Journal of Theoretical and Applied Information Technology
30th April 2022. Vol.100. No 8

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

ASYMPTOTIC OF SOLVING A DYNAMIC PROBLEM OF ELASTICITY THEORY FOR AN INCOMPRESSIBLE MEDIUM

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ABSTRACT

In this paper, the behavior of the solution of the dynamic problem and the theory of elasticity as $\lambda \to \infty$ for the second boundary value problem is studied. An unimprovable estimate of the rate of convergence of the solution for a compressible medium to an incompressible parameter $1/\lambda$ is obtained.

In [1], the following question was considered, approximations of the solution of the problem for an incompressible medium by the solution of the problem for compressible media as $\lambda \to \infty$, as well as the possibility justification for using the difference schemes proposed in [2] to obtain a solution to the problem under study. In [3,4], the dynamic problem of contact of compressible and incompressible media was considered, theorems on the existence and uniqueness of a generalized solution were proved, and estimates were obtained for the proximity of the solution of a contact problem to solutions of problems for compressible and incompressible media.

In this paper, we have studied the stability of the difference scheme proposed by A.N. Kanavalov for solving the dynamic problem of elasticity theory. The approximation analysis allows to select the optimal grid steps associated with the parameter λ.

Keywords: *Incompressible Medium, Deformations, Displacements, E Task Of The Stokes, Theory Of Elasticity.*

1. INTRODUCTION

Stationary linearized equations of a slightly compressible liquid have the following form

$-\nu\Delta\bar{u}_\varepsilon - \varepsilon^{-1}\text{grad div}\,\bar{u}_\varepsilon = \bar{f}$, in Ω, (1)

$\bar{u}_\varepsilon = 0$ on $\partial\Omega$, for $\varepsilon > 0$ (2)

Equations (1), (2) are also stationary Lame equations from the theory of elasticity. In [5] it is shown that the task (1), (2) has a unique solution $\bar{u}_\varepsilon$ for every fixed $\varepsilon > 0$ and that $\bar{u}_\varepsilon$ converges to the solution $\bar{u}_\varepsilon$ of the Stokes problem at $\varepsilon \to 0$. Initially, equations (1), (2) were used as "approximating" for the Stokes equations, one of the ways to overcome the difficulty of "$\text{div}\bar{u} = 0$"was to solve equations (1), (2) with a sufficiently small ε in order to solve the Stokes equations themselves. This idea can also be applied to the dynamic Stokes problem by choosing as $\varepsilon = \frac{1}{\lambda}$, for a fixed μ; λ, μ are the Lame coefficients. In this case, $\nu = \frac{\lambda}{2(\lambda+\mu)}$ and $\lambda \to \infty, \nu \to \frac{1}{2}$, i.e. the case of an incompressible medium.

2. MATERIALS AND METHODS OF RESEARCH

The method of a priori estimates shows the asymptotic proximity of solutions of compressible and incompressible media at $\lambda \to \infty$. At the end of the article, an analysis of the difference scheme for solving an incompressible medium is carried out.

Let $D \subset R^3$ be a bounded simply connected domain with boundary ϒ. The solution of the dynamic task of the theory of elasticity for an incompressible medium satisfies the equation of motion [5,6,7,8,9,10,11]

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \mu \Delta \bar{u} - \nabla p + \bar{f} = 0, x \in D, \tag{3}$$

incompressibility condition

$$div\, \bar{u} = 0, \ \ x \in D, \tag{4}$$

displacement-strain ratio

$$2\varepsilon_{ik} = \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), i,k = 1,2,3, \tag{5}$$

state equations

$$\sigma_{ik} = -\delta_{ik} p + 2\mu\varepsilon_{ik} \tag{6}$$

initial conditions

$$\bar{u}(x,0) = \bar{\varphi}(x), \ \left.\frac{\partial \bar{u}}{\partial t}\right|_{t=0} = \bar{\psi}(x), \tag{7}$$

and boundary conditions

$$\sum_{k=1}^{3} \left[\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \delta_{ik} p\right]_{n_k} = 0, \ \ x \in ɣ, \quad t \in [0,T] \tag{8}$$

For task (3) - (8), we assume that the corresponding conditions are met for the matching of the initial and boundary conditions. Task (3) - (8) is called task I. Along with task I, we will consider dynamic task II.[6]

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \mu\Delta\bar{u} + (\lambda+\mu)\nabla div\, \bar{u} + \bar{f}, \quad x \in D,$$

$$\bar{u}|_{t=0} = \bar{\varphi}(x), \qquad \left.\frac{\partial \bar{u}}{\partial t}\right|_{t=0} = \bar{\psi}(x),$$

$$\sigma_{ik} = \lambda\delta_{ik}\theta + 2\mu\varepsilon_{ik},$$

$$\theta = \sum_{k=1}^{3} \varepsilon_{kk},$$

$$2\varepsilon_{ik} = \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), i,k = 1,2,3,$$

$$\sum_{k=1}^{3} \sigma_{ik} n_k = 0, x \in ɣ, t \in [0,T] \tag{9}$$

We obtain a priori uniform estimates for the parameter $1/\lambda$ for solving task II.

