_2} = \bar{B}_1^{3[V1]_2} R_1^{3[V1]_2} + \bar{B}_2^{3[V1]_2} R_2^{3[V1]_2} + h^2\left[\bar{D}^{3[V1]_2} R_3^{3[V1]_2} + \bar{E}^{3[V1]_2} R_4^{3[V1]_2}\right] \quad (9)$$

which is equivalent to

$$y_{n+p} = y_n + hpy'_n + h^2[\bar{D}_{11}^{(V1)} f_n + \bar{E}_{11}^{(V1)} f_{n+p} + \bar{E}_{12}^{(V1)} f_{n+q} + \bar{E}_{13}^{(V1)} f_{n+1} + \bar{E}_{14}^{(V1)} f_{n+2} + \bar{E}_{15}^{(V1)} f_{n+3}]$$

$$y_{n+q} = y_n + hqy'_n + h^2[\bar{D}_{21}^{(V1)} f_n + \bar{E}_{21}^{(V1)} f_{n+p} + \bar{E}_{22}^{(V1)} f_{n+q} + \bar{E}_{23}^{(V1)} f_{n+1} + \bar{E}_{24}^{(V1)} f_{n+2} + \bar{E}_{25}^{(V1)} f_{n+3}]$$

$$y_{n+1} = y_n + hy'_n + h^2[\bar{D}_{31}^{(V1)} f_n + \bar{E}_{31}^{(V1)} f_{n+p} + \bar{E}_{32}^{(V1)} f_{n+q} + \bar{E}_{33}^{(V1)} f_{n+1} + \bar{E}_{34}^{(V1)} f_{n+2} + \bar{E}_{35}^{(V1)} f_{n+3}]$$

$$y_{n+2} = y_n + 2hy'_n + h^2[\bar{D}_{41}^{(V1)} f_n + \bar{E}_{41}^{(V1)} f_{n+p} + \bar{E}_{42}^{(V1)} f_{n+q} + \bar{E}_{43}^{(V1)} f_{n+1} + \bar{E}_{44}^{(V1)} f_{n+2} + \bar{E}_{45}^{(V1)} f_{n+3}]$$

$$y_{n+3} = y_n + 3hy'_n + h^2[\bar{D}_{51}^{(V1)} f_n + \bar{E}_{51}^{(V1)} f_{n+p} + \bar{E}_{52}^{(V1)} f_{n+q} + \bar{E}_{53}^{(V1)} f_{n+1} + \bar{E}_{54}^{(V1)} f_{n+2} + \bar{E}_{55}^{(V1)} f_{n+3}]$$

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where the elements of $\bar{D}_{11}^{(V1)}, \bar{D}_{21}^{(V1)}, \dots, \bar{D}_{51}^{(V1)}$ and $\bar{E}_{11}^{(V1)}, \bar{E}_{12}^{(V1)}, \dots, \bar{E}_{55}^{(V1)}$ are given below

$$\bar{D}_{11}^{(V1)} = \frac{p^2(4p^4+840q-7p^3(6+q)+14p^2(11+6q)-35p(6+11q))}{2520q},$$

$$\bar{D}_{21}^{(V1)} = \frac{q^2(-7p(-120+55q-12q^2+q^3)+2q(-105+77q-21q^2+2q^3))}{2520p},$$

$$\bar{D}_{31}^{(V1)} = \frac{(53-147q+7\ \ (-21+97q))}{2520pq},$$

$$\bar{D}_{41}^{(V1)} = \frac{2(4-21q+7p(-3+14q))}{315pq}, \bar{D}_{51}^{(V1)} = \frac{3(9-21q+7p(-3+13\ \ ))}{280pq},$$

$$\bar{E}_{11}^{(V1)} = \frac{p^2(10p^4+420q-14p^3(6+q)+21p^2(11+6q)-35p(6+11q))}{420(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{12}^{(V1)} = \frac{p^4(-105+77p-21p^2+2p^3)}{210(p-q)q(-6+11q-6\ \ ^2+q^3)},$$

$$\bar{E}_{13}^{(V1)} = -\frac{p^4(4p^3-210q-7p^2(5+q)+14p(6+5q))}{840(-1+p)(-1+q)},$$

$$\bar{E}_{14}^{(V1)} = \frac{p^4\left(4p^3-105q-7p^2(4+q)+14p(3+4q)\right)}{840(-2+p)(-2+q)},$$

$$\bar{E}_{15}^{(V1)} = -\frac{p^4(4p^3-70q-7p^2(3+q)+14p(2+3q))}{2520(-3+p)(-3+q)},$$

$$\bar{E}_{21}^{(V1)} = \frac{q^4(105-77q+21\ \ ^2-2q^3)}{210p(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{22}^{(V1)} = \frac{q^2(q(210-231q+84q^2-10q^3)+7p(55q-60-18q^2+2q^3))}{420(p-q)(-6+11q-6q^2+q^3)},$$

$$\bar{E}_{23}^{(V1)} = \frac{q^4\left(q(-84+35q-4q^2)+7p(30-10q+q^2)\right)}{840(-1+p)(-1+q)},$$

$$\bar{E}_{24}^{(V1)} = \frac{q^4(-7p(15-8q+\ \ ^2)+2q(21-14q+2q^2))}{840(-2+p)(-2+q)},$$

$$\bar{E}_{25}^{(V1)} = \frac{q^4\left(q(-28+21q-4q^2)+7p(10-6q+q^2)\right)}{2520(-3+p)(-3+q)},$$

$$\bar{E}_{31}^{(V1)} = \frac{(-53+147q)}{420p(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{32}^{(V1)} = \frac{(53-147p)}{420(p-q)q(-6+11q-6\ \ ^2+q^3)},$$

$$\bar{E}_{33}^{(V1)} = \frac{(66-119q+7p(-17+38q))}{840(-1+p)(-1+q)}, \bar{E}_{34}^{(V1)} = \frac{(p(35-91q)-17+35q)}{840(-2+p)(-2+q)},$$

$$\bar{E}_{35}^{(V1)} = \frac{(10-21q+7p(-3+8q))}{2520(p-3)(q-3)},$$

$$\bar{E}_{41}^{(V1)} = \frac{4(-4+21q)}{105p(11p-6-6p^2+p^3)(p-q)},$$

$$\bar{E}_{42}^{(V1)} = \frac{4(4-21p)}{105(p-q)q(11q-6-6q^2+q^3)},$$

$$\bar{E}_{43}^{(V1)} = \frac{2(52-56q+7p(-8+11q))}{105(-1+p)(-1+q)},$$

$$\bar{E}_{44}^{(V1)} = -\frac{2(-18+7q+7p(1+q))}{105(-2+p)(-2+q)}, \bar{E}_{45}^{(V1)} = \frac{2(-4+7pq)}{315(-3+p)(-3+q)},$$

$$\bar{E}_{51}^{(V1)} = \frac{27(7q-3)}{140p(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{52}^{(V1)} = \frac{27(3-7p)}{140(p-q)q(-6+11q-6q^2+q^3)},$$

$$\bar{E}_{53}^{(V1)} = \frac{27(18-21q+7p(-3+4q))}{280(-1+p)(-1+q)},$$

$$\bar{E}_{54}^{(V1)} = \frac{27(45+7p(-3+q)-21q)}{280(-2+p)(-2+q)}, \bar{E}_{55}^{(V1)} = \frac{3(54-21q+7p(-3+2q))}{280(-3+p)(-3+q)}.$$

