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THE EFFECTIVENESS OF ROBUST MIXED EWMA-CUSUM CONTROL CHART ON *G*-AND-*H* DISTRIBUTION

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ABSTRACT

Cumulative sum (CUSUM) chart and exponentially weighted moving average (EWMA) chart are popularly used in statistical process control (SPC) as they can quickly detect small shifts in the process mean. Recently, a Mixed EWMA-CUSUM (MEC) control chart was introduced for better detection of small shifts. Like the EWMA and CUSUM control charts, the MEC chart relies on the normality assumption for optimal performances. In the case that the underlying distribution of the data is non-normal, the chart may no longer be effective in signaling a true out-of-control process. Therefore, this paper proposed the use of a median estimator for Phase II monitoring of location via the MEC chart. The performance of this robust MEC control chart was tested on various *g*-and-*h* distributions in terms of the average run length (ARL). It has been found to perform well regardless of the distributional shapes compared to the standard MEC chart which uses the mean as the estimator.

Keywords: *Average Run Length, Mixed EWMA-CUSUM Control Chart, Normality, Robust Estimator, Statistical Process Control,*

1. INTRODUCTION

A control chart is a powerful tool for monitoring changes in the process. One of the earliest and most popular types of control charts is the Shewhart chart, created by Walter A. Shewhart [1]. However, this chart is not effective for identifying small or moderate shifts in process parameters as it only uses the most recent observation in the process [3-6]. Since then, CUSUM and EWMA control charts have been proposed for better detection of small and moderate shifts.

Unlike the Shewhart chart, the CUSUM and EWMA control charts are designed to include not only the most recent observation but also past observations in the process. As a result, they are more sensitive and effective than the Shewhart chart in detecting small to moderate shifts. Moreover, Abbas et al. [7] introduce the mixed EWMA-CUSUM (MEC) control chart, which combines the benefits of the EWMA and CUSUM charts to increase the performance of both in detecting small and moderate shifts. By combining the two charts' salient features, the MEC chart can detect shift quicker than the EWMA and CUSUM charts [8].

It is known that standard estimators, i.e., the mean and standard deviation, are sensitive to outliers, which can lead to inaccurate process monitoring results. Using robust estimators, such as the median, can mitigate this problem and improve the performance of control charts. The median is a robust estimator with the highest possible breakdown point, i.e., 50%, meaning it can withstand a large proportion of outliers without significantly disrupting the estimation [9]. In the context of the MEC control chart, using the median instead of the mean estimator can improve the robustness of the chart and make it more reliable in the presence of outliers This approach has been explored in previous studies on CUSUM and

EWMA charts, which have shown that using median-based estimators can improve the performance of the charts under non-normality and provide better protection against small and moderate shifts in process parameters [10-13]. By extending the use of median estimators to the MEC control chart, it may be possible to improve further its performance in detecting small and moderate process shifts.

The following sections provide more details on the structures of the MEC chart and its underlying CUSUM and EWMA components for process location monitoring.

2. METHODOLOGY

2.1 Cumulative sum control chart

The CUSUM control chart is particularly effective in detecting small shifts in a process and was first introduced by Page [13] in 1954. The CUSUM chart is based on analyzing the cumulative data points of present and previous samples rather than analyzing the mean of each subgroup individually. This chart can provide a more accurate indication of changes in the process, as it considers the overall trend of the data rather than just the mean of each subgroup which is the measurements of the samples at a specified time.

The tabular CUSUM scheme uses two CUSUM statistics to control the process location parameters. These statistics are expressed as follows:

$$C_i^+ = max[0, (\hat{\theta}_i - \theta_0) - K_{\hat{\theta}} + C_{i-1}^+], \text{ for } i = 1,2, \dots, m \quad (1)$$

and

$$C_i^- = max[0, -(\hat{\theta}_i - \theta_0) - K_{\hat{\theta}} + C_{i-1}^-], \text{ for } i = 1,2, \dots, m \quad (2)$$

with i representing the sample number and $\hat{\theta}$ the location estimator. The statistics' initial value is usually fixed to be equivalent to the target value. The two CUSUM statistics (C_i^+ and C_i^-) are compared against $H_{\hat{\theta}}$ where $H_{\hat{\theta}}$ is the control limit. The process is deemed out-of-control when one of the two CUSUM statistics exceeds $H_{\hat{\theta}}$. The standardized CUSUM control chart parameters are $K_{\hat{\theta}} = k \times \sigma_{\hat{\theta}}$ and $H_{\hat{\theta}} = h \times \sigma_{\hat{\theta}}$ where the value of constants k and h are chosen for a specific value of the in-control ARL.