Multiply (9) by $\frac{\partial \bar{u}}{\partial t}$ scalarly in $L_2(D)$, and we have

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \bar{u}}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\mu E(\bar{u},\bar{u}) + \frac{1}{2}\lambda\frac{d}{dt}\|div\, \bar{u}\|^2 = \tag{10}$$

$$= \int_D \bar{f}\frac{\partial \bar{u}}{\partial t} dx,$$

Where

$$E(\bar{u},\bar{v}) = \frac{1}{2}\mu \int_D \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx.$$

Further, evaluating successively the right part (10) we get

$$\left|\int_D f u_t dx\right| \le \|u_t\| \cdot \|f\|_{L_2(D)} \le \delta\|u_t\|^2 + C_\delta\|f\|^2,$$

Where $\delta > 0, C_\delta > 0$ are constants [12,13,14,15].

Using the Gronwall lemma [16], we obtain

$$\|\bar{u}_t\|^2_{L_\infty(0,T;\, L_2(D))} + \mu\|\bar{u}\|^2_{W_2^1(D)} + \lambda\|div\, \bar{u}\|^2 \le \tag{11}$$

$$\le \lambda\|div\, \bar{\varphi}\|^2 + C,$$

Suppose that $\bar{\varphi}(x)$ such that $div\, \bar{\varphi} = 0$.

Differentiate the equation of motion in (9) with respect to t, then multiply by $\bar{u}_{tt}$ scalarly in $L_2(D)$ and we will have

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

Applying the Cauchy inequality to the right side, then ε −inequality, using the Gronwall lemma, and assuming that div $\overline{\psi} = 0$, we get

$$\|\bar{u}_{tt}\|^2_{L_\infty(0,T;\,L_2(D))} + \|\bar{u}_t\|^2_{L_\infty(0,T;w_2^1(D)} + \\ +\lambda\|div\ \bar{u}_t\|^2_{L_\infty(0,T;\,L_2(D))} \quad \le C < \infty, \tag{13}$$

where C depends on the norms $\|\bar{\varphi}\|, \|\bar{\psi}\|, \|f_t\|_{L_2(D)}$

We introduce the following notation

$$L_{u_i} = \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k},$$

multiplying the first equation (9) by L_{u_t}scalarly $L_2(D)$, taking into account the boundary conditions in (9) and by virtue of

$$\int_D f L_{u_t}\, dx = \frac{\partial}{\partial t}\int_D f L_u dx - \int_D f_t\, L_u dx, \tag{14}$$

$$\|\bar{u}_t\|^2_{L_\infty(0,T;\,L_2(D))} + \lambda\|div\ \bar{u}_t\|^2_{L_\infty(0,T;\,L_2(D))} \\ + \|\bar{u}\|^2_{L_\infty\left(0,T;\,w_2^2(D)\right)} + \\ +\lambda^2\|grad\ div\ \bar{u}\|^2_{L_\infty(0,T;\,L_2(D))} \quad \le C < \infty \tag{15}$$

Here the assessment of coercivity is taken into account. Further, by differentiating the first equation (9) with respect to t, we multiply it by $L_{u_{tt}}$scalarly in $L_2(D)$, after simple transformations we get the estimate

$$\left\|\bar{u}_{tt_x}\right\|^2_{L_\infty(0,T;\,L_2(D))} \\ + \frac{1}{2}\lambda\|div\ \bar{u}_{tt}\|^2_{L_\infty(0,T;\,L_2(D))} \\ + \|\bar{u}_t\|^2_{L_2\left(0,T;\,w_2^1(D)\right)} + \\ +\lambda^2\|div\bar{u}_t\|^2_{L_\infty\left(0,T;w_2^1(D)\right)} \\ \le C(\left\|\bar{u}_{tt_x}(x,0)\right\|^2 \\ + \frac{1}{2}\|div\bar{u}_{tt}(x,0)\|^2 + \\ +\left\|\bar{\psi}(x)\right\|^2_{w_2^2(D)} + \lambda^2\left\|div\ \bar{\psi}\right\|^2 \\ + \left\|\bar{f}_{tt}\right\|^2_{L_2(0,T;\,L_2(D))}), \tag{16}$$

Setting that $div\ \bar{u}_{tt}(x,0) = 0$, we can simplify the estimate (16), this condition can be satisfied.

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}\mu E(\bar{u}_t,\bar{u}_t) \\ + \frac{1}{2}\lambda\frac{d}{dt}\|div\bar{u}_t\|^2 = \\ = \int_D f_t u_{tt}\, dx, \tag{12}$$

Indeed, we take the divergence from the first equation (9) at $t = 0$, and we get

$$\frac{\partial^2}{\partial t^2} div\ \bar{u}\Big|_{t=0} = div\ \bar{f}\Big|_{t=0}$$

That is, if the vector$\bar{f}$- is solenoidal, then we have from the last equality

$$\frac{\partial^2}{\partial t^2} div\ \bar{u}\Big|_{t=0} = 0.$$

If we assume that $\|\bar{\varphi}\|, \left\|\bar{\psi}\right\|, \bar{f}(x,0)$-are smooth solenoid vectors, then

$$\frac{\partial^k}{\partial t^k} div\ \bar{u}\Big|_{t=0} = 0{,}1\ , k = 1{,}2, \ldots$$

So we have proven.