Combining Equation (7) at $x_{n+\mu}$ ($\mu = p, q, 1,2,3$) and Equation (9) gives the first derivative of the block as below

$$y'_{n+p} = y'_n + h[\acute{D}_{11}^{(V1)} f_n + \acute{E}_{11}^{(V1)} f_{n+p} + \acute{E}_{12}^{(V1)} f_{n+q} + \acute{E}_{13}^{(V1)} f_{n+1} + \acute{E}_{14}^{(V1)} f_{n+2} + \acute{E}_{15}^{(V1)} f_{n+3}]$$

$$y'_{n+q} = y'_n + h[\acute{D}_{21}^{(V1)} f_n + \acute{E}_{21}^{(V1)} f_{n+p} + \acute{E}_{22}^{(V1)} f_{n+q} + \acute{E}_{23}^{(V1)} f_{n+1} + \acute{E}_{24}^{(V1)} f_{n+2} + \acute{E}_{25}^{(V1)} f_{n+3}]$$

$$y'_{n+1} = y'_n + h[\acute{D}_{31}^{(V1)} f_n + \acute{E}_{31}^{(V1)} f_{n+p} + \acute{E}_{32}^{(V1)} f_{n+q} + \acute{E}_{33}^{(V1)} f_{n+1} + \acute{E}_{34}^{(V1)} f_{n+2} + \acute{E}_{35}^{(V1)} f_{n+3}]$$

$$y'_{n+2} = y'_n + h[\acute{D}_{41}^{(V1)} f_n + \acute{E}_{41}^{(V1)} f_{n+p} + \acute{E}_{42}^{(V1)} f_{n+q} + \acute{E}_{43}^{(V1)} f_{n+1} + \acute{E}_{44}^{(V1)} f_{n+2} + \acute{E}_{45}^{(V1)} f_{n+3}]$$

$$y'_{n+3} = y'_n + h[\acute{D}_{51}^{(V1)} f_n + \acute{E}_{51}^{(V1)} f_{n+p} + \acute{E}_{52}^{(V1)} f_{n+q} + \acute{E}_{53}^{(V1)} f_{n+1} + \acute{E}_{54}^{(V1)} f_{n+2} + \acute{E}_{55}^{(V1)} f_{n+3}]$$

where the elements of $\acute{D}_{11}^{(V1)}, \acute{D}_{21}^{(V1)}, \dots, \acute{D}_{51}^{(V1)}$ and $\acute{E}_{11}^{(V1)}, \acute{E}_{12}^{(V1)}, \dots, \acute{E}_{55}^{(V1)}$ are given as

$$\acute{D}_{11}^{(V1)} = \frac{p(2p^4+180q-3p^3(6+q)+5p^2(11+6q)-10p(6+11q))}{360q},$$

$$\acute{D}_{21}^{(V1)} = \frac{q^2(q(q-2)(2q-5)+(q+3)(-20-3(q-5)q))+p(15(12+(q-6)q))}{360p},$$

$$\acute{D}_{31}^{(V1)} = \frac{(17-38q+p(-38+135q))}{360pq}, \acute{D}_{41}^{(V1)} = \frac{(p(15q-2)-2(1+q))}{45pq},$$

$$\acute{D}_{51}^{(V1)} = \frac{(162-81(3+p+q)+45(2+3q+p(3+q))-60q-60p-30pq)}{40pq},$$

$$\acute{E}_{11}^{(V1)} = \frac{p\left(10p^4+180q-12p^3(6+q)+15p^2(11+6q)-20p(6+11q)\right)}{60(-3+p)(2-3p+p^2)(p-q)},$$

$$\acute{E}_{12}^{(V1)} = \frac{p^3(55p-60-18\ \ ^2+2p^3)}{60(p-q)(-3+q)q(2-3q+q^2)},$$

$$\acute{E}_{13}^{(V1)} = \frac{p^3\left(-2p^3+60q+3\ \ ^2(5+q)-5p(6+5q)\right)}{120(-1+p)(-1+q)},$$

$$\acute{E}_{14}^{(V1)} = \frac{p^3(2p^3-30q-3\ \ ^2(4+q)+5p(3+4q))}{120(-2+p)(-2+q)},$$

$$\acute{E}_{15}^{(V1)} = \frac{p^3\left(20q-2\ \ ^3+3p^2(3+q)-5p(2+3q)\right)}{360(3-p)(3-q)},$$

$$\acute{E}_{21}^{(V1)} = \frac{q^3(60-55q+18\ \ ^2-2q^3)}{60(-3+p)p(2-3p+p^2)(p-q)},$$

$\acute{E}_{22}^{(V1)} = \frac{q^2(120+q(2(36-5q)q-165))+p(q(40+3q(4q-15))-45(q-2)^2)}{60(p-q)(q-3)(q-2)(q-1)}$,

$\acute{E}_{23}^{(V1)} = \frac{q^3\left(q((15-2q)q-30)+p(60-25q+3q^2)\right)}{120(-1+p)(-1+q)}$,

$\acute{E}_{24}^{(V1)} = \frac{q^3(p(15(-2+q)+(5-3q)q)+q(15-12q+2q^2))}{120(-2+p)(-2+q)}$,

$\acute{E}_{25}^{(V1)} = \frac{q^3((2-q)q(-5+2q)+p(20+3(-5+q)q))}{360(3-p)(3-q)}$,

$\acute{E}_{31}^{(V1)} = \frac{(-17+38q)}{60(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{32}^{(V1)} = \frac{(17-38p)}{60(p-q)(-3+q)q(2-3q+q^2)}$,

$\acute{E}_{33}^{(V1)} = \frac{(40-57q+p(-57+95q))}{120(-1+p)(-1+q)}$, $\acute{E}_{34}^{(V1)} = \frac{(-7+p(12-25q)+12q)}{120(-2+p)(-2+q)}$,

$\acute{E}_{35}^{(V1)} = \frac{(4-7p-7q+15pq)}{360(3-p)(3-q)}$, $\acute{E}_{41}^{(V1)} = \frac{4(1+q)}{15(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{42}^{(V1)} = -\frac{4(1+p)}{15(p-q)(-3+q)q(2-3q+q^2)}$,

$\acute{E}_{43}^{(V1)} = \frac{2(10-9q+p(-9+10q))}{15(-1+p)(-1+q)}$, $\acute{E}_{44}^{(V1)} = \frac{(2(11-6q)+p(-12+5q))}{15(-2+p)(-2+q)}$,

$\acute{E}_{45}^{(V1)} = \frac{2(-2+p+q)}{45(3-p)(3-q)}$, $\acute{E}_{51}^{(V1)} = \frac{9(-3+2q)}{20(-3+p)(-2+p)(-1+p)p(p-q)}$,

$\acute{E}_{52}^{(V1)} = \frac{9(3-2p)}{20(p-q)(-3+q)(-2+q)(-1+q)q}$,

$\acute{E}_{53}^{(V1)} = \frac{9\left(-54+20pq+2\ (2+p+q)-15\left(2q+p(2+q)\right)\right)}{40(-1+p)(-1+q)}$,

$\acute{E}_{54}^{(V1)} = \frac{9(54-10pq-27(1+p+q)+15(p+q+p\ \ ))}{40(-2+p)(-2+q)}$,

$\acute{E}_{55}^{(V1)} = \frac{(810+60pq-324(3+p+q)+135(2+3q+p(3+q))-60(2q+p(2+3q)))}{40(-3+p)(-3+q)}$.

