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SOLVING FUZZY FRACTIONAL NUCLEAR DECAY EQUATIONS BY REPRODUCING KERNEL HILBERT SPACE METHOD

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ABSTRACT

This paper deals with the solutions of fuzzy fractional nuclear decay equations (FNDEs) under Caputo H-differentiability by Reproducing Kernel Hilbert Space Method (RKHSM). The equations were reformulated under the influence of fuzzy logic. The Residual Power Series has been applied in solving differential equations with fuzzy initial conditions. This method is illustrated by solving two examples. The solution method has shown high efficiency and accuracy in the case of comparison with the exact solution and another method.

Keywords: *Nuclear Decay Equations, Reproducing Kernel Hilbert Space Method.*

1. INTRODUCTION

The initial form of the fuzzy differential equations appeared through the suggestions of Chang and Zadeh (Sheldon& Zadeh, 1972) these equations quickly grew in the applications of many sciences, physics, engineering, energy, and other sciences, at the present time, the fuzzy differential equations have overlapped with the fractional differential equations with many applications (Allahviranloo, 2021), and deep complications have emerged to find solutions to many equations, one of the equations with important applications is what is known as the Nuclear Decay Equations (NDEs) as a kind of fractional Riccati differential equations (Khodadadi & Karabacak, 2016 Marakhtanov & Okunev,2018) the solution of this form of equations is extremely important because of its applications in the field of nuclear energy, so in this paper we will put a solution to this type of equations with fuzzy conditions by the RKHSM. Therefore, we consider the next form of NDEs as the following:

$$\begin{cases} D_t^{\beta} y(t) = -ay(t), \ 0 < t \le 1, 0 \le \beta \le 1, \\ y(0) = d \end{cases} \tag{1}$$

Where D_t^{β} is the Caputo fractional derivative for order β, a is a variable, and is the count of total radionuclide's existing in any radioactive, It can be observed that Eq.1 is a general formulation of where the initial value d .

According to the organization of this paper, the first section will include a simple introduction that contains the general form of the equations, while the second section includes the main definitions. Third section include the formulation of the NDEs under fuzzy conditions, and in the fourth section the solution steps will be formulated, while the fifth section includes examples to demonstrate the effectiveness of solution, and the conclusion will be in the final section.

2. MAIN DEFINITIONS

In this section, the most important definitions that will be used in this research are presented, which include the main definitions of Caputo fractional derivatives and fuzzy numbers.

The definition of Caputo is one of the most important definitions of the fractional derivation

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procedure, Caputo definition of fractional derivatives includes the following cases:

Definition 1 (Albzeirat et al., 2019). The left Caputo fraction derivatives is defined as:

$$D^{c,\beta}_{t\in[a,b]}y(t) = \frac{1}{\Gamma(\lceil \beta \rceil - \beta} \int_a^t (t-\tau)^{\lceil \beta \rceil - \beta - 1} y^{(\lceil \beta \rceil)}\tau \ d\tau . \quad (2)$$

Definition 2 (Albzeirat et al., 2019). The right Caputo fraction derivatives is defined as:

$$D^{c,\beta}_{t\in[a,b]}y(t) = \frac{(-1)^{\lceil \beta \rceil}}{\Gamma(\lceil \beta \rceil - \beta} \int_a^t (t-\tau)^{\lceil \beta \rceil - \beta - 1} y^{(\lceil \beta \rceil)}\tau \ d\tau . \quad (3)$$

A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function.

Definition 3 (Albzeirat et al., 2018): Let $u_F(t)\in R_F^n$ and $r\in[0,\ 1]$. The $r-\text{cut}$ of $u_F(t)$ is the crisp set $[u_F(t)]$ that contains all elements with membership degree in $u_F(t)$ greater than or equal to r, that is $[u_F(t)]^r = \{t\in R : u_F(t)\geq r\}$. For a fuzzy interval $u_F(t)$, its $r-\text{cut}$ is closed and bounded in R. We denoted them by:

$[u_F(t)]^r = [u_{1,1r}(t),\ u_{1,2r}(t)]$ where $u_{1,1r} = \min\{t : t\in [u_F(t)]^r\}$ and $u_{1,2r} = \max\{t : t\in [u_F(t)]^r\}$ for each $r\in [0,\ 1]$.