Lemma 1. Let $\|\bar{\varphi}\|, \left\|\bar{\psi}\right\|, \bar{f}(x,0)$ be smooth solenoidal vectors. Then for the solution of the problem (9) there is an estimate

$$\left\|\frac{\partial \bar{u}}{\partial t^k}\right\|^2_{L_\infty\left(0,T;\,w_2^1(D)\right)} \\ + \frac{1}{2}\lambda\left\|\frac{\partial^k}{\partial t^k} div\bar{u}\right\|^2_{L_\infty(0,T;\,L_2(D))} + \\ + \left\|\frac{\partial^{k-1}\bar{u}}{\partial t^{k-1}}\right\|^2_{L_2\left(0,T;\,w_2^2(D)\right)} + \\ +\lambda^2\left\|\frac{\partial^{k-1}\bar{u}}{\partial t^{k-1}} div\bar{u}\right\|^2_{L_\infty(0,T;w_2^1(D))} \le C < \infty, \tag{17}$$

Considering the estimate (17), we proceed to the limit as $\lambda \to \infty$ in task (9). Since as $\lambda \to \infty$, there is a relation $\bar{u} \to \bar{u}_0$ weakly in $w_2^2\left(0,T;w_2^2(D)\right)$ as $\lambda \to \infty$ λ div $\bar{u} \to p$ weakly in $L_2\left(0,T;w_2^1(D)\right)$, where $\bar{u}_0, p$ is the solution to task I. Now we estimate the proximity of the solution of the task (9) to the solution of task I. Let $\overline{w} = \bar{u} - \bar{u}_0, \lambda$ div $\bar{u} - p = \pi$, where $\bar{u}$ is the solution of task (9), $\bar{u}_0$, p is the solution of task I, we get the

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

problem

$$\frac{\partial^2 \overline{w}}{\partial t^2} = \mu\Delta\overline{w} + \mu\nabla div\overline{w} + \nabla\pi, x \in D, \quad (18)$$

$$\overline{w}|_{t=0} = 0, \quad \left.\frac{\partial \overline{w}}{\partial t}\right|_{t=0} = 0, x \in \overline{D}, \quad (19)$$

$$\sum_{i=1}^{3}\left[\mu\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) + \delta_{ij}\pi\right]_{n_j} = 0,\ x \in \gamma,\ t \in [0, T] \quad (20)$$

Multiply (18) by $\overline{w}_t$ scalarly in $L_2(D)$, and we get

$$\frac{1}{2}\frac{d}{dt}\|\overline{w}_t\|^2 + \frac{\mu}{2}\frac{d}{dt}E(\overline{w}, w) - \int_D \pi\ div\ \overline{w}_t\ dx = 0, \quad (21)$$

Estimating the right-hand side of (21) using the Cauchy inequality, theε-inequality, and using the Gronwall lemma, we arrive at the estimate

$$\|\overline{w}_t\|^2_{L_2(0,T;\ L_2(D))} + \|\overline{w}\|^2_{L_\infty(0,T;\ w_2^1(D))} \leq C \cdot \lambda^{-2}, \quad (22)$$

Differentiating equation (18) with t, then multiplying it by $\overline{w}_{tt}$ scalarly in $L_2(D)$, using the same argument as in obtaining estimate (13) we will have

$$\|\overline{w}_{tt}\|^2_{L_\infty(0,T;\ L_2(D))} + \|\overline{w}_t\|^2_{L_\infty(0,T;\ w_2^1(D))} \leq C \cdot \lambda^{-2}, \quad (23)$$

By virtue of (18) (20) the inequality holds true

$$\|\pi\|^2_{L_\infty(0,T;\ L_2(D))} \leq C \cdot \lambda^{-2}, \quad (24)$$

Turning to (21), taking into account the estimates (22), (24), we finally get

$$\|\overline{w}_t\|^2_{L_\infty(0,T;\ L_2(D))} + \|\overline{w}\|^2_{L_\infty\left(0,T;\ w_2^1(D)\right)} + \|\pi\|^2_{L_\infty(0,T;\ L_2(D))} \leq C \cdot \lambda^{-2}, \quad (25)$$

That is, the following is proved

Theorem 2. Let the conditions of Lemma 1 be fulfilled, then the estimate is fair.

$$\left\|\frac{\partial^{k+2}\overline{w}}{\partial t^{k+2}}\right\|^2_{L_\infty(0,T;\ L_2(D))} + \left\|\frac{\partial^{k+1}\overline{w}}{\partial t^{k+1}}\right\|^2_{L_\infty(0,T;\ L_2(D))} + \left\|\frac{\partial^{k}\pi}{\partial t^{k}}\right\|^2_{L_\infty(0,T;\ L_2(D))} \leq C \cdot \lambda^{-2}.$$

Further it is possible to formulate

Lemma 2. Estimation of the proximity (25) of solutions of tasks I and II is best possible with respect to the parameter λ.