2.1.1 Properties of 3S2P1 Block Method

The convergence property will be discussed for all the developed block methods in this article. This implies that the conditions for consistency and zero-stability will be investigated.

By using Taylor series expansion about $x_n$, it is found that the order of the 3S2P1 block method is $[6,6,6,6,6]^T$. Hence, the new 3S2P1 block method is consistent since its order is greater than 1.

To check if the 3S2P1 block method is zero-stable, we consider its first characteristic polynomial written as

$$\pi(\omega) = \left|\omega I_{5\times5} - \bar{B}_1^{3[V1]_2}\right|$$

$$= \left|\omega\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{pmatrix}\right|.$$

Setting $\pi(\omega) = 0$, we have

$$\omega^4(\omega-1) = 0$$

which implies $\omega = 0,0,0,0,1$. As a result, the newly developed method is zero-stable.

The same strategy as discussed above is adopted to derive the other two three-step hybrid block methods with two-off-step points located in sub-intervals $[x_{n+1}, x_{n+2}]$, and $[x_{n+2}, x_{n+3}]$.

**2.2 Derivation of Three-Step HBM with Two-Off-Step Points Located in $[x_{n+1}, x_{n+2}]$**

Using the previous procedure mentioned in Section 2.1, and substituting the obtained values $\alpha_j$'s into Equation (2) yields another different continuous implicit hybrid three-step scheme below

$$y(x) = \sum_{j=1,p} \alpha_j(x) y_{n+j} + \sum_{j=0}^{3} \beta_j(x) f_{n+j} + \sum_{j=p,q} \beta_j(x) f_{n+j} \quad (10)$$

whose first derivative is

$$y'(x) = \sum_{j=1,p} \frac{\partial}{\partial x}\alpha_j(x) y_{n+j} + \sum_{j=0}^{3} \frac{\partial}{\partial x}\beta_j(x) f_{n+j} + \sum_{j=p,q} \frac{\partial}{\partial x}\beta_j(x) f_{n+j}. \quad (11)$$

Equation (10) is then evaluated at the non-interpolating points $x_{n+\gamma}$, $(\gamma = 0, q, 2, 3)$ and Equation (11) at $x_n$ which is combined to obtain the second three-step block hybrid method (3S2P2) given by

$$A^{3[V2]_2} Y^{3[V2]_2} = B_1^{3[V2]_2} R_1^{3[V2]_2} + B_2^{3[V2]_2} R_2^{3[V2]_2} + h^2\left[D^{3[V2]_2} R_3^{3[V2]_2} + E^{3[V2]_2} R_4^{3[V2]_2}\right]. \quad (12)$$

Multiplying Equation (12) with inverse of $A^{3[V2]_2}$ leads to

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$$I_{5\times5}Y^{3[V2]_2} = \bar{B}_1^{3[V2]_2}R_1^{3[V2]_2} + \bar{B}_2^{3[V2]_2}R_2^{3[V2]_2} + h^2\left[\bar{D}^{3[V2]_2}R_3^{3[V2]_2} + \bar{E}^{3[V2]_2}R_4^{3[V2]_2}\right] \quad (13)$$

which can be written as

$$y_{n+1} = y_n + hy'_n + h^2[\bar{D}_{11}^{(V2)}f_n + \bar{E}_{11}^{(V2)}f_{n+1} + \bar{E}_{12}^{(V2)}f_{n+p} + \bar{E}_{13}^{(V2)}f_{n+q} + \bar{E}_{14}^{(V2)}f_{n+2} + \bar{E}_{15}^{(V2)}f_{n+3}]$$
$$y_{n+p} = y_n + hpy'_n + h^2[\bar{D}_{21}^{(V2)}f_n + \bar{E}_{21}^{(V2)}f_{n+1} + \bar{E}_{22}^{(V2)}f_{n+p} + \bar{E}_{23}^{(V2)}f_{n+q} + \bar{E}_{24}^{(V2)}f_{n+2} + \bar{E}_{25}^{(V2)}f_{n+3}]$$
$$y_{n+q} = y_n + hqy'_n + h^2[\bar{D}_{31}^{(V2)}f_n + \bar{E}_{31}^{(V2)}f_{n+1} + \bar{E}_{32}^{(V2)}f_{n+p} + \bar{E}_{33}^{(V2)}f_{n+q} + \bar{E}_{34}^{(V2)}f_{n+2} + \bar{E}_{35}^{(V2)}f_{n+3}]$$
$$y_{n+2} = y_n + 2hy'_n + h^2[\bar{D}_{41}^{(V2)}f_n + \bar{E}_{41}^{(V2)}f_{n+1} + \bar{E}_{42}^{(V2)}f_{n+p} + \bar{E}_{43}^{(V2)}f_{n+q} + \bar{E}_{44}^{(V2)}f_{n+2} + \bar{E}_{45}^{(V2)}f_{n+3}]$$
$$y_{n+3} = y_n + 3hy'_n + h^2[\bar{D}_{51}^{(V2)}f_n + \bar{E}_{51}^{(V2)}f_{n+1} + \bar{E}_{52}^{(V2)}f_{n+p} + \bar{E}_{53}^{(V2)}f_{n+q} + \bar{E}_{54}^{(V2)}f_{n+2} + \bar{E}_{55}^{(V2)}f_{n+3}].$$

The elements of $\bar{D}_{11}^{(V2)}, \bar{D}_{21}^{(V2)}, \dots, \bar{D}_{51}^{(V2)}$ and $\bar{E}_{11}^{(V2)}, \bar{E}_{12}^{(V2)}, \dots, \bar{E}_{55}^{(V2)}$ are provided below