The ARL is typically used to measure the performance of control charts. It is defined as the expected number of plotted chart statistics before a point is seen exceeding the control limit [14]. Under an in-control state, the ARL measures the false alarm rate. Meanwhile, under an out-of-control state, the ARL measures the true detection of changes in the process. Henceforth, the in-control ARL is denoted by ARL_0 and the out-of-control ARL by ARL_1. Ideally, the value of the ARL_0 should be large while the value of the ARL_1 should be small. This way, the chart is likely to give false signals but quick in detecting true out-of-control conditions in the process.

2.2 Exponentially weighted moving average control chart

The EWMA control chart was developed by Roberts [15] in 1959 where the chart's statistic is calculated by taking a weighted average of the present and all past subgroup values, with more weight given to recent data and decreasing weight for the rest of the older data. The EWMA statistic is expressed as:

$$Z_i = \lambda\hat{\theta}_i + (1-\lambda)Z_{i-1}, \qquad \text{for } i = 1,2, \dots, m \quad (3)$$

with λ as a smoothing constant that ranges between 0 and 1 ($0 < \lambda \leq 1$), the starting value, Z_0, represents the target mean set to be equal to θ_0. The EWMA statistic, Z_i, is plotted against the upper control limit (UCL) and lower control limit (LCL).

The center line (CL) and control limits of the EWMA control chart are defined as:

$$UCL_i = \theta_0 + L_{\hat{\theta}}\sqrt{Var(\hat{\theta})\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})} \quad (4)$$

$$CL = \theta_0$$

$$LCL_i = \theta_0 - L_{\hat{\theta}}\sqrt{Var(\hat{\theta})\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})} \quad (5)$$

where $L_{\hat{\theta}}$ is the positive coefficient value. This coefficient is usually set at a value that yields the pre-determined ARL_0. If the EWMA statistic falls outside either of the limits, it signals that the process may be out-of-control.

Both the CUSUM and EWMA control charts have been improved consistently [16-17]. Abbas et al. [7] made one of the most recent contributions, who combined the two memory-type control charts discussed above to create the MEC

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chart.

2.3 Mixed EWMA-CUSUM control chart

The MEC chart combines the CUSUM and EWMA components into one new control chart structure. The plotting statistics defined in the MEC control chart are as follows:

$$MEC_i^+ = \max\,[0, (Z_i - \theta_0) - K_{\hat{\theta}} + MEC_{i-1}^+],\ \text{for } i = 1,2,\dots,m \quad (6)$$

and

$$MEC_i^- = \min\,[0, (Z_i - \theta_0) + K_{\hat{\theta}} + MEC_{i-1}^-],\ \text{for } i = 1,2,\dots,m \quad (7)$$

where i is the sample number until m subgroups, and Z_i is the EWMA statistic in (3).

In this study, $\hat{\theta}_i$ was computed using the sample median. The calculation is as:

$$\hat{\theta}_i = \begin{cases} X_{\frac{n+1}{2}} , & \text{if } n \text{ is odd} \\ \frac{1}{2}\left(X_{\frac{n}{2}} + X_{\frac{n+2}{2}}\right), & \text{if } n \text{ is even} \end{cases} \quad (8)$$

The MEC_0^+ is the upper CUSUM statistic and MEC_0^- is the lower CUSUM statistic. Both statistics are initially set to 0, meanwhile, $K_{\hat{\theta}}$ is the time-varying reference value in the MEC chart. In equation (8), the value of $\lambda \in (0,1]$ and the initial value of Z_i is usually set to be the same as the target value $(Z_0 = \theta_0)$. The variance of Z_i is defined as:

$$Var(Z_i) = \sigma_{\hat{\theta}}^2 \left[\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})\right] \quad (9)$$

There are two standardized parameters, $K_{\hat{\theta},i} = k \times \sqrt{Var(Z_i)}$ and $H_{\hat{\theta},i} = h \times \sqrt{Var(Z_i)}$, where the notation $H_{\hat{\theta}}$ represents the control limit, k, and h are the constants comparable to the one utilized in the standard CUSUM chart. The k and h are the reference value and decision limit, respectively, chosen based on the pre-determined ARL_0. When i in (9) approaching infinity $(i \to \infty)$, $Var(Z_i) = \sigma_{\hat{\theta}}^2 \left[\frac{\lambda}{2-\lambda}\right]$. Then the two quantities become $K_{\hat{\theta}} = k \times \sigma_{\hat{\theta}}\sqrt{\frac{\lambda}{2-\lambda}}$ and $H_{\hat{\theta}} = h \times \sigma_{\hat{\theta}}\sqrt{\frac{\lambda}{2-\lambda}}$.