Definition 4 (Albzeirat et al., 2018): $u_F(t)\in R_F$, u_F is triangular if its membership function has the following form:

$$u_F(t)=\begin{cases} 0, & t<a, \\ \dfrac{t-a}{b-a}, & a\leq t\leq b, \\ \dfrac{c-t}{c-b}, & b\leq t\leq c, \\ 0, & t>c. \end{cases} \quad (4)$$

Where it's r-cut is simply $[u_F(t)]^r = [a+r(b-a),\ c\text{-}r(c\text{-}b)]$, for any $r\in[0,\ 1]$.

3. FUZZY FRACTIONAL NUCLEAR DECAY EQUATIONS

Radioactive decay occur naturally or are artificially produced in nuclear reactors, cyclotrons, particle accelerators, or radionuclide generators (Yatsevich & Honda, 1997). What happens naturally does not pose a danger because it occurs under natural conditions and within a gradual time that does not cause massive energy, as the force of energy emission gradually fades with time, while the danger appears when these emissions occur abnormally in nuclear reactors, cyclotrons, particle accelerators, or radionuclide generators which may cause disasters. Therefore it is imperative to control the variables and their fuzzy conditions to make nuclear applications less dangerous and more accurate in control. The logic of reformulating Eq.1 into the fuzzy formula comes through the nature of the data, where the equation represents an atom that contains extra energy that makes the behavior of the reaction unstable according to the change of materials, time, or surrounding conditions, and all these data are considered within the Fuzzy description of radionuclides. Based on this logic, NDEs Eq.1 can be formulated within a fuzzy formula into FDEs.

By calling the previous definitions of fuzzy, the equation will be reformulated according to definitions as follows:

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If NDEs under effect of fuzzy initial condition:

$$\begin{cases} D_t^{\beta} y(t) = -ay(t),\ 0<t\le 1, 0\le\beta\le 1, \\ y(0)=d \end{cases} \quad (5)$$

Where D_t^{β} is the Caputo fractional derivative for order β , a is a variable, and $y(t)$ is the count of total radionuclide's existing in any radioactive, It can be observed that Eq.5 is a general formulation of FNDEs where the initial value d is a fuzzy numbers.

By fuzzy properties, Eq.7 takes the following form:

$$\begin{cases} D_{t,r1}^{\beta} y_{r1}(t) = -ay_{r1}(t),\ 0<t\le 1, 0\le\beta\le 1, \\ D_{t,r2}^{\beta} y(t) = -ay_{r1}(t),\ 0<t\le 1, 0\le\beta\le 1, \\ y_{r1}(0)=r1, \\ y_{r2}(0)=r2, \end{cases} \quad (6)$$

Where $D_{t,r1}^{\beta}, D_{t,r2}^{\beta}$ is the Caputo fractional derivative for order β , a is a variable, and $y_{r1}(t)$, and $y_{r2}(t)$ is the count of total radionuclide's existing in any radioactive, where the initial value d is a fuzzy numbers.

4. REPRODUCING KERNEL HILBERT SPACE METHOD

In this part of the study the main steps of the solution using Reproducing Kernel Hilbert Space Method will be laid out.

Step 1: After homogenising the initial conditions in Equation (6) and using $r-$cut definition, we apply the Riemann-Liouville for both sides of equations, which leads to new form for Equation (6) as follow:

$$\begin{aligned} &I_{R,[a,z]}^{\beta} D_{c,[a,z]}^{\beta} y_{1,1r}^{*}(z) = I_{R,[a,z]}^{\beta} g_{1,1r}^{*}(z, y_{1,1r}^{*}(z), y_{1,2r}^{*}(z)), \\ &y_{1,1r}^{*}(z_0) = 0,\ y_{1,2r}^{*}(z) = 0 \end{aligned} \quad (7)$$

$$\begin{aligned} &I_{RL,[a,z]}^{\beta} D_{c,[a,z]}^{\beta} y_{1,2r}^{*}(z) = I_{RL,[a,z]}^{\beta} g_{1,2r}^{*}(z, y_{1,1r}^{*}(z), y_{1,2r}^{*}(z)), \\ &y_{1,1r}^{*}(z_0) = 0,\ y_{1,2r}^{*}(z) = 0 \end{aligned} \quad (8)$$

By $I_{RL}^{\alpha} D_c^{\alpha} y(t) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{z^k}{k!}$, $z>0,\ n-1<\alpha\le n.$