Let us suppose that the proof is the contrary that means

$$\|\overline{w}_t\|^2_{L_\infty(0,T;\ L_2(D))} + \|\overline{w}\|^2_{L_\infty\left(0,T;\ w_2^1(D)\right)} + \|\pi\|^2_{L_\infty(0,T;\ L_2(D))} \leq C \cdot \lambda^{-(2+\alpha)}$$

$\overline{w} = \overline{u} - \overline{u}_0, \pi = \lambda\ div\,\overline{u} - p,\ \ \alpha > 0$

is constant, perhaps small enough, therefore we have

$$\|\lambda\ div\ \overline{u} - p\| \geq \|p\| - \lambda\|div\ \overline{u}\|,$$

$$\|p\| \leq \lambda\|div\ \overline{u}\| + \|\lambda\ div\ \overline{u} - p\| \leq C\left(\|\pi\| + \frac{\lambda}{\lambda^{1+\frac{\alpha}{2}}}\right), \quad (26)$$

In inequality (26) let us pass to the limit as $\lambda \to \infty$, when we obtain by virtue of (25)

$$\|p\|_{L_2(D)} = 0,\ i.e.\ p = 0.$$

Thus, for $\overline{u}_0$andpwe get the following task

$$\frac{\partial^2 \overline{u}_0}{\partial t^2} = \mu\Delta\overline{u}_0 - \nabla p + \bar{f}, x \in D$$

$$div\,\overline{u}_0 = 0,\ p = 0,\ \sum_{k=1}^{3}\sigma_{ik}n_k = 0,\ on\ \gamma.$$

This contradicts our assumption, since the last problem is unsolvable (the original task I is correct, and that was required).

It is shown above that the solution of the dynamic problem of elasticity theory for an incompressible medium

$$\frac{\partial^2 \overline{u}}{\partial t^2} = \mu\Delta\overline{u} - \nabla p + \bar{f} = 0,\ x \in D \quad (27)$$

$$\bar{u}|_{t=0} = \bar{u}_0(x),\ \frac{\partial \bar{u}}{\partial t}\Big|_{t=0} = \bar{u}_1(x),\ x \in D$$

$$\sum_{k=1}^{3} \sigma_{ik}(x,t)\, n_k = 0, x \in ɣ, t \in [0,T]$$

where $\sigma_{ik} = -\delta_{ik} p + 2\mu\varepsilon_{ik}$ is possible to obtain by passing to the limit at $\lambda \to \infty$, in the solution of the dynamic problem of elasticity theory

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \mu\Delta\bar{u}_0 + (\lambda+\mu)\nabla \operatorname{div} \bar{u} + \bar{f}, x \in D,$$

$$\bar{u}|_{t=0} = \bar{u}_0(x), \qquad \frac{\partial \bar{u}}{\partial t}\Big|_{t=0} = \bar{u}_1(x) x \in D \qquad (29)$$

$$\sum_{k=1}^{3} \sigma_{ik}(x,t) n_k = 0, x \in ɣ, t \in [0,T]$$

where $\sigma_{ik} = \lambda\delta_{ik} \operatorname{div}\bar{u} + 2\mu\varepsilon_{ik}$

For the numerical solution of problem (27) in a parallelepiped D=$\{0 \le x_i \le l_i,\ i = 1,2,3\}$ in [17] difference schemes in voltages are proposed.

$$2\frac{\partial^2 \varepsilon_{ij}}{\partial t^2} = \frac{\partial\ L_i\bar{\sigma}}{\partial x_j} + \frac{\partial\ L_j\bar{\sigma}}{\partial x_i} + g_{ij}, i,$$
$$j = 1,2,3, x \in D$$

$$\sum_{i=1}^{3}\left[\frac{\partial\ L_i\bar{\sigma}}{\partial x_i} + g_{ii}\right] = 0, x \in D \qquad (29)$$

$$\sum_{k=1}^{3} \sigma_{ik}(x,t) n_k = 0, x \in ɣ, t \in [0,T]$$

где $L_i\sigma = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$, $\sigma_{ij} = -\delta_{ij} p + 2\mu\varepsilon_{ij}$, $g_{ij} = \frac{1}{2}\left(\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right)$

The initial conditions take the form

$$\varepsilon_{ij}(x,0) = \alpha_{ij}(x), \frac{\partial \varepsilon_{ij}}{\partial t}\Big|_{t=0} = \beta_{ij}(x), x \in D$$

$$\alpha_{ij} = \frac{1}{2}\left(\frac{\partial u_{0i}}{\partial x_j} + \frac{\partial u_{0j}}{\partial x_i}\right),$$
$$\beta_{ij} = \frac{1}{2}\left(\frac{\partial u_{1j}}{\partial x_i} + \frac{\partial u_{1i}}{\partial x_j}\right) \qquad (30)$$

Equation (29) is obtained from (27) for the displacement $\bar{u}$ using the displacement-strain ratio $\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$. In addition, we assume that the initial conditions are such that

$$div\ \bar{u}_0(x) = div\ \bar{u}_1(x) = 0, \qquad (31)$$

is easy to show that problem (27) and problem (29) are equivalent to [2].

Similarly, excluding the displacement $\bar{u}$ in problem (28) we come to the statement of the dynamic problem of the theory of elasticity in stresses

$$2\frac{\partial^2 \varepsilon_{ij}}{\partial t^2} = \frac{\partial L_i\bar{\sigma}}{\partial x_j} + \frac{\partial L_i\bar{\sigma}}{\partial x_j} + g_{ij}, x \in D$$

$$\varepsilon_{ij}(x,0) = \propto_{ij}^{\lambda}(x), \frac{\partial \varepsilon_{ij}}{\partial t}\Big|_{t=0} = \beta_{ij}^{\lambda}(x),\ x \in \bar{D} \qquad (32)$$

$$\sum_{j=1}^{3} \sigma_{ij}(x,t) n_j = 0, x \in ɣ, t \in [0,T]$$

where $\propto_{ij}^{\lambda}(x) = \propto_{ij}(x) - \frac{p\delta_{ij}}{3\lambda+2\mu}$, $\beta_{ij}^{\lambda}(x) = \beta_{ij}(x)$.