$$\bar{D}_{11}^{(V2)} = \frac{(53-147q+7p(-21+97q))}{2520pq},$$
$$\bar{D}_{21}^{(V2)} = \frac{p^2(4p^4+840q-7p^3(6+q)+14p^2(11+6q)-35p(6+11q))}{2520q},$$
$$\bar{D}_{31}^{(V2)} = \frac{q^2\left(-7p(-120+55q-12q^2+q^3)+2q(-105+77q-21q^2+2q^3)\right)}{2520},$$
$$\bar{D}_{41}^{(V2)} = \frac{2(4-21q+7p(-3+14q))}{315pq}, \quad \bar{D}_{51}^{(V2)} = \frac{3(9-21q+7p(-3+13q))}{280pq},$$
$$\bar{E}_{11}^{(V2)} = \frac{(66-119q+7p(-17+38q))}{840(-1+p)(-1+q)},$$
$$\bar{E}_{12}^{(V2)} = \frac{(-53+147q)}{420p(-6+11p-6\ {}^2+p^3)(p-q)},$$
$$\bar{E}_{13}^{(V2)} = \frac{(53-14\quad)}{420(p-q)q(-6+11q-6q^2+q^3)},$$
$$\bar{E}_{14}^{(V2)} = \frac{(p(35-91q)-17+35q)}{840(-2+p)(-2+q)}, \bar{E}_{15}^{(V2)} = \frac{(10-21q+7p(-3+8q))}{2520(p-3)(q-3)},$$
$$\bar{E}_{21}^{(V2)} = -\frac{p^4(4p^3-210q-7p^2(5+q)+14p(6+5q))}{840(-1+p)(-1+q)},$$
$$\bar{E}_{22}^{(V2)} = \frac{p^2(10p^4+420q-14p^3(6+q)+21p^2(11+6q)-35p(6+11q))}{420(-6+11p-6p^2+p^3)(p-q)},$$
$$\bar{E}_{23}^{(V2)} = \frac{p^4(-105+77p-21p^2+2p^3)}{210(p-q)q(-6+11q-6q^2+q^3)},$$
$$\bar{E}_{24}^{(V2)} = \frac{p^4(4p^3-105q-7p^2(4+q)+14p(3+4q))}{840(-2+p)(-2+q)},$$
$$\bar{E}_{25}^{(V2)} = -\frac{p^4\left(4p^3-70q-7p^2(3+q)+14p(2+3q)\right)}{2520(-3+p)(-3+q)},$$
$$\bar{E}_{31}^{(V2)} = \frac{q^4(q(-84+35q-4q^2)+7p(30-10q+q^2))}{840(-1+p)(-1+q)},$$
$$\bar{E}_{32}^{(V2)} = \frac{q^4(105-77q+21\ {}^2-2q^3)}{210p(-6+11p-6p^2+p^3)(p-q)},$$
$$\bar{E}_{33}^{(V2)} = \frac{q^2(q(210-231q+84q^2-10q^3)+7p(55q-60-18q^2+2q^3))}{420(p-q)(-6+11q-6\ {}^2+q^3)},$$
$$\bar{E}_{34}^{(V2)} = \frac{q^4\left(-7p(15-8q+q^2)+2q(21-14q+2q^2)\right)}{840(-2+p)(-2+q)},$$
$$\bar{E}_{35}^{(V2)} = \frac{q^4\left(q(-28+21q-4q^2)+7p(10-6q+q^2)\right)}{2520(-3+p)(-3+q)},$$
$$\bar{E}_{41}^{(V2)} = \frac{2(52-56q+7p(-8+11q))}{105(-1+p)(-1+q)},$$
$$\bar{E}_{42}^{(V2)} = \frac{4(-4+21\quad)}{105p(11p-6-6p^2+p^3)(p-q)},$$
$$\bar{E}_{43}^{(V2)} = \frac{4(4-21p)}{105(p-q)q(11q-6-6q^2+q^3)},$$
$$\bar{E}_{44}^{(V2)} = -\frac{2(-18+7q+7p(1+q))}{105(-2+p)(-2+q)}, \bar{E}_{45}^{(V2)} = \frac{2(-4+7pq)}{315(-3+p)(-3+q)},$$
$$\bar{E}_{51}^{(V2)} = \frac{27(18-21q+7p(-3+4q))}{280(-1+p)(-1+q)},$$
$$\bar{E}_{52}^{(V2)} = \frac{27(7q-3)}{140p(-6+11p-6\ {}^2+p^3)(p-q)},$$
$$\bar{E}_{53}^{(V2)} = \frac{27(3-7p)}{140(p-q)q(-6+11q-6q^2+q^3)},$$
$$\bar{E}_{54}^{(V2)} = \frac{27(45+7p(-3+q)-21q)}{280(-2+p)(-2+q)}, \bar{E}_{55}^{(V2)} = \frac{3(54-21q+7p(-3+2q))}{280(-3+p)(-3+q)}.$$

To get the corresponding first derivative of the 3S2P2 block method, Equation (11) is evaluated at $x_{n+\mu}$ $(\mu = 1, p, q, 2, 3)$ and then combined with Equation (13) to produce

$$y'_{n+1} = y'_n + h[\acute{D}_{11}^{(V2)}f_n + \acute{E}_{11}^{(V2)}f_{n+1} + \acute{E}_{12}^{(V2)}f_{n+p} + \acute{E}_{13}^{(V2)}f_{n+q} + \acute{E}_{14}^{(V2)}f_{n+2} + \acute{E}_{15}^{(V2)}f_{n+3}]$$
$$y'_{n+p} = y'_n + h[\acute{D}_{21}^{(V2)}f_n + \acute{E}_{21}^{(V2)}f_{n+1} + \acute{E}_{22}^{(V2)}f_{n+p} + \acute{E}_{23}^{(V2)}f_{n+q} + \acute{E}_{24}^{(V2)}f_{n+2} + \acute{E}_{25}^{(V2)}f_{n+3}]$$
$$y'_{n+q} = y'_n + h[\acute{D}_{31}^{(V2)}f_n + \acute{E}_{31}^{(V2)}f_{n+1} + \acute{E}_{32}^{(V2)}f_{n+p} + \acute{E}_{33}^{(V2)}f_{n+q} + \acute{E}_{34}^{(V2)}f_{n+2} + \acute{E}_{35}^{(V2)}f_{n+3}]$$
$$y'_{n+2} = y'_n + h[\acute{D}_{41}^{(V2)}f_n + \acute{E}_{41}^{(V2)}f_{n+1} + \acute{E}_{42}^{(V2)}f_{n+p} + \acute{E}_{43}^{(V2)}f_{n+q} + \acute{E}_{44}^{(V2)}f_{n+2} + \acute{E}_{45}^{(V2)}f_{n+3}]$$
$$y'_{n+3} = y'_n + h[\acute{D}_{51}^{(V2)}f_n + \acute{E}_{51}^{(V2)}f_{n+1} + \acute{E}_{52}^{(V2)}f_{n+p} + \acute{E}_{53}^{(V2)}f_{n+q} + \acute{E}_{54}^{(V2)}f_{n+2} + \acute{E}_{55}^{(V2)}f_{n+3}]$$

where $\acute{D}_{11}^{(V2)}, \acute{D}_{21}^{(V2)}, \dots, \acute{D}_{51}^{(V2)}$ and $\acute{E}_{11}^{(V2)}, \acute{E}_{12}^{(V2)}, \dots, \acute{E}_{55}^{(V2)}$ are given below

$$\acute{D}_{11}^{(V2)} = \frac{(17-38q+p(-38+135q))}{360pq},$$
$$\acute{D}_{21}^{(V2)} = \frac{p(2p^4+180q-3p^3(6+q)+5p^2(11+6q)-10p(6+11q))}{360q},$$

$\acute{D}_{31}^{(V2)} = \frac{q^2(q(q-2)(2q-5)+(q+3)(-20-3(q-5)q))+p(15(12+(q-6)q))}{360}$,

$\acute{D}_{41}^{(V2)} = \frac{(p(15q-2)-2(1+q))}{45pq}$,

$\acute{D}_{51}^{(V2)} = \frac{(162-81\,(3+p+q)+45\,(2+3q+p(3+q))-60q-60p-30pq)}{40pq}$,

$\acute{E}_{11}^{(V2)} = \frac{(40-57q+p(-57+95q))}{120(-1+p)(-1+q)}$,

$\acute{E}_{12}^{(V2)} = \frac{(-17+38q)}{60(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{13}^{(V2)} = \frac{(17-38p)}{60(p-q)(-3+q)q(2-3q+q^2)}$,

$\acute{E}_{14}^{(V2)} = \frac{(-7+p(12-25q)+12q)}{120(-2+p)(-2+q)}$, $\acute{E}_{15}^{(V2)} = \frac{(4-7p-7q+15pq)}{360(3-p)(3-q)}$,

$\acute{E}_{21}^{(V2)} = \frac{p^3(-2p^3+60q+3p^2(5+q)-5p(6+5q))}{120(-1+p)(-1+q)}$,

$\acute{E}_{22}^{(V2)} = \frac{p\left(10p^4+180q-12p^3(6+q)+15p^2(11+6q)-20p(6+11q)\right)}{60(-3+p)(2-3p+p^2)(p-q)}$,