That's leads to next form:

$$\begin{aligned} &y_{1.1r}^{*}(z) - y_{1,1r}^{*}(z_0) - y_{1,2r}^{*}(z_0) \\ &= I_{RL,[a,z]}^{\beta} g_{1,1r}^{*}(z, y_{1,1r}^{*}(z), y_{1,2r}^{*}(z)) \end{aligned} \quad (9)$$

$$\begin{aligned} &y_{1.2r}^{*}(z) - y_{1,1r}^{*}(z_0) - y_{1,2r}^{*}(z_0) \\ &= I_{RL,[a,z]}^{\beta} g_{1,2r}^{*}(z, y_{1,1r}^{*}(z), y_{1,2r}^{*}(z)) \end{aligned} \quad (5.5)$$

Sine $y_{1,1r}^{*}(z_0)=0,\ y_{1,2r}^{*}(z)=0$ that leads to the next form:

$$\begin{aligned} &y_{1.1r}^{*}(z) = I_{RL,[a,z]}^{\beta} g_{1,1r}^{*}(z, y_{1,1r}^{*}(z), y_{1,2r}^{*}(z)) \\ &y_{1.2r}^{*}(z) = I_{RL,[a,z]}^{\beta} g_{1,2r}^{*}(z, y_{1,1r}^{*}(z), y_{1,2r}^{*}(z)) \end{aligned} \quad (10)$$

Therefore,

$$y^*_{1.1r}(z) = \frac{1}{\Gamma(\alpha)}\int_a^z (z-t)^{\alpha-1}\left(g^*_{1,1r}(t, y^*_{1,1r}(t), y^*_{1,2r}(t))\right)dt, \; z > a$$
$$y^*_{1.2r}(z) = \frac{1}{\Gamma(\alpha)}\int_a^z (z-t)^{\alpha-1}\left(g^*_{1,2r}(t, y^*_{1,1r}(t), y^*_{1,2r}(t))\right)dt, \; z > a \qquad (11)$$

Then,

$$\begin{aligned} y^*_{1.1r}(z) &= G^*_{1,1r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z)) \\ y^*_{1.2r}(z) &= G^*_{1,2r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z)) \\ y^*_{1,1r}(z_0) &= 0, \; y^*_{1,2r}(z) = 0 \end{aligned} \qquad (12)$$

That's led to the next form for the solution of $y_{1,1r}(z)$ and $y_{1,2r}(z)$ as the following

$$\begin{aligned} y_{1,1r}(z) &= G^*_{1,1r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z)) + z_{1,1r} \\ y_{1,2r}(z) &= G^*_{1,2r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z)) + z_{1,2r} \end{aligned} \qquad (13)$$

Now, to solve the last system by implements of the reproducing kernel theory, we adhere to the following steps;

Step 2: By using reproducing kernel theory tools for the first order need to construct the space $FS_2^{m+1}[a, b]$, where F mention for first and S for space. In the equation (5.1) $m = 1$, that's leads to constructing the space $FS_2^2[a, b]$.

Step 3: The inner product in $FS_2^2[a, b]$ is expressed by:

$$\langle y, v\rangle_{FS_2^2[a,b],} = \sum_{i=0}^{1} y^{(i)}(a)\, v^{(i)}(a) + \int_0^1 y^{(2)}(z)\, v^{(2)}(z)\, dz, \; \forall u, \qquad (14)$$
$v \in FS_2^2[a, b]$,

And the norm is given by:

$$\|y(z)\|_{FS_2^2[a,b]} = \sqrt{\langle y(z), y(z)\rangle_{FS_2^2[a,b]}} \qquad (15)$$

Step 4: space $FS_2^2[a, b]$ is a reproducing kernel Hilbert space, that leads to the next result: for each fixed $z \in [a, b]$, there exists $R_2K_z(y) \in FS_2^2[a, b]$ such that $\langle y(y), R_2K_z(y)\rangle_{FS_2^2[a,b]} = y(z)$ for any $y(y) \in FS_2^2[a, b]$ and $y \in [a, b]$.