As we know, in [2] is shown that problem (28) is equivalent to problem (32). Next, we consider an explicit difference scheme for problem (32), following [3,4]

$$2\varepsilon_{ij,\bar{t}t}(L_{ih}\bar{\sigma}_h^n + f_{ih}^n)_{x_j} + (L_{jh}\bar{\sigma}_h^n + f_{jh}^n)_{x_i}$$

$$\sigma_{ij,h}^0 = \propto_{ij}, (\sigma_{ij,h}^1)_{\bar{t}} = \tilde{\beta}_{ij}$$

$$\sigma_{ij,h}^n = 0, x_i = 0, l_i, i,j = 1,2,3 \qquad (33)$$

where

$$\propto_{ij} = \mu(u_{0i}, x_j + u_{0j}, x_i) + \lambda\delta_{ij}\sum_{k=1}^{3} u_{0k}, x_k$$

$$\hat{\beta}_{ij} = \beta_{ij} + \frac{\tau^2}{2}\frac{\partial^2 \sigma_{ij}}{\partial t^2}\Big|_{t=0},$$

$$\beta_{ij} = \mu(u_{1i}, x_j + u_{1j}, x_i) + \lambda\delta_{ij}\sum_{k=1}^{3} u_{1k}, x_k$$

$$\frac{\partial^2 \delta_{ij}}{\partial t^2}\Big|_{t=0} \mu\left[(L_{ih}\bar{\sigma}_h^0 + f_{ih}^0)_{x_j} + \left(L_{jh}\bar{\sigma}_h^0 + f_{jh}^0\right)_{x_i}\right] + $$
$$+ \lambda\delta_{ij}\sum_{k=1}^{3}(L_{kh}\bar{\sigma}_h^0 + f_{kh}^0)_{x_k},$$

$$L_{ih}\bar{\sigma}_h = (\sigma_{i1,h})_{\bar{x}_1} + (\sigma_{i2,h})_{\bar{x}_2} + (\sigma_{i3,h})_{\bar{x}_3}$$

$$\sigma_{i1,h} = \lambda\delta_{ij}\sum_{k=1}^{3}\varepsilon_{kk,h} + 2\mu\varepsilon_{ij,h}$$

$$f^n = f(n\tau), \sigma f_t = f(t+\tau) - f(\tau), \tau f_{\bar{t}t} = f_t - f_{\bar{t}}$$

The solution of the scheme (33) is defined in the grid area $D = \{(lh_1, mh_2, kh_3),\ l =$

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

$0, 1, \dots, N_1, m = 0, 1, \dots, N_2, k = 0,1, \dots N_3, h_i \cdot N_i = l_i, \tau N = T\}$. The structure of the difference scheme (33) is such that the difference analogs of the corresponding theorems for the dynamic problem of elasticity theory are fulfilled for it. The difference scheme (33) approximates the original differential problem with an accuracy of $0(\tau^2 + |h|)$.

Let's write the scheme (33) in canonical form [3]

$$B\overline{\sigma}_t^0 + \tau^2 R\overline{\sigma}_{tt} + A\overline{\sigma} = \overline{g}\,, \tag{34}$$

$\overline{\sigma}^0, \overline{\sigma}^1$ set, in our case B=0, $\overline{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^T$

$$r^2R = \left\|\begin{matrix} a & b & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ b & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{matrix}\right\|, \quad a = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)},$$

$$b = -\frac{\lambda}{2\mu(3\lambda+2\mu)},$$

$$c = \frac{1}{\mu} - A =$$

$$= \left\|\begin{matrix} \Lambda_{11} & 0 & 0 & \Lambda_{12} & \Lambda_{13} & 0 \\ 0 & \Lambda_{22} & 0 & \Lambda_{21} & 0 & \Lambda_{23} \\ 0 & 0 & \Lambda_{33} & 0 & \Lambda_{31} & \Lambda_{32} \\ \Lambda_{21} & \Lambda_{12} & 0 & \Lambda_{11}+\Lambda_{22} & \Lambda_{23} & \Lambda_{13} \\ \Lambda_{31} & 0 & \Lambda_{13} & \Lambda_{32} & \Lambda_{11}+\Lambda_{33} & \Lambda_{12} \\ 0 & \Lambda_{32} & \Lambda_{23} & \Lambda_{31} & \Lambda_{21} & \Lambda_{22}+\Lambda_{33} \end{matrix}\right\|$$

$$\Lambda_{ij}(\cdot) = (\cdot)_{x_i\bar{x}_j}.$$

Operators A and R of scheme (32) such that $A = A^* > 0, R = R^* > 0$, and $R - \frac{1}{4}A \geq 0$,

If

$$\frac{\tau^2}{h_i^2} \leq \frac{1}{3(3\lambda + 2\mu)}, \tag{35}$$

Indeed, by virtue of the Koshi-Bunyakovskii's inequality and the ε-inequality, with subsequent application of the difference analog of the embedding theorem [4].