$\acute{E}_{23}^{(V2)} = \frac{p^3(55p-60-18p^2+2p^3)}{60(p-q)(-3+q)q(2-3q+q^2)}$,

$\acute{E}_{24}^{(V2)} = \frac{p^3\left(2p^3-30q-3p^2(4+q)+5p(3+4q)\right)}{120(-2+p)(-2+q)}$,

$\acute{E}_{25}^{(V2)} = \frac{p^3\left(20q-2p^3+3p^2(3+q)-5p(2+3q)\right)}{360(3-p)(3-q)}$,

$\acute{E}_{31}^{(V2)} = \frac{q^3\left(q((15-2q)q-30)+p(60-25q+3q^2)\right)}{120(-1+p)(-1+q)}$,

$\acute{E}_{32}^{(V2)} = \frac{q^3(60-55q+18q^2-2q^3)}{60(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{33}^{(V2)} = \frac{q^2(120+q\,(2(36-5q)\,q-165))+p\,(q(40+3q\,(4q-15))-45(q-2)^2)}{60\,(p-q)\,(q-3)\,(q-2)(q-1)}$,

$\acute{E}_{34}^{(V2)} = \frac{q^3\left(p(15(-2+q)+(5-3q)q)+q(15-12q+2q^2)\right)}{120(-2+p)(-2+q)}$,

$\acute{E}_{35}^{(V2)} = \frac{q^3((2-q)q(-5+2q)+p(20+3(-5+q)q))}{360(3-p)(3-q)}$,

$\acute{E}_{41}^{(V2)} = \frac{2(10-9q+p(-9+10q))}{15(-1+p)(-1+q)}$,

$\acute{E}_{42}^{(V2)} = \frac{4(1+q)}{15(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{43}^{(V2)} = -\frac{4(1+p)}{15(p-q)(-3+q)q(2-3q+q^2)}$,

$\acute{E}_{44}^{(V2)} = \frac{(2(11-6q)+p(-12+5q))}{15(-2+p)(-2+q)}$, $\acute{E}_{45}^{(V2)} = \frac{2(-2+p+q)}{45(3-p)(3-q)}$,

$\acute{E}_{51}^{(V2)} = \frac{9\left(-54+20pq+27(2+p+q)-15(2q+p(2+q))\right)}{40(-1+p)(-1+q)}$,

$\acute{E}_{52}^{(V2)} = \frac{9(-3+2q)}{20(-3+p)(-2+p)(-1+p)p(p-q)}$,

$\acute{E}_{53}^{(V2)} = \frac{9(3-2p)}{20(p-q)(-3+q)(-2+q)(-1+q)q}$,

$\acute{E}_{54}^{(V2)} = \frac{9(54-10pq-27(1+p+q)+15(p+q+pq))}{40(-2+p)(-2+q)}$,

$\acute{E}_{55}^{(V2)} = \frac{(810+60pq-324(3+p+q)+135(2+3q+p(3+q))-60(2q+p(2+3q)))}{40(-3+p)(-3+q)}$

.

2.2.1 Properties of 3S2P2 Method

It was found that the 3S2P2 block method has same order $[6,6,6,6,6]^T$ and shares the same first characteristic polynomial as 3S2P1 which implies $\omega = 0,0,0,0,1$. Therefore, this method is also zero-stable. Furthermore, it is convergent because it fulfils the consistency and zero stability definitions.

**2.3 Derivation of Three-Step HBM with Two-Off-Step Points Located in $[x_{n+2}, x_{n+3}]$**

Similarly, replacing the values $\alpha_j$'s in Equation (2) gives the third continuous implicit hybrid three-step scheme

$$y(x) = \sum_{j=p,q} \alpha_j(x) y_{n+j} + \sum_{j=0}^{3} \beta_j(x) f_{n+j} + \sum_{j=p,q} \beta_j(x) f_{n+j}. \quad (14)$$

Differentiating Equation (14) once results in

$$y'(x) = \sum_{j=p,q} \frac{\partial}{\partial x} \alpha_j(x) y_{n+j} + \sum_{j=0}^{3} \frac{\partial}{\partial x} \beta_j(x) f_{n+j} + \sum_{j=p,q} \frac{\partial}{\partial x} \beta_j(x) f_{n+j}. \quad (15)$$

Combining the evaluation of Equation (14) at the non-interpolating points $x_{n+\gamma}$, $(\gamma = 0,1,2,3)$ and Equation (15) at $x_n$ produces the last three-step block hybrid method (3S2P3) as follows

$$A^{3[V3]_2} Y^{3[V3]_2} = B_1^{3[V3]_2} R_1^{3[V3]_2} + B_2^{3[V3]_2} R_2^{3[V3]_2} + h^2 \left[ D^{3[V3]_2} R_3^{3[V3]_2} + E^{3[V3]_2} R_4^{3[V3]_2} \right], \quad (16)$$

which can be simplified as

$$I_{5\times5} Y^{3[V3]_2} = \bar{B}_1^{3[V3]_2} R_1^{3[V3]_2} + \bar{B}_2^{3[V3]_2} R_2^{3[V3]_2} + h^2 \left[ \bar{D}^{3[V3]_2} R_3^{3[V3]_2} + \bar{E}^{3[V3]_2} R_4^{3[V3]_2} \right] \quad (17)$$

after being multiplied with the inverse of $A^{3[V3]_2}$.

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Expressing Equation (17) as a system of equations gives

$$y_{n+1} = y_n + hy'_n + h^2[\bar{D}_{11}^{(V3)} f_n + \bar{E}_{11}^{(V3)} f_{n+1} + \bar{E}_{12}^{(V3)} f_{n+2} + \bar{E}_{13}^{(V3)} f_{n+p} + \bar{E}_{14}^{(V3)} f_{n+q} + \bar{E}_{15}^{(V2)} f_{n+3}]$$

$$y_{n+2} = y_n + 2hy'_n + h^2[\bar{D}_{21}^{(V3)} f_n + \bar{E}_{21}^{(V3)} f_{n+1} + \bar{E}_{22}^{(V3)} f_{n+2} + \bar{E}_{23}^{(V3)} f_{n+p} + \bar{E}_{24}^{(V3)} f_{n+q} + \bar{E}_{25}^{(V3)} f_{n+3}]$$

$$y_{n+p} = y_n + hpy'_n + h^2[\bar{D}_{31}^{(V3)} f_n + \bar{E}_{31}^{(V3)} f_{n+1} + \bar{E}_{32}^{(V3)} f_{n+2} + \bar{E}_{33}^{(V3)} f_{n+p} + \bar{E}_{34}^{(V3)} f_{n+q} + \bar{E}_{35}^{(V3)} f_{n+3}]$$

$$y_{n+q} = y_n + hqy'_n + h^2[\bar{D}_{41}^{(V3)} f_n + \bar{E}_{41}^{(V3)} f_{n+1} + \bar{E}_{42}^{(V3)} f_{n+2} + \bar{E}_{43}^{(V3)} f_{n+p} + \bar{E}_{44}^{(V3)} f_{n+q} + \bar{E}_{45}^{(V2)} f_{n+3}]$$

$$y_{n+3} = y_n + 3hy'_n + h^2[\bar{D}_{51}^{(V3)} f_n + \bar{E}_{51}^{(V3)} f_{n+1} + \bar{E}_{52}^{(V3)} f_{n+2} + \bar{E}_{53}^{(V3)} f_{n+p} + \bar{E}_{54}^{(V3)} f_{n+q} + \bar{E}_{55}^{(V3)} f_{n+3}]$$