Both MEC_i^+ and MEC_i^- are plotted against the control limit $H_{\hat{\theta}}$. to identify an out-of-control process. The procedure yields statistical control if the two plotting statistics are dispersed between 0 and $H_{\hat{\theta}}$. The process is said to be out-of-control if either of the plotting statistics exceeds $H_{\hat{\theta}}$.

To increase the sensitivity of the chart to a small shift (of about 1σ) k is typically set to $1/2$ [7]. Practitioners could achieve the desired ARL_0 by making careful decisions about h, δ (amount of shift), and λ by pairing these with a fixed value of k. For instance, Abbas et al. [7] set the ARL_0 at 168, 400, and 500 with $k = \frac{1}{2}$ to obtain several combinations of suitable h, δ, and λ values that result in the desired ARL.

2.4 Simulation Procedures

Generally, a control chart is constructed based on the assumption that both parameters of the process are known. For example, studies by Haq and Khoo [18] on the EWMA, Dunbar [19] on the CUSUM, and Anwar et al. [20] on the MEC chart.

In this study, the Monte Carlo simulation approach was used to model and evaluate the performance of the MEC control chart based on the median estimator. Data were generated using SAS 9.4 software by manipulating several variables in assessing the strengths and weaknesses of the robust MEC chart. Specifically, 10,000 datasets were generated to determine the ARL values.

The charting constants for the MEC charts in this study were determined for $k = 0.5$ and $\lambda = 0.13$ with ARL_0 set at 370 under $g = 0$, $h = 0$ distribution, corresponding to a standard normal distribution. Table 1 lists the charting constants of two sample sizes (n).

Table 1: The charting constants for different sample sizes(n).

n	*Mean*	*Median*
5	28.02	28.30
9	27.85	28.13

The g-and-h distribution was applied to manipulate the distributional shapes. Each of the distributions was combined with a different sample ($n = 5, 9$) and various shift values ($\delta = 0, 0.25, 0.5, 0.75, 1, 1.5, 2, 3$).The performance of the charts was examined based on the ARL.

The following steps must be followed to generate data for the *g*-and-*h* distributions:

i. Generate standard normal variates, Z_{ij}
ii. Convert the standard normal variates to random variables via equation

$$Y_{ij} = \begin{cases} \frac{\exp(gZ_{ij})-1}{g}\exp\left(\frac{hZ_{ij}^2}{2}\right), & g\neq 0 \\ Z_{ij}\exp\left(\frac{hZ_{ij}^2}{2}\right), & g=0 \end{cases} \tag{10}$$

where *g* and *h* control the skewness and elongation in the tail of the distribution, respectively.

A typical normal distribution is represented with $g = 0$ and $h = 0$, $Y_{ij} = Z$. The tails of the distribution get heavier as *h* increases. The same is true for *g*, which regulates the skewness. Table 2 shows four different *g*-and-*h* distributions that resulted from varying *g*-and-*h* statistic values. The performance of the robust MEC chart was observed under all four *g*-and-*h* distributions in this study.

Table 2: Condition of g-and-h distribution

(*g*,*h*)	*Description*
(0,0)	Normal
(0,0.5)	Symmetric heavy tail
(0.5,0)	Skewed normal tail
(0.5,0.5)	Skewed with a heavy tail

3. RESULTS AND DISCUSSION

This study compared the robust MEC charts based on the median estimator with the standard MEC chart which is based on the sample mean. Table 3 presents the results obtained based on the ARL. As mentioned earlier, a good control chart should demonstrate a large ARL_0 and a small ARL_1 value.

When concentrating on the normal distribution, ($g = 0$, $h = 0$), all charts produce $ARL_0 \approx 370$ as designed in this study. All charts perform similarly in shift detection, and as *n* increases, the values of ARL_1 for both charts decrease, indicating better shift detection.