$$\|y_{\bar{x}}\|^2 \leq \frac{4}{h^2}\|y\|^2$$

It will have

$$(A\overline{\sigma}_h, \overline{\sigma}_h) \leq \frac{12}{h_i^2}(\overline{\sigma}_h, \overline{\sigma}_h),$$

on the other hand

$$(\tau^2 R\overline{\sigma}_h, \overline{\sigma}_h) \geq k(\overline{\sigma}_h, \overline{\sigma}_h), \text{где } k = \frac{1}{3\lambda+2\mu}$$

the smallest eigenvalue of the matrix $\tau^2 R$ there fore $R - \frac{1}{4}A \geq 0$, if the condition (35) is done for the solution $\overline{\sigma}_h$ of scheme (34) a priori estimate [16] is valid.

$$\|\overline{\sigma}^{n+1}\|_\xi \leq \|\overline{\sigma}^0\|_\xi + \|\bar{y}^0\|_{A^{-1}} + \\ + \sum_{k=1}^{n} \tau \|\bar{g}_t^k\|_{A^{-1}} + \|\bar{y}^{n+1}\|_{A^{-1}},$$

Where

$$\|\bar{\sigma}^{n+1}\|_\xi^2 = \frac{1}{4}\|\bar{\sigma}^{n+1} - \bar{\sigma}^n\|_A^2 + \\ + \|\bar{\sigma}^{n+1} - \bar{\sigma}^n\|_{R-\frac{1}{4}A}^2 \tag{36}$$

We note that from the last equality, considering condition (35), it follows

$$\|\bar{\sigma}^n\|_\xi \leq \|\bar{\sigma}^n\|_A + \|\bar{\sigma}_t^n\|_{\tau^2 R}, \tag{37}$$

If $\overline{\sigma}^n$ is the solution of the difference scheme (34), and $\overline{\sigma}^\lambda$ is the solution of problem (32), then taking into account the method of approximation of problem (32) by scheme (34) from (35) it follows

$$\|\bar{y}^0\|_\xi = \frac{1}{\sqrt{\mu}}\|\bar{y}_t^0\| = 0(\tau^2 + h_i),$$

where $\overline{y}^n = \overline{\sigma}^n - \overline{\sigma}^\lambda$.

Hereafter,

$(A\overline{y}^n, \overline{y}^n \geq 2\mu(\tau^2 R\overline{y}^n, \overline{y}^n) \geq \frac{2\mu}{3\lambda+2\mu}\|\overline{y}^n\|^2$ i.e. $A \geq \frac{2\mu}{3\lambda+2\mu}E$ or $(A)^{-1} \geq 3\frac{\lambda+2\mu}{2\mu}E$

E is the identity matrix. Thus

$$\|\bar{y}^n\|_{A^{-1}} \leq \sqrt{\frac{3\lambda + 2\mu}{2\mu}}\|\bar{y}^n\| = O\,(\lambda^{\frac{1}{2}}h_i + \lambda^{\frac{1}{2}}t^2)$$

$$\sum_{k=1}^{n} \tau \|\bar{y}_t^k\|_{A^{-1}} = O\,(\lambda^{\frac{1}{2}}h_i + \lambda^{\frac{1}{2}}\tau^2)$$

where

$\|\bar{y}^n\|^2 = (\bar{y}^n, \bar{y}^n), (u,v) = \sum_{h_1}^{l_1}\sum_{h_2}^{l_2}\sum_{h_3}^{l_3} u \cdot v \cdot H$

H isa volume element which sets in accordance with the point M (x_1, x_2, x_3), consequently $H = h_1 \cdot h_2 \cdot h_3$, if the point M is internal;

$H = 0{,}5 \cdot h_1 \cdot h_2 \cdot h_3$, if M is a boundary; $H = 0{,}25 \cdot h_1 \cdot h_2 \cdot h_3$ if the point M lies on the edge and $H = 0{,}125 \cdot h_1 \cdot h_2 \cdot h_3$. Returning to (36), we come to an estimate for the discrepancy

$$\|\bar{y}^n\|_\xi = O\left(\lambda^{\frac{1}{2}}\tau^2 + \lambda^{\frac{1}{2}}h_i\right) \tag{38}$$

Having an estimate of (38), we consider the behavior of the solution of the scheme (34) when λ→ ∞.

Theorem. The solution of the difference

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

problem (34) $\overline{\sigma}_h^{\lambda}$ converges to the solution of the dynamic problem of elasticity theory for an incompressible medium $\overline{\sigma}$ (29) when $\lambda \to \infty$, $\tau \to 0$, $h_i \to 0$, $\frac{\tau^2}{h_i^2} \leq \frac{1}{3(3\lambda+2\mu)}$ and estimate is fair

$$\left\|\bar{\sigma}_h^{\lambda}-\bar{\sigma}\right\|_{\xi} \leq O\left(\lambda^{\frac{1}{2}}h_i+\lambda^{\frac{1}{2}}\tau^2+\lambda^{-1}\right) \quad (39)$$

Proof.