The $R_2K_z(y)$ given by:

$$R_2K_z(y) = \begin{cases} 1 + zy + \dfrac{yz^2}{2} - \dfrac{z^3}{6}, & y \le z \\ 1 + zy + \dfrac{zy^2}{2} - \dfrac{y^3}{6}, & y > z \end{cases} \qquad (16)$$

To solve Equation (5.8) as asystem by reproducing kernel theory need to define a differential operator

$$L_{m,jr} : FS_2^2[a, b] \to FS_2^1[a, b] \text{ where } m = 1, j = 1, 2. \qquad (17)$$

In this step the equations (5.8) converted into next form:

$$\begin{aligned} L_{1,mr} y^*_{1,1r}(z) &= G^*_{1,1r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z)), \\ y^*_{1,1r}(z_0) &= 0, \; y^*_{1,2r}(z) = 0 \end{aligned} \qquad (18)$$

Where
$z \in [a, b]$, $y^*_{1,1r}(z)$ and $y^*_{1,2r}(z) \in FS_2^2[a, b]$,
$G^*_{1,1r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z))$
and
$G^*_{1,2r}(z, y^*_{1,1r}(z), y^*_{1,2r}(z)) \in FS_2^2[a, b]$.

Step 5: By apply the differential operator:

$$L_{m,jr} : FS_2^2[a, b] \to FS_2^1[a, b] \text{ where } m = 1, j = 1, 2. \qquad (19)$$

Then equation (19) converted into the next form:

$$L_{1,1r}y^*_{1,1r}(z)=G^*_{1,1r}(z,\, y^*_{1,1r}(z),\, y^*_{1,2r}(z)),$$
$$L_{1,2r}y^*_{1,2r}(z)=G^*_{1,2r}(z,\, y^*_{1,1r}(z),\, y^*_{1,2r}(z)), \quad (20)$$
$$y^*_{1,1r}(z_0)=0,\; y^*_{1,2r}(z)=0$$

Where $z\in[a,\,b]$, $y^*_{1,1r}(z)$ and $y^*_{1,1r}(z)\in FS^2_2[a,\,b]$,

$G^*_{1,1r}(z,\, y^*_{1,1r}(z),\, y^*_{1,2r}(z))$ and

$G^*_{1,12r}(z,\, y^*_{1,1r}(z),\, y^*_{1,2r}(z)) \in FS^1_2[a,b]$.

Step 6: In this step need to construct an orthogonal function system of $FS^1_2[a,b]$, for do that, let $\{z_k\}^{\infty}_{k=1}$ be a countable dense set in $[a,\,b]$.

Let $e_{k,m,jr}(z)=R_1K_{z_k}(z)$ where $R_1K_z(z)$ is the RK of $FS^1_2[a,\,b]$, that is given $\forall\, y^*_{m,jr}(z)\in FS^1_2[a,\,b]$, $j=1,\,2$.

It follows:

$$\langle y^*_{1,jr}(z),\, e_{k,1,1r}(z)\rangle_{FS^1_2[a,\,b]}=\langle y^*_{1,jr}(z),\, RK_{z_k}(z)\rangle$$

Additionally, let $L^{ad}_{1,jr}e_{k,1,jr}(z)=\Psi_{k,1,jr}(z)$ where $L^{ad}_{1,jr}$ is an adjoint operator of $L_{1,jr}$, $j=1,2$.

Thus, $\langle y^*_{k,1,jr}(z),\, \Psi_{k,1,jr}(z)\rangle_{FS^2_2[a,\,b]}$

$$=\langle y^*_{k,1,jr}(z),\, L^{ad}_{1,jr}e_{k,1,jr}(z)\rangle_{FS^2_2[a,\,b]}$$
$$=\langle L_{1,jr}y^*_{k,1,jr}(z),\, e_{k,1,jr}(z)\rangle_{FS^1_2[a,\,b]}$$
$$=L_{1,jr}y^*_{1,jr}(z_k),\; j=1,\,2,\; k=1,\,2,\,\ldots$$

Hence, $\Psi_{k,1,jr}(z)$ expressed by the next form:

$$\Psi_{k,1,jr}(z)=L^{ad}_{1,jr}e_{k,1,jr}(z)=$$
$$=\langle L^{ad}_{1,jr}e_{k,1,jr}(z),\, R_2K_{z_k}(z)\rangle_{FS^2_2[a,\,b]}$$
$$=\langle e_{k,1,jr}(z),\, L_{1,jr}R_2K_{z_k}(z)\rangle_{FS^1_2[a,\,b]}$$
$$=L_{1,jr}\, R_2K_{z_k}(z)\big|_{y=z_k},\; j=1,\,2,\; k=1,\,2,\,\ldots$$

Theorem 1. For the Equations (5.15), suppose that inverse operator for $L_{1,jr}$, $j=1,\,2$ exists, there for, if $\{z_k\}^{\infty}_{k=1}$ is dense in $[a,\,b]$, then $\{\Psi_{k,m,jr}(z)\}^{(\infty,1,2r)}_{(k,m,jr)=(1,1,1r)}$ is the complete function of $FS^2_2[a,\,b]$.