The in-control performance of the robust MEC chart remains unaffected when $g = 0$ and $h = 0.5$ (i.e., a symmetric heavy-tailed distribution). The values of the ARL_0 are demonstrated in Table 3, which are not much different than the expected value (370). In contrast, both sample sizes show that the value of the ARL_0 for the standard MEC chart is significantly larger than 370. Nonetheless, both charts perform similarly in an out-of-control situation as indicated by the value of the ARL_1 in Table 3.

When the underlying process data is skewed, i.e., $g = 0.5$ and $h = 0$, the in-control performances of both standard and robust charts are unaffected. Table 3 demonstrates that both charts yielded $ARL_0 \approx 370$ for $n = 5$ and $n = 9$, respectively. Additionally, under this data scenario, the out-of-control performances of the MEC charts are comparable to the normal data distribution.

Lastly, under the extreme data condition where both skewness and heavy-tailed occurred, that is, $g = 0.5$ and $h = 0.5$, the ARL_0 of the standard MEC chart is highly affected by this type of data distribution. The bold values in Table 3 are significantly higher than the nominal value of 370. Conversely, the robust MEC chart can still produce the ARL_0 close to the nominal value despite the extreme data condition. Thus, even when both charts perform similarly in an out-of-control condition, the performance of the standard chart is questionable due to its large ARL_0.

Overall, the results show that the in-control performance of the standard MEC chart performs well for skewed distribution but is highly affected under heavy-tailed distribution. Contrarily, the robust MEC chart, based on the sample median, consistently outperforms the standard MEC chart in terms of in-control performance regardless of the distributional shape of the data. Furthermore, this robust chart can detect shifts as effectively as a standard MEC chart across all distributions considered in this investigation.

Table 3: ARL values for MEC charts with k = 0.5 at ARL_0 = 370

(g.h)	n	Methods	0	0.25	0.5	0.75	1	1.5	2	3
(0,0)	5	Mean	370.153	27.524	14.525	10.601	8.572	6.453	5.256	4.010
		Median	369.980	27.629	14.604	10.679	8.631	6.490	5.288	4.019
	9	Mean	370.091	20.349	11.484	8.527	6.925	5.220	4.222	3.067
		Median	370.025	20.582	11.557	8.584	6.970	5.250	4.252	3.086
(0,0.5)	5	Mean	**916.526**	26.590	14.342	10.501	8.516	6.364	5.108	4.007
		Median	369.248	27.585	14.601	10.663	8.632	6.483	5.274	4.018
	9	Mean	**976.950**	19.921	11.351	8.455	6.961	5.084	4.070	3.016
		Median	369.284	20.555	11.573	8.583	6.977	5.244	4.236	3.082
(0.5,0)	5	Mean	372.452	27.588	14.532	10.596	8.584	6.451	5.257	3.990
		Median	372.962	27.739	14.625	10.673	8.633	6.488	5.300	3.995
	9	Mean	365.538	20.408	11.472	8.513	6.948	5.212	4.223	3.055
		Median	372.261	20.545	11.586	8.571	6.970	5.250	4.257	3.072
(0.5,0.5)	5	Mean	**1455.208**	26.837	14.302	10.531	8.621	6.413	5.001	3.991
		Median	385.235	27.799	14.655	10.693	8.607	6.484	5.291	3.989
	9	Mean	**2320.710**	19.864	11.341	8.506	6.951	4.994	4.000	2.998
		Median	370.140	20.592	11.568	8.578	6.976	5.240	4.249	3.064

4. APPLICATION

Actual data regarding anticipated rainfall (in milliliters, mm) for Kedah in Northwest Malaysia from 2019 to 2020 was used to illustrate the application of the proposed robust MEC chart. Figure 1 shows the rainfall data which were collected from 104 samples of size 7. Phase I uses the first half of the data to construct control limits, while Phase II uses the second half to monitor out-of-control samples. For n = 7, when the values of λ and k are fixed at 0.13 and 0.5, respectively (as in the simulated study), h becomes 27.86 for the standard MEC chart and robust MEC chart. The outputs of the investigated charts are shown in Figures 2 and 3 where both plotting statistics (MEC_i^+ and MEC_i^-), are plotted against the control limits, H and H^-, respectively. The process is out-of-control if the MEC_i^+ exceeds the H and/or MEC_i^- exceeds the H^-. Otherwise, the process is deemed to be in-control.

The output for the standard MEC chart is shown in Figure 2. The chart displays 29 out-of-control signals in samples 14 to 41. Figure 3, which shows the robust MEC chart's result, shows 31 out-of-control signals from samples 11 through 41. This result suggests that the robust MEC chart performs better than the standard MEC chart because it can quickly identify shifts and alert out-of-control samples.