where the elements of $\bar{D}_{11}^{(V3)}, \bar{D}_{21}^{(V3)}, \dots, \bar{D}_{51}^{(V3)}$ and $\bar{E}_{11}^{(V3)}, \bar{E}_{12}^{(V3)}, \dots, \bar{E}_{55}^{(V3)}$ are given below

$$\bar{D}_{11}^{(V3)} = \frac{(53-147q+7p(-21+97q))}{2520pq},$$

$$\bar{D}_{21}^{(V3)} = \frac{2(4-21q+7p(-3+14\ \ ))}{315pq},$$

$$\bar{D}_{31}^{(V3)} = \frac{p^2(4p^4+840q-7p^3(6+q)+14p^2(11+6q)-35p(6+11q))}{2520q},$$

$$\bar{D}_{41}^{(V3)} = \frac{q^2(-7p(-120+55q-12q^2+q^3)+2q(-105+77q-21q^2+2q^3))}{2520p},$$

$$\bar{D}_{51}^{(V3)} = \frac{3(9-21q+7p(-3+13\ \ ))}{280pq}, \quad \bar{E}_{11}^{(V3)} = \frac{(66-119q+7p(-17+38q))}{840(-1+p)(-1+q)}, \bar{E}_{12}^{(V3)} = \frac{(p(35-91\ \ )-17+3\ \ \ )}{840(-2+p)(-2+q)},$$

$$\bar{E}_{13}^{(V3)} = \frac{(-53+147q)}{420p(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{14}^{(V3)} = \frac{(53-147p)}{420(p-q)q(-6+11q-6q^2+q^3)},$$

$$\bar{E}_{15}^{(V3)} = \frac{(10-21q+7p(-3+8q))}{2520(p-3)(q-3)}, \bar{E}_{21}^{(V3)} = \frac{2(52-56q+7p(-8+11q))}{105(-1+p)(-1+q)},$$

$$\bar{E}_{22}^{(V3)} = -\frac{2(-18+7q+7p(1+q))}{105(-2+p)(-2+q)},$$

$$\bar{E}_{23}^{(V3)} = \frac{4(-4+21\ \ )}{105p(11p-6-6p^2+p^3)(p-q)},$$

$$\bar{E}_{24}^{(V3)} = \frac{4(4-21p)}{105(p-q)q(11q-6-6q^2+q^3)}, \bar{E}_{25}^{(V3)} = \frac{2(-4+7p\ \ )}{315(-3+p)(-3+q)},$$

$$\bar{E}_{31}^{(V3)} = -\frac{p^4(4p^3-210q-7p^2(5+q)+14p(6+5q))}{840(-1+p)(-1+q)},$$

$$\bar{E}_{32}^{(V3)} = \frac{p^4(4p^3-105q-7p^2(4+q)+14p(3+4q))}{840(-2+p)(-2+q)},$$

$$\bar{E}_{33}^{(V3)} = \frac{p^2(10p^4+420q-14p^3(6+q)+21p^2(11+6q)-35p(6+11q))}{420(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{34}^{(V3)} = \frac{p^4(-105+77p-21\ \ ^2+2p^3)}{210(p-q)q(-6+11q-6q^2+q^3)},$$

$$\bar{E}_{35}^{(V3)} = -\frac{p^4(4p^3-70q-7p^2(3+q)+14p(2+3q))}{2520(-3+p)(-3+q)},$$

$$\bar{E}_{41}^{(V3)} = \frac{q^4(q(-84+35q-4q^2)+7p(30-10q+q^2))}{840(-1+p)(-1+q)},$$

$$\bar{E}_{42}^{(V3)} = \frac{q^4(-7p(15-8q+q^2)+2q(21-14q+2q^2))}{840(-2+p)(-2+q)},$$

$$\bar{E}_{43}^{(V3)} = \frac{q^4(105-77q+21q^2-2q^3)}{210p(-6+11p-6\ \ ^2+p^3)(p-q)},$$

$$\bar{E}_{44}^{(V3)} = \frac{q^2(q(210-231q+84q^2-10q^3)+7p(55q-60-18\ \ ^2+2q^3))}{420(p-q)(-6+11q-6q^2+q^3)},$$

$$\bar{E}_{45}^{(V3)} = \frac{q^4(q(-28+21q-4q^2)+7p(10-6q+\ \ ^2))}{2520(-3+p)(-3+q)},$$

$$\bar{E}_{51}^{(V3)} = \frac{27(18-21q+7p(-3+4q))}{280(-1+p)(-1+q)}, \bar{E}_{52}^{(V3)} = \frac{27(45+7p(-3+q)-21q)}{280(-2+p)(-2+q)},$$

$$\bar{E}_{53}^{(V3)} = \frac{27(7q-3)}{140p(-6+11p-6p^2+p^3)(p-q)},$$

$$\bar{E}_{54}^{(V3)} = \frac{27(3-7p)}{140(p-q)q(-6+11q-6q^2+q^3)},$$

$$\bar{E}_{55}^{(V3)} = \frac{3(54-21q+7p(-3+2q))}{280(-3+p)(-3+q)}.$$

Combining Equation (15) at $x_{n+\mu}$, $(\mu = 1,2,p,q,3)$ and Equation (17) gives the first derivative of the 3S2P3 block method as

$$y'_{n+1} = y'_n + h[\acute{D}_{11}^{(V3)} f_n + \acute{E}_{11}^{(V3)} f_{n+1} + \acute{E}_{12}^{(V3)} f_{n+2} + \acute{E}_{13}^{(V3)} f_{n+p} + \acute{E}_{14}^{(V3)} f_{n+q} + \acute{E}_{15}^{(V3)} f_{n+3}]$$

$$y'_{n+2} = y'_n + h[\acute{D}_{21}^{(V3)} f_n + \acute{E}_{21}^{(V3)} f_{n+1} + \acute{E}_{22}^{(V3)} f_{n+2} + \acute{E}_{23}^{(V3)} f_{n+p} + \acute{E}_{24}^{(V3)} f_{n+q} + \acute{E}_{25}^{(V3)} f_{n+3}]$$

$$y'_{n+p} = y'_n + h[\acute{D}_{31}^{(V3)} f_n + \acute{E}_{31}^{(V3)} f_{n+1} + \acute{E}_{32}^{(V3)} f_{n+2} + \acute{E}_{33}^{(V3)} f_{n+p} + \acute{E}_{34}^{(V3)} f_{n+q} + \acute{E}_{35}^{(V3)} f_{n+3}]$$

$$y'_{n+q} = y'_n + h[\acute{D}_{41}^{(V3)} f_n + \acute{E}_{41}^{(V3)} f_{n+1} + \acute{E}_{42}^{(V3)} f_{n+2} + \acute{E}_{43}^{(V3)} f_{n+p} + \acute{E}_{44}^{(V3)} f_{n+q} + \acute{E}_{45}^{(V3)} f_{n+3}]$$

$$y'_{n+3} = y'_n + h[\acute{D}_{51}^{(V3)} f_n + \acute{E}_{51}^{(V3)} f_{n+1} + \acute{E}_{52}^{(V3)} f_{n+2} + \acute{E}_{53}^{(V3)} f_{n+p} + \acute{E}_{54}^{(V3)} f_{n+q} + \acute{E}_{55}^{(V3)} f_{n+3}]$$

where $\acute{D}_{11}^{(V3)}, \acute{D}_{21}^{(V3)}, \dots, \acute{D}_{51}^{(V3)}$ and $\acute{E}_{11}^{(V3)}, \acute{E}_{12}^{(V3)}, \dots, \acute{E}_{55}^{(V3)}$ are given below

$$\acute{D}_{11}^{(V3)} = \frac{(17-38q+p(-38+135q))}{360pq}, \acute{D}_{21}^{(V3)} = \frac{(p(15q-2)-2(1+q))}{45pq},$$

$$\acute{D}_{31}^{(V3)} = \frac{p(2p^4+180q-3p^3(6+q)+5p^2(11+6q)-10p(6+11q))}{360q},$$

$$\acute{D}_{41}^{(V3)} = \frac{q^2(q(q-2)(2q-5)+(q+3)(-20-3(q-5)q))+p(15(12\ \ (q-6)q))}{360p},$$

$$\acute{D}_{51}^{(V3)} = \frac{(162-8\ \ (3+p+q)+45(2+3q+p(3+q))-60q-60p-30pq)}{40pq},$$