By the triangle inequality

$$\left\|\bar{\sigma}_h^{\lambda}-\bar{\sigma}\right\|_{\xi} \leq \left\|\bar{\sigma}_h^{\lambda}-\bar{\sigma}\right\|_{\xi}+\left\|\bar{\sigma}_h^{\lambda}-\bar{\sigma}\right\|_{\xi} \quad (40)$$

As $\overline{\sigma}_h^{\lambda}-\overline{\sigma}^{\lambda}=\overline{y}^n$, then for the first syllable of the right part (40), the score (38) is valid. Let us denote

$$\bar{\omega}=\overline{\sigma}^{\lambda}-\overline{\sigma}$$

Then

$$\left\|\bar{\omega}^{n+1}\right\|_{\theta} \leq \left\|\bar{\omega}^{n+1}\right\|_{A}+\|\bar{\omega}_t^{n+1}\|_{\tau^2 R} \quad (41)$$

We have

$$\left\|\bar{\omega}^{n+1}\right\|_{\theta}=\sqrt{\textstyle\sum_{i=1}^{3}\left\|L_{ih}\bar{\omega}^{n+1}\right\|^2}=O(\lambda^{-1}+h_i)$$

Further

$$\|\bar{\omega}^{n+1}\|_{\tau^2 R}^2=\frac{1}{M}\sum_{i=1}^{3}\left\|\omega_{ij,t}^{n+1}\right\|^2+$$

$$+\frac{1}{(3\lambda+2\mu)}\{\sum_{i=1}^{3}\left\|\omega_{ii,t}^{n+1}\right\|^2+$$

$$+\frac{\lambda}{2\mu}\textstyle\sum_{i,j=1}^{3}\left\|\omega_{ii,t}^{n+1}-\omega_{jj,t}^{n+1}\right\|^2\ \}, \text{ as}$$

$$\left\|\omega_{ij,t}^{n+1}\right\|^2=\left\|2\mu\varepsilon_{ij,t}^{n+1}\right\|^2=O(\lambda^{-2}+h_i)$$

$$\left\|\omega_{ii,t}^{n+1}-\omega_{jj,t}^{n+1}\right\|^2=\left\|2\mu(\varepsilon_{ii,t}^{n+1}-\varepsilon_{jj,t}^{n+1}\right\|^2=O(\lambda^{-2}+h_i)$$

$$\|\bar{\omega}_t^{n+1}\|^2{}_{\tau^2 R}=O(\lambda^{-2}+h_i^2),$$

for (41) we will have

$$\left\|\overline{\sigma}^{\lambda}-\overline{\sigma}\right\|_{\theta}=O(\lambda^{-1}+h_i)$$

Continuing (40), using (35) and the last estimate, we will get (41). Let us note that due to condition (35), the estimate (38) can be written as

$$\left\|\overline{\sigma}_h^{\lambda}-\overline{\sigma}^{\lambda}\right\|_{\xi}=O\left(\lambda^{\frac{1}{2}}h_i+\lambda^{-1}\right) \quad (42)$$

The construction of the estimate (42) shows that at a fixed λ, with respect to refinement of the grid step h_i, starting from a certain one, it cannot be expected increases in accuracy.

A similar situation occurs if you increase λ for a fixed h_i. It would appear reasonable to require that the terms of the evaluation (42) have the same order. Then, assuming that $h_i=O\left(\lambda^{-3/2}\right)$, the rate of convergence of the solution of the difference scheme (34) to the solution of the dynamic problem of elasticity theory for an incompressible medium (29) can be expressed in terms of

λ: $\left\|\overline{\sigma}_h^{\lambda}-\overline{\sigma}\right\|_{\xi}=O\left(\frac{1}{\lambda}\right)$ if we take $\lambda=O\left(h_i^{-\frac{2}{3}}\right)$, the estimate takes the next view

$$\left\|\bar{\sigma}_h^{\lambda}-\bar{\sigma}\right\|_{\xi}=O\left(h_i^{\frac{2}{3}}\right), \quad (43)$$

By changing the method of approximation of volume forces vector $\bar{f}$ in scheme (34) so that

$$(f_{ih}^*)_{xj}+\left(f_{jh}^*\right)_{xi}=\frac{\partial f_i}{\partial x_j}+\frac{\partial f_j}{\partial x_i}+O(h_i^2)$$

Then the difference scheme will have the order $O(\tau^2+h_i^2)$. Therefore, the rate of convergence of its solutions to the solution of the problem of elasticity theory for an incompressible material is expressed by the estimate

$$\left\|\overline{\sigma}_h^{\lambda}-\overline{\sigma}\right\|_{\xi}=O\left(h_i^{\frac{4}{3}}\right).$$

3. CONCLUSIONS

In general, for approximation of the problem (32), a difference scheme can be constructed with the approximation order $O\left(\tau^{\propto}+h_i^{\beta}\right)$, where the parameters τ and h_i are subordinated to the condition $\tau^{\gamma_1} \leq c h_i^{\gamma_1}$, then from (40) follows

$$\left\|\overline{\sigma}_h^{\lambda}-\overline{\sigma}\right\|_{\xi} \leq \left\|\overline{\sigma}_h^{\lambda}-\overline{\sigma}^{\lambda}\right\|_{\xi}+\left\|\overline{\sigma}^{\lambda}-\overline{\sigma}\right\|_{\xi}=$$

$$=O\left[\left(\tau^{\propto}+h_i^{\beta}\right)\cdot\lambda^{\frac{1}{2}}+\lambda^{-1}\right]=$$

$$=O\left(\left(h_i^{\frac{\propto\gamma_2}{\gamma_1}}+h_i^{\beta}\right)\cdot\lambda^{\frac{1}{2}}+\lambda^{-1}\right),$$

if now it has completed

$\beta \leq \frac{\propto\gamma_2}{\gamma_1}$, by taking $\lambda=O\left(h_i^{-\frac{2\beta}{3}}\right)$, in place of (43) we will find

$$\left\|\overline{\sigma}_h^{\lambda}-\overline{\sigma}\right\|_{\xi}=\left(h_i^{\frac{2}{3}\beta}\right).$$