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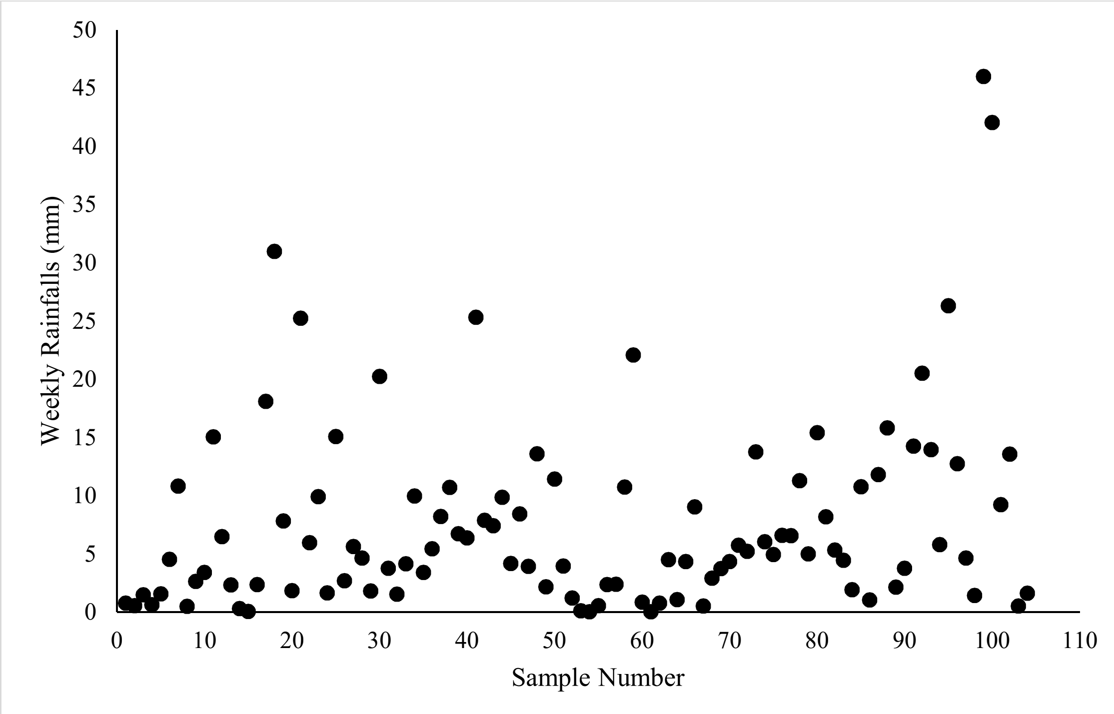