$\acute{E}_{11}^{(V3)} = \frac{(40-57q+p(-57+95q))}{120(-1+p)(-1+q)}$, $\acute{E}_{12}^{(V3)} = \frac{(-7+p(12-25q)+12q)}{120(-2+p)(-2+q)}$,

$\acute{E}_{13}^{(V3)} = \frac{(-17+38q)}{60(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{14}^{(V3)} = \frac{(17-38\ \ )}{60(p-q)(-3+q)q(2-3q+q^2)}$, $\acute{E}_{15}^{(V3)} = \frac{(4-7p-7q+15p\ \ )}{360(3-p)(3-q)}$,

$\acute{E}_{21}^{(V3)} = \frac{2(10-9q+p(-9+10q))}{15(-1+p)(-1+q)}$, $\acute{E}_{22}^{(V3)} = \frac{(2(11-6q)+p(-12+5q))}{15(-2+p)(-2+q)}$,

$\acute{E}_{23}^{(V3)} = \frac{4(1+q)}{15(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{24}^{(V3)} = -\frac{4(1+p)}{15(p-q)(-3+q)q(2-3q+q^2)}$, $\acute{E}_{25}^{(V3)} = \frac{2(-2+p+q)}{45(3-p)(3-q)}$,

$\acute{E}_{31}^{(V3)} = \frac{p^3(-2p^3+60q+3p^2(5+q)-5p(6+5q))}{120(-1+p)(-1+q)}$,

$\acute{E}_{32}^{(V3)} = \frac{p^3(2p^3-30q-3p^2(4+q)+5p(3+4q))}{120(-2+p)(-2+q)}$,

$\acute{E}_{33}^{(V3)} = \frac{p(10p^4+180q-12p^3(6+q)+15p^2(11+6q)-20p(6+11q))}{60(-3+p)(2-3p+p^2)(p-q)}$,

$\acute{E}_{34}^{(V3)} = \frac{p^3(55p-60-18\ \ ^2+2p^3)}{60(p-q)(-3+q)q(2-3q+q^2)}$,

$\acute{E}_{35}^{(V3)} = \frac{p^3(20q-2\ \ ^3+3p^2(3+q)-5p(2+3q))}{360(3-p)(3-q)}$,

$\acute{E}_{41}^{(V3)} = \frac{q^3(q((15-2q)q-30)+p(60-25q+3q^2))}{120(-1+p)(-1+q)}$,

$\acute{E}_{42}^{(V3)} = \frac{q^3(p(15(-2+q)+(5-3q)q)+q(15-12q+2q^2))}{120(-2+p)(-2+q)}$,

$\acute{E}_{43}^{(V3)} = \frac{q^3(60-55q+18q^2-2q^3)}{60(-3+p)p(2-3p+p^2)(p-q)}$,

$\acute{E}_{44}^{(V3)} = \frac{q^2(120+q(2(36-5q)q-165))+p(q(40+3q(4q-15))-45(q-2)^2)}{60(p-q)(q-3)(q-2)(q-1)}$,

$\acute{E}_{45}^{(V3)} = \frac{q^3((2-q)q(-5+2q)+p(20+3(-5+q)q))}{360(3-p)(3-q)}$,

$\acute{E}_{51}^{(V3)} = \frac{9(-54+20pq+2\ \ (2+p+q)-15(2q+p(2+q)))}{40(-1+p)(-1+q)}$,

$\acute{E}_{52}^{(V3)} = \frac{9(54-10pq-27(1+p+q)+15(p+q+p\ \ ))}{40(-2+p)(-2+q)}$,

$\acute{E}_{53}^{(V3)} = \frac{9(-3+2q)}{20(-3+p)(-2+p)(-1+p)p(p-q)}$,

$\acute{E}_{54}^{(V3)} = \frac{9(3-2p)}{20(p-q)(-3+q)(-2+q)(-1+q)q}$,

$\acute{E}_{55}^{(V3)} = \frac{(810+60pq-324(3+p+q)+135(2+3q+p(3+q))-60(2q+p(2+3q)))}{40(-3+p)(-3+q)}$.

2.3.1 Properties of 3S2P3 Method

It is also found that the order of this 3S2P3 block method is the same as the previous two methods which implies its consistency. In addition, the method satisfies zero-stability condition as its roots are $\omega = 0,0,0,0,1$. Consequently, it is convergent.

## 3. NUMERICAL EXPERIMENT

To demonstrate the capability of the newly developed methods in estimating numerical solution, they are employed to solve second order IVPs considered by previous methods in literature for the purpose of comparison. The accuracy of methods in terms of absolute error is shown in Tables 1 – 4.

### 3.1 Tested Problems

**Problem 1:** $y'' - y = 0,\ y(0) = 1, y'(0) = 1, 0 < x < 1$ with $h = \frac{1}{10}$.
Exact solution: $y(x) = e^x$. Source: [14, 15].

**Problem 2:** $y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0,\ y(1) = 1, y'(1) = 1$ with $h = \frac{1}{320}$.
Exact solution: $y(x) = \frac{5}{3x} - \frac{4}{x^2}$. Source: [16, 17].

**Problem 3:** $y'' - x(y')^2 = 0,\ y(0) = 1, y'(0) = \frac{1}{2}, 0 < x < 1$ with $h = \frac{1}{10}$.
Exact solution: $y(x) = 1 + \frac{1}{2} ln\left|\frac{2+x}{2-x}\right|$. Source: [18, 19].

**Problem 4:** $y'' - y' = 0,\ y(0) = 1, y'(0) = -1, 0 < x < 1$ with $h = \frac{1}{10}$.
Exact solution: $y(x) = 1 - e^x$. Source: [20, 21].