Proof :

$\forall$ fixed $y^*_{1,1r}(z)$ and $y^*_{1,2r}(z)\in FS^2_2[a\; b]$. let,

$$\langle y^*_{1,jr}(z), L^{ad}_{1,jr}\; e_{k,1,jr}(z)\rangle_{FS^2_2}$$
$$=\langle L_{1,jr}y^*_{1,jr}(z),\; e_{k,1,jr}\rangle_{FS^1_2}=0,\; j=1,\;2.$$

Note that $\{z_k\}^{\infty}_{k=1}$ is dense in $[a,\,b]$.

Therefore, $L_{1,1r}\, y^*_{1,1r}(z)=0,\; L_{1,2r}\, y^*_{1,2r}(z)=0$. $y^*_{1,1r}(z)=0$ and $y^*_{1,2r}(z)=0$, from the $L^{ad}_{1,1r}$ and $L^{ad}_{1,2r}$, and the continuity of $y^*_{1,1r}(z)$ and $y^*_{1,2r}(z)$. proof is complete.

Now, by using Gram-Schmidt orthonormalization construct an orthonormal system $\{\overline{\Psi}_{k,q,sr}\}^{(\infty,1,jr)}_{(k,q,sr)=(1,1,1r)}$ of $FS^2_2[a,\,b]$, via the Gram-Schmidt orthogonalization process where

$$\overline{\psi}_{k,m}(z)=\sum_{l=1}^{k}\sum_{m=1}^{2}\beta_{l,m}\psi_{l,m}(z) \qquad (21)$$

Where $\beta_{k,m}$ is orthogonalization coefficient.

We rewrite all equation by mention $1,1r=1$ and $1,2r=2$.

Theorem 2. Let $\{z_k\}^{\infty}_{k=1}$ be dense in $[a,\,\mathrm{b}]$, and the solution (20) is unique on $FS^2_2[a,\,b]$. Then the exact solution is given by

$$\sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{i}\sum_{q=1}^{m}\beta_{l,\mathrm{q}}G^*_m\left(z_l,\, y^*_m(z_l)\right)\overline{\psi_{k,m}}(z)\,(22)$$

The approximate solution $y^*_{m.jr}(z)$ is obtained by taking finitely many terms for (5.17) in the representation form of

$$y^{*N}_m=\sum_{k=1}^{N}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,\mathrm{q}}G^*_l\left(z_l,\, y^*_m(z_l)\right)\overline{\psi_{k,m}}(z) \quad (23)$$

Proof: By applying Theorem 1, we can see that. $\{\Psi_{k,m}(z)\}^{(\infty,2)}_{(k,m)=(1,1)}$ is the complete orthonormal basis,

$$\langle y^*_m(z),\, e_m(z)\rangle_{FS^1_2[a,\,b]}=\langle y^*_m(z),\, R_1K_{z_k}(z)\rangle=y^*_m(z_k)$$

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For each

$$y_m^*(z_k)\, y_m^*(z) \in FS_2^2[a,\ b],\ \sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \overline{\psi}_{k,mr}(z)\right\rangle \overline{\psi}_{k,m}(z)$$

is the Fourier series expansion of $\left\{\Psi_{k,m}(z)\right\}_{(k,m)=(1,1)}^{(\infty,2)}$, the series is convergent

$$\sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \overline{\psi}_{k,m}(z)\right\rangle \overline{\psi}_{k,m}(z) \tag{24}$$

$$y_m^*(z) = \sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \overline{\psi}_{k,m}(z)\right\rangle \overline{\psi}_{k,m}(z) \tag{25}$$

$$y_{m,jr}^*(z) = \sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\psi_{l,q}(z)\right\rangle \overline{\psi}_{k,m}(z) \tag{26}$$

$$y_{m,jr}^*(z) = \sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left\langle y_m^*(z),\ L_m^{ad} e_{l,q}(z)\right\rangle \overline{\psi}_{k,m}(z) \tag{27}$$

$$y_{m,jr}^*(z) = \sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left\langle L_m y_m^*(z),\ e_{l,q}(z)\right\rangle \overline{\psi}_{k,m}(z) \tag{28}$$

I $FS_2^2[a,\ b]$.

$$y_{m,jr}^*(z) = \sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q} G_q^*\left(z,\ y_m^*(z_l)\right) \overline{\psi}_{k,m}(z) \tag{29}$$

$$y_{m,jr}^{*N}(z) = \sum_{k=1}^{N}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q} G_q^*\left(z,\ y_q^*(z_l)\right) \overline{\psi}_{k,m}(z) \tag{30}$$

proof is complete.