Figure 1: The Scatter Plot of Weekly Rainfalls

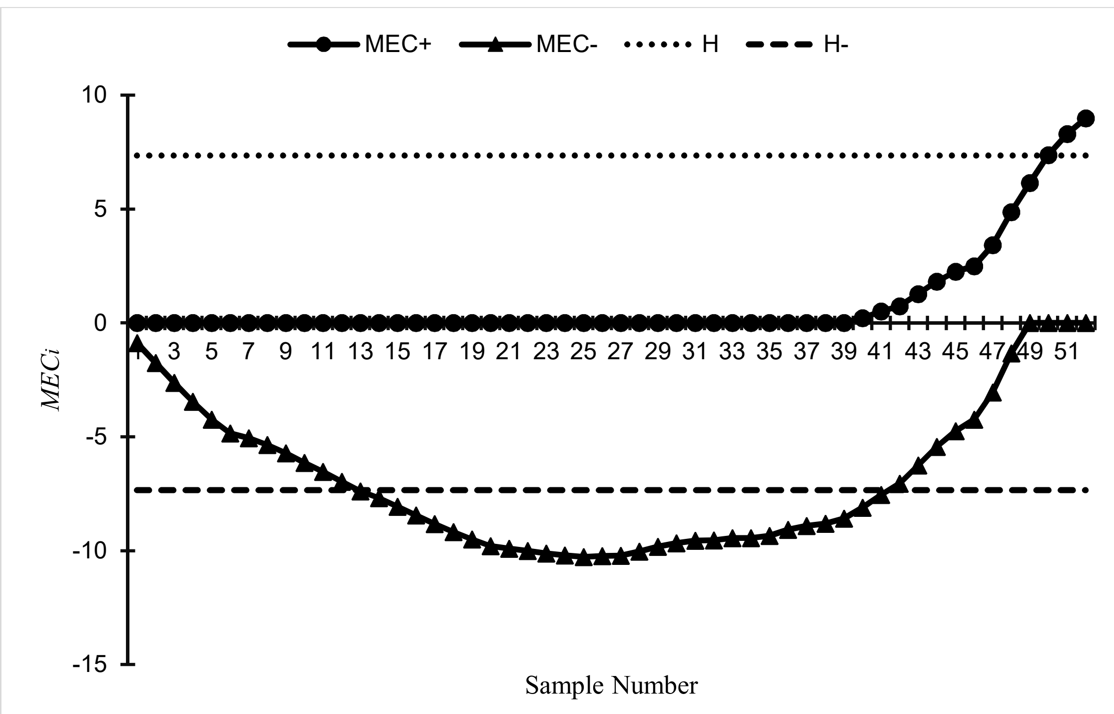

Figure 2. The Standard MEC chart

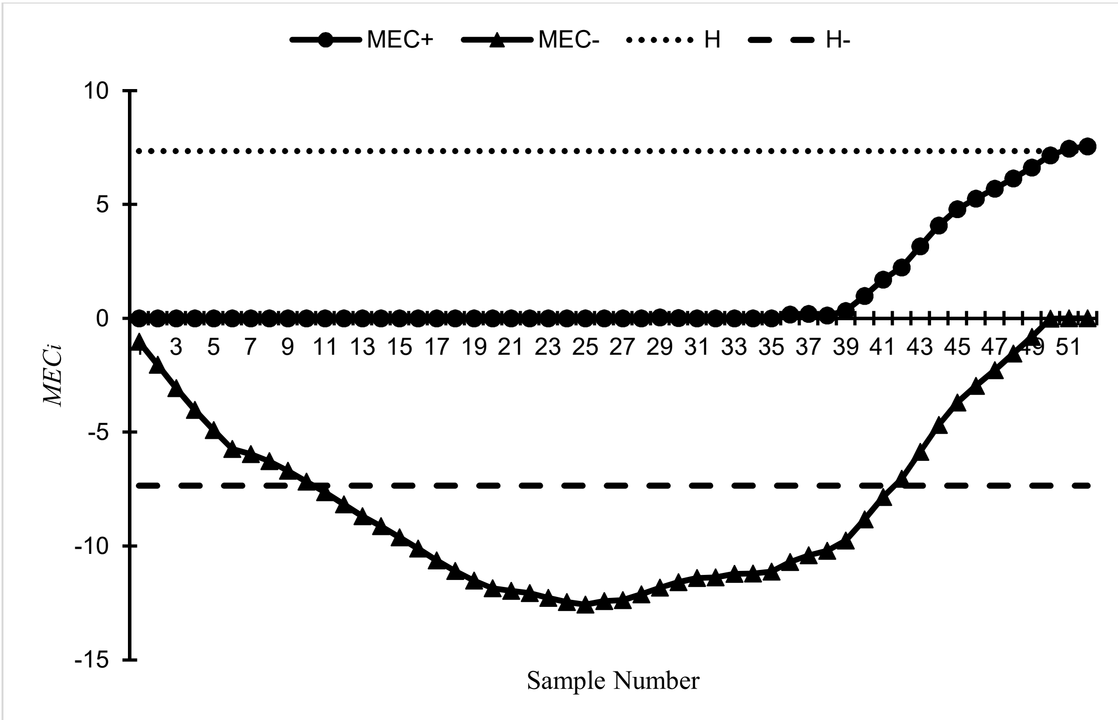

Figure 3. The Robust MEC chart

5. CONCLUSION

The MEC control chart was initially introduced to enhance the performance of both EWMA and CUSUM control charts under normality. This study proposed the use of a median estimator in Phase II for monitoring of location via the MEC charts and subsequently, compared its performance with the standard MEC chart under non-normality data scenarios. The findings indicate that the robust MEC chart has good control of the false alarm rate (by producing the value of the ARL_0 close to the nominal value) across varying distributional shapes. Therefore, the chart can be confidently used for detecting true out-of-conditions. Conversely, the standard MEC chart does not have good control of the false alarm rate under heavy-tailed scenarios. This finding is crucial since without proper in-control robustness, the shift detection capability of the standard MEC chart becomes questionable.

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