*Table 1. Comparison Of Absolute Errors Obtained By New Hybrid Block Methods With The Existing Method For Problem 1*

| $x$ | 3S2P1 $p = \frac{1}{16}, q = \frac{1}{3}$ | 3S2P2 $p = \frac{17}{16}, q = \frac{5}{4}$ | 3S2P3 $p = \frac{7}{3}, q = \frac{5}{2}$ | Kuboye et al. [14], $h = 0.01$ | Sagir [15] |
|---|---|---|---|---|---|
| 0.1 | 2.411404e-13 | 3.638867e-12 | 2.431033e-11 | 6.143119e-11 | - |
| 0.2 | 3.061329e-12 | 8.805845e-12 | 6.186562e-11 | 1.111704e-10 | - |
| 0.3 | 1.063993e-11 | 2.118661e-11 | 9.805579e-11 | 1.508866e-10 | 5.7600e-10 |
| 0.4 | 5.192646e-11 | 4.530643e-11 | 1.611751e-10 | 1.820369e-10 | 1.6413e-09 |

| | | | | | |
|---|---|---|---|---|---|
| 0.5 | 8.894574e-11 | 7.189360e-11 | 2.434584e-10 | 2.058921e-10 | 1.7001e-09 |
| 0.6 | 1.498492e-10 | 1.088187e-10 | 3.254994e-10 | 2.235584e-10 | 2.3905e-09 |
| 0.7 | 2.493450e-10 | 1.623928e-10 | 4.458243e-10 | 2.359977e-10 | 3.4705e-09 |
| 0.8 | 3.448744e-10 | 2.203104e-10 | 5.943015e-10 | 2.440449e-10 | 4.4925e-09 |
| 0.9 | 4.748948e-10 | 2.934168e-10 | 7.451115e-10 | 2.484232e-10 | 4.1569e-09 |
| 1.0 | 6.597363e-10 | 3.904628e-10 | 9.506582e-10 | 2.497582e-10 | 4.4590e-09 |

*Table 2. Comparison Of Absolute Errors Obtained By New Hybrid Block Methods With The Existing Method For Problem 2*

| $x$ | 3S2P1 $p=\frac{95}{100}, q=\frac{99}{100}$ | 3S2P2 $p=\frac{1001}{1000}, q=\frac{3}{2}$ | 3S2P3 $p=\frac{641}{320}, q=\frac{11}{5}$ | Badmus [16] | Anake [17] |
|---|---|---|---|---|---|
| 1.0031 | 3.108624e-15 | 2.886580e-15 | 2.153833e-14 | 8.30000e-08 | 7.6998408e-10 |
| 1.0063 | 8.881784e-15 | 7.771561e-15 | 5.329071e-14 | 1.16000e-06 | 4.1327297e-08 |
| 1.0094 | 1.065814e-14 | 1.043610e-14 | 7.949197e-14 | 6.63000e-06 | 1.0432657e-07 |
| 1.0125 | 1.265654e-14 | 1.332268e-14 | 9.925394e-14 | 9.49100e-06 | 1.8729474e-07 |
| 1.0156 | 1.798561e-14 | 1.754152e-14 | 1.294520e-13 | 1.95350e-06 | 2.8933289e-07 |
| 1.0188 | 1.887379e-14 | 2.020606e-14 | 1.536549e-13 | 9.41600e-06 | 4.0957480e-07 |
| 1.0219 | 2.042810e-14 | 2.264855e-14 | 1.727507e-13 | 4.65050e-05 | 5.4718601e-07 |
| 1.0250 | 2.442491e-14 | 2.708944e-14 | 2.007283e-13 | 4.71220e-05 | 7.0136241e-07 |
| 1.0281 | 2.509104e-14 | 2.930989e-14 | 2.238210e-13 | 1.86926e-04 | 8.7132936e-07 |
| 1.0313 | 2.597922e-14 | 3.175238e-14 | 2.415845e-13 | 4.43321e-04 | 9.9304354e-07 |

*Table 3. Comparison Of Absolute Errors Obtained By New Hybrid Block Methods With The Existing Method For Problem 3*

| $x$ | 3S2P1 $p=\frac{1}{20}, q=\frac{1}{5}$ | 3S2P2 $p=\frac{101}{100}, q=\frac{19}{10}$ | 3S2P3 $p=\frac{7}{3}, q=\frac{5}{2}$ | Ukpebor [18] | Adeniyi and Alabi [19] |
|---|---|---|---|---|---|
| 0.1 | 3.429257e-12 | 5.658540e-11 | 2.914147e-10 | 1.689804e-11 | 0.1708719055e-09 |
| 0.2 | 2.525757e-11 | 1.484519e-10 | 7.425338e-10 | 8.055976e-10 | 0.6836010114e-08 |
| 0.3 | 1.186042e-10 | 2.934510e-10 | 1.175723e-09 | 8.615662e-09 | 0.1555757709e-07 |
| 0.4 | 4.941714e-10 | 7.443595e-10 | 3.131032e-09 | 4.024075e-08 | 0.2880198295e-07 |
| 0.5 | 6.762457e-10 | 1.424695e-09 | 6.066527e-09 | 1.406582e-07 | 0.4802328029e-07 |
| 0.6 | 1.903629e-09 | 2.406246e-09 | 8.970254e-09 | 3.816098e-07 | 0.7628531256e-07 |
| 0.7 | 1.062657e-09 | 1.584668e-09 | 1.714980e-08 | 9.264709e-07 | 0.1157914170e-06 |
| 0.8 | 1.021616e-09 | 1.890521e-09 | 3.124468e-08 | 2.002368e-06 | 0.1727046080e-06 |
| 0.9 | 2.313574e-09 | 3.505337e-09 | 4.510866e-08 | 4.118350e-06 | 0.2561456831e-06 |
| 1.0 | 2.140793e-08 | 1.449912e-08 | 1.084217e-07 | 7.978886e-06 | 0.3815695118e-06 |

*Table 4. Comparison Of Absolute Errors Obtained By New Hybrid Block Methods With The Existing Method For Problem 4*

| $x$ | 3S2P1, $p=\frac{1}{16}, q=\frac{1}{3}$ | 3S2P2 $p=\frac{17}{16}, q=\frac{5}{4}$ | 3S2P3 $p=\frac{201}{100}, q=\frac{101}{50}$ | Abhulimen and Aigbiremhon [20] | Mohammed et.al [21] |
|---|---|---|---|---|---|
| 0.1 | 2.441242e-13 | 3.774661e-12 | 5.805935e-10 | 7.281999e-08 | 5.7260e-06 |
| 0.2 | 3.075817e-12 | 9.462542e-12 | 1.465301e-09 | 3.221988e-07 | 6.6391e-06 |
| 0.3 | 8.389622e-12 | 2.105421e-11 | 2.158797e-09 | 7.844950e-07 | 7.0283e-06 |
| 0.4 | 1.251230e-10 | 1.283023e-10 | 2.792141e-09 | 1.502293e-06 | 7.4539e-06 |
| 0.5 | 2.775233e-10 | 2.913204e-10 | 3.820902e-09 | 2.523574e-06 | 7.8935e-06 |
| 0.6 | 4.255221e-10 | 4.643212e-10 | 4.573249e-09 | 3.904267e-06 | 8.1942e-06 |
| 0.7 | 7.864138e-10 | 8.744203e-10 | 5.221467e-09 | 5.709429e-06 | 8.1810e-06 |
| 0.8 | 1.191802e-09 | 1.324888e-09 | 6.381199e-09 | 8.012196e-06 | 8.1810e-06 |
| 0.9 | 1.612248e-09 | 1.813063e-09 | 7.144361e-09 | 1.089732e-05 | 8.1730e-06 |
| 1.0 | 2.343279e-09 | 2.648069e-09 | 7.750545e-09 | 1.446283e-05 | 8.1650e-06 |

## 5. CONCLUSION

This article has proposed three new HBMs with generalized two off-step points for solving second IVPs. The aim of identifying which interval produces the best results led to the results showing comparable performance between all three HBMs. Furthermore, the convergence properties of each HBM was investigated and the block methods were found to consistent, zero-stable, and hence, convergent. In addition, the new HBMs performed better in terms of absolute error in comparison to methods of higher order and methods using smaller step-size in their implementation such as [14]. Overall, this article has successfully introduced new numerical methods for the solution of second order IVPs with better accuracy.

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