As results:

1:- The approximate solution $y_m(z),\ m=1,\ 2.$ by using Riemann-Liouville integral operator can be presented by:

$$\begin{bmatrix} \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\frac{1}{\Gamma(\beta)}\int_a^{z_l}(z_l - t_l)^{\alpha-1}\left(g_{1,2r}^*(t,\ y_{1,1r}^*(t_l),\ y_{1,2r}^*(t_l))\right)dt_l * \overline{\psi}_{km}(z)\right] \\ +z_{1.1r}, \\ \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\frac{1}{\Gamma(\beta)}\int_a^{z_l}(z_l - t_l)^{\alpha-1}\left(g_{1,1r}^*(t,\ y_{1,1r}^*(t_l),\ y_{1,2r}^*(t_l))\right)dt_l * \overline{\psi}_{km}(z)\right] \\ +z_{1,1r} \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}G_q^*\left(z_l,\ y_{F1}^*(z_l)\right)\overline{\psi}_{km}(z)\right] + z_{1.1r}, \\ \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}G_q^*\left(z_l,\ y_{F1}^*(z_l)\right)\overline{\psi}_{km}(z)\right] + z_{1,1r} \end{bmatrix} = \left[y_{1r}^N(z),\ y_{2r}^N(z)\right] \tag{32}$$

$$y_{mr}^{NR}(z) = \begin{pmatrix} N - \text{terms by Riemann-Liouville integral operator}, \\ y_{odd(m)r_q}^*(z) + z_{1,1r},\ y_{even(m)r_q}^*(z) + z_{1,2r} \end{pmatrix} = \left[y_{1,1r}(z),\ \mathrm{y}_{1,2r}(z)\right] \tag{33}$$

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2: The approximate solution $y_m(z),\ m=1,\ 2.$ by using Caputo fractional derivative operator can be presented by:

$$\begin{bmatrix} \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left(g^*_{1,1r}(t,\ y^*_{1,1r}(t_l),\ y^*_{1,2r}(t_l))\right)*\overline{\psi}_{km}(z)\right] \\ +z_{1.1r}, \\ \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left(g^*_{1,2r}(t,\ y^*_{1,1r}(t_l),\ y^*_{1,2r}(t_l))\right)*\overline{\psi}_{km}(z)\right] \\ +z_{1,1r} \end{bmatrix} \tag{34}$$

$$=\left[y^N_{1r}(z),\ y^N_{2r}(z)\right]$$

$$y^{NC}_{mr}(z)=\begin{pmatrix} N-\text{terms by Caputo fractional derivative operator}, \\ y^*_{odd(m)r_q}(z)+z_{1,1r},\ y^*_{even(m)r_q}(z)+z_{1,2r} \end{pmatrix} \tag{35}$$

$$=\left[y_{1,1r}(z),\ y_{1,2r}(z)\right]$$

5. NUMERICAL EXAMPLES

Consider the following Fuzzy Fractional Nuclear Decay Equation

$$\begin{cases} D_t^{\beta}y(t)=-ay(t),\ 0<t\le 1,\ 0\le\beta\le 1, \\ y(0)=d=[0.5+0.5*r,\ 1.5\text{-}0.5*r] \end{cases} \tag{36}$$

Where D_t^{β} is the Caputo fractional derivative for order $\beta=0.5$, $a=1$, and $y(t)$ is the count of total radionuclide's existing in any radioactive, and the initial value d is a fuzzy numbers, and $r=0,5$.

Where $r=0.5$ then $d=[0.75,\ 1.25]$. Table 1. and Table 2. Show the solution $y_{r1}(t)$, and $y_{r2}(t)$ after applied the last steps.