We present the results of numerical calculations. Numerical experiments were carried out to test the possibilities of practical use of the difference scheme (32, 33) in solving the dynamic problem of elasticity theory for an incompressible medium (3-

Journal of Theoretical and Applied Information Technology
30th April 2022. Vol.100. No 8

ISSN: 1992-8645 www.jatit.org E-ISSN: 1817-3195

8). The exact solution is chosen as the test

$$\bar{u} = \{(x-1)\sin t; (1-y)\sin t\}$$
$$\sigma_{11} = 2\mu(1-x^2-y^2)\sin t,$$
$$\sigma_{22} = -2\mu(1+x^2+y^2)\sin t,$$
$$\sigma_{12} = 0$$

a plane dynamic problem of elasticity theory for an incompressible medium

$$\frac{\partial^2 Ui}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} + f_i$$

$\operatorname{div} \bar{u} = 0.$

$\bar{f} = \{(4\mu x + 1 - x)\sin t;\ (4\mu y + y - 1)\sin t\},$

$u_i(x,0) = 0, \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = x-1, \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = 1-y;$

In the area $D_T = \{0 \le x, y \le 1\} \times [0 \le t \le 0{,}5]$

As the approximate solution of the formulated problem was the solution of the difference scheme (32, 33).

Calculations were carried out for dimensionless Lame coefficients,

$\bar{\lambda} = \frac{\lambda}{E}, \bar{\mu} = \frac{\mu}{E}$ for E=50 MPa. For $\bar{\mu} = 0{,}5$ solutions of the difference scheme were obtained at different values of the $\bar{\lambda}$, on the grids 10× 10, 20× 20, 40× 40. The time step of the τ was chosen based on the stability conditions of the difference scheme. The analysis of the numerical results shown on the tables showed the validity of the evaluation $\bar{\lambda} = O(h^{-4/3})$, definitely on the grid 10× 10 a with step h =0,1 value $\bar{\lambda}$, according to this formula $\bar{\lambda} \approx 20$, and the score for smaller $\bar{\lambda}$ or larger 20 as can be seen from the table leads to an increase in the error rate. A similar situation occurs on a 20× 20 grid.

h =0,05, the value of the $\bar{\lambda} \approx 50$ according to the formula is optimal because the norm of discrepancy is minimal. And finally, on the 40× 40 grid, h = =0,025 the optimal value of the $\bar{\lambda} \approx 150$, which confirms the validity of the relationship between $\bar{\lambda}$ and h.

The tables show the error $W = \|u^{\lambda} - u\|_{L_{i,h}}$ a depending on $\bar{\lambda}$.

$$W = \|u^{\lambda} - u\| = \sqrt{\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(u_{ij}^{\lambda,n} - u_{ij}^{n})^2 h^2}$$

for a fixed time, layer i.e., n=15, n=100, n=120, $t_n = n\tau$, n- number of a certain layer $h_1 = h_2 = h = 0{,}1$.

Table 1: Grid 10× 10, $\lambda_{opt} \approx 20$.

h	N × N	n	λ	W
0,1	10× 10	15	10	0, 0836
0,1	10× 10	100	10	0, 0942
0,1	10× 10	120	10	0, 0967
0,1	10× 10	15	50	0, 0996
0,1	10× 10	100	50	0, 0987
0,1	10× 10	120	50	0, 0998
0,1	10× 10	15	20	0, 0112
0,1	10× 10	100	20	0, 0108
0,1	10× 10	120	20	0, 0124

Table 2: Grid 20× 20, $\lambda_{opt} \approx 50$

h	N × N	n	λ	W
0,05	20× 20	15	20	0, 0847
0,05	20× 20	100	20	0, 0975
0,05	20× 20	120	20	0, 0963
0,05	20× 20	15	50	0, 0114
0,05	20× 20	100	50	0, 0110
0,05	20× 20	120	50	0, 0107
0,05	20× 20	15	100	0, 0765
0,05	20× 20	100	100	0, 0836
0,05	20× 20	120	100	0, 0912

Table 3: Grid 40× 40, $\lambda_{opt} \approx 150$.

h	N × N	n	λ	W
0,025	40× 40	15	100	0, 0938
0,025	40× 40	100	100	0, 0879
0,025	40× 40	120	100	0, 0798
0,025	40× 40	15	150	0, 0142
0,025	40× 40	100	150	0, 0132
0,025	40× 40	120	150	0, 0127
0,025	40× 40	15	250	0, 0786
0,025	40× 40	100	250	0, 0862
0,025	40× 40	120	250	0, 0987

4. CONCLUSION

Then, with numerical realizations, schemes are indicated, the order of approximation error has the form proposed above $\|\bar{\sigma}_h^{\lambda} - \bar{\sigma}\|_{\xi} = \left(h_i^{\frac{2}{3}\beta}\right)$, which allows calculations to be carried out with such an error.

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