The exact solution where $\beta=1$ give by:

$$y_{r1}(t)=0.125\mathrm{e}^{-x}$$
$$y_{r2}(t)=1.25\mathrm{e}^{-x}$$

Table 1 Approximate Solution of $y_{r1}(t)$ Compared with the Exact Solution

X	*Exact Solution* $\beta=1$	Approximate Solution $\beta=1$	Approximate Solution $\beta=0.5$	Approximate Solution by VIM (Khodadadi, & Karabacak,2015) $\beta=0.5$
0.1	0.75	0.75	0.75	
0.2	0.6786280635269697	0.6786307869	0.5501221366	0.5427
0.3	0.6140480648084864	0.6140531539	0.487463391	0.4828
0.4	0.5556136655112884	0.5556206471	0.4471287043	0.4440
0.4	0.5027400345267294	0.5027485034	0.4175262196	0.4152
0.5	0.45489799478447507	0.4549076052	0.3942151239	0.3924

Table 2 Approximate Solution Of $y_{r2}(t)$ Compared With The Exact Solution

t	*Exact Solution* $\beta = 1$	Approximate Solution $\beta = 1$	Approximate Solution $\beta = 0.5$	Approximate Solution by VIM (Khodadadi, & Karabacak,2015) $\beta = 0.5$
0.1	1. 25	1. 25	1. 25	0.0
0.2	1.1310467725449493	1.128630787	0.9168702277	0.9045
0.3	1.0234134413474774	1.014053154	0.812438985	0.8047
0.4	0.9260227758521473	0.9056206471	0.7452145072	0.7400
0.4	0.8379000575445492	0.8027485034	0.6958770327	0.6920
0.5	0.7581633246407917	0.7049076052	0.6570252065	0.6540

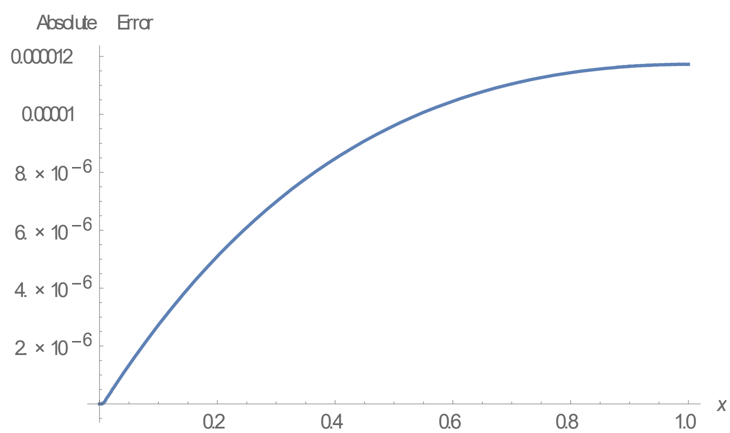

Figure 1 Error Between Exact Solution And Approximate Solution For $y_{r1}(t)$

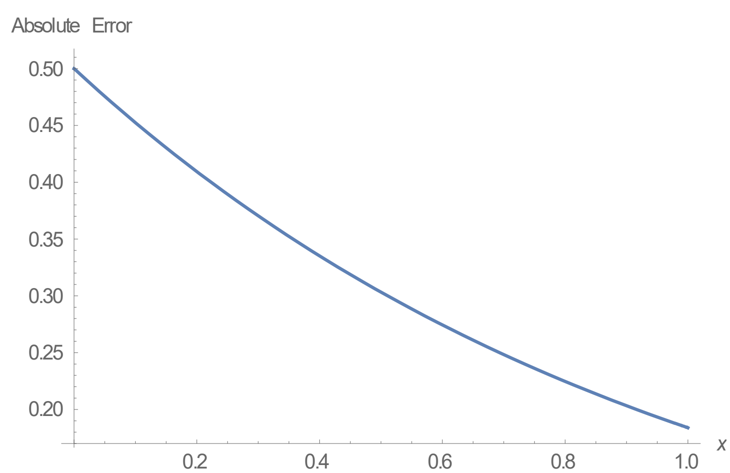

Figure 2 Error Between Exact Solution And Approximate Solution For $y_{r2}(t)$

The Tables 1, and 2 show the accuracy of the solution as the ratio of the approach between the solution and the solution in (Khodadadi, & Karabacak,2015) is very high. The small error percentage appears between the approximate solution and the exact solution shown in Fig 1, and Fig 2.

6. CONCLUSION

In this paper, the RKHSM was investigated to solve FNDEs. Three models of fuzzy equations have been formulated by combining fuzzy logic with the NDEs. The final numerical findings and comparisons for examples in tables above indicate that the RKHSM is an efficient instrument to solving FNDEs